Linear Systems (034032)

tutorial 12

1 Topics

Response to initial conditions, Modal response, Lyapunov stability.

2 Background

2.1 State-Space solution with initial conditions

Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Its the solution in $t > 0$ is

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s$$

In the unforced (autonomous) case, e.g. under $u \equiv 0$, the state evolves according to

$$x(t) = \mathrm{e}^{At}x_0$$

2.2 Lyapunov stability: nonlinear case

An equilibrium $x_{\mathrm{eq}} \in \mathbb{R}^n$ of autonomous dynamics $\dot{x} = f(x)$ is said to be

- **stable** if for every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0) - x_{\mathrm{eq}}\| < \delta \implies \|x(t) - x_{\mathrm{eq}}\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- **asymptotically stable** if it is stable and there is $\delta > 0$ such that

$$\|x(0) - x_{\mathrm{eq}}\| < \delta \implies \lim_{t\to\infty} \|x(t) - x_{\mathrm{eq}}\| = 0$$

The **region of attraction** of an asymptotically stable equilibrium is the set of initial conditions $x(0)$ that generate states x converging to x_{eq}. If the region of attraction is the whole $\mathbb{R}^n$, then the equilibrium is said to be **globally asymptotically stable**.

2.3 Lyapunov stability: linear case

Theorem 1. *An equilibrium of the autonomous dynamics $\dot{x} = Ax$ is*

- *stable iff* $\operatorname{spec}(A) \in \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ *and every imaginary eigenvalue is simple.*
- *asymptotically stable iff* $\operatorname{spec}(A) \in \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$ *and those properties are global.*

2.4 Lyapunov's indirect method

Theorem 2. *Let $\dot{x} = f(x)$ for a continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}^n$, $x_{\text{eq}} \in \mathbb{R}^n$ be its equilibrium, and*

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_{\text{eq}}}$$

be the corresponding Jacobian matrix.

- *If* $\operatorname{spec}(A) \in \mathbb{C} \setminus \bar{\mathbb{C}}_0$, *then* x_{eq} *is asymptotically stable.*
- *If* A *has at least one eigenvalue in* $\mathbb{C}_0$, *then* x_{eq} *is unstable.*

If the rightmost eigenvalue of the Jacobian matrix is on the imaginary axis, then the stability conclusion is ambiguous.

2.5 Modal decomposition

If A is diagonalizable and all its eigenvalues are real, then the initial condition response can be decomposed as

$$x(t) = \sum_{i=1}^{n} \eta_i e^{\lambda_i t} \mu_i(x_0), \quad \mu_i(x_0) := \upsilon_i' x_0 \in \mathbb{R}$$

where $\lambda_i \in \mathbb{R}$ is an eigenvalue of A, $\eta_i \in \mathbb{R}^n$ is the corresponding right eigenvector, and $\upsilon_i \in \mathbb{R}^n$ is the transpose of the ith row of $\begin{bmatrix} \eta_1 & \cdots & \eta_n \end{bmatrix}^{-1}$ such that $\upsilon_i' \eta_j = \delta_{ij}$ (the Dirac delta). The signals $\boldsymbol{\eta_i}$ $\mathbf{exp}_{\boldsymbol{\lambda_i}}$ are known as **modes** of the system and scalars $\boldsymbol{\mu_i(x_0)}$ are their **degrees of excitation**.

2.6 Matlab commands

Some Matlab commands to diagonalize matrices.

- `[T,Lambda] = eig(A);` diagonalizes a square matrix `A`, returning a diagonal matrix `Lambda` comprising the eigenvalues of `A` and the corresponding square similarity transformation matrix `T`.
- `[TR,LambdaR] = cdf2rdf([T,Lambda]);` converts complex diagonal form (the output of `eig`) to a real block diagonal form.

3 Problems

Question 1. Consider the following state space equations.

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} x(t)$$

1. Is the system Lyapunov stable?
2. Find a transformation diagonalizing the matrix A.
3. Carry out the modal decomposition of the system with respect to any initial condition $x(0)$.
4. Find the response for the following two initial conditions. Draw the responses in a phase portrait.

- $x(0) = \begin{bmatrix} 1.05 \\ -1 \end{bmatrix}$
- $x(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Question 2. Given the autonomous dynamics

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x(t)$$

Is their equilibria Lyapunov stable? Carry out the modal decompositions of the responses.

Question 3. Consider the following systems:

1. $\dot{x}(t) = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix} x(t)$
2. $\dot{x}(t) = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix} x(t)$
3. $\dot{x}(t) = \begin{bmatrix} -2 & 8 \\ -8 & -2 \end{bmatrix} x(t)$

For each of them determine whether it is stable and carry out the modal decompositions of the responses.

Question 4. Given the free system in figure 1,

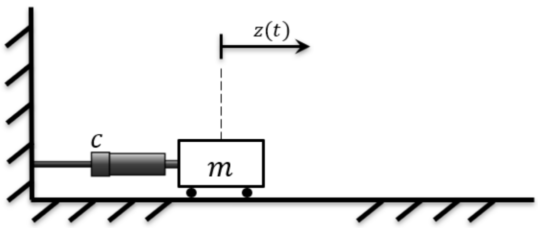

Fig. 1: Mass damper system.

The system is a mass damper system without a spring. The parameter values are $m = 1,\ c = 2$. Also, the state variables are defined

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$$

Is the system stable? Find the response of the system to the initial conditions $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ via the modes.

Question 5. Consider the system shown in Fig. 2.

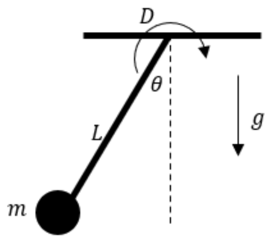

Fig. 2: Pendulum system.

It consists of a pendulum with a point mass m at the end of a massless rod of length L. On the axis of the pendulum, there is a torque due to viscous friction proportional to the speed of rotation with the friction coefficient D.

- Use the state vector

  $$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

  to derive the physical realization of this system.

- Find the equilibrium points and linearize the system around these points.

- Take numerical values of the system parameters as

  $$m = 1\,[\mathrm{kg}]\,, \quad L = 5\,[\mathrm{m}]\,, \quad \text{and} \quad g = 10\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]$$

  and determine if the nonlinear system is stable at the respective equilibrium points via Lyapunov's indirect method for two cases:

  – $D = 50\,[\mathrm{N\,m\,s}]$ (damped system)
  – $D = 0\,[\mathrm{N\,m\,s}]$ (undamped system)