Linear Systems (034032)

TUTORIAL 11

1 Topics

Solving state equations, canonical realizations, and linearization.

2 Background

2.1 State space representation

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$, the state space representation of a system of a continuous-time system $G : u \mapsto y$ is given by

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} .$$

One can show that the impulse response (i.e. $u = \delta$) of this system is given by

$$g(t) = Ce^{At}B\mathbb{1}(t) + D\delta(t).$$

With the impulse response known, we can get the response for any input by using the convolution property of LTI systems.

$$y = g * u$$

2.2 State space to transfer function

Going from a state space representation to a transfer function representation is very simple.

$$G(s) = C(sI - A)^{-1}B + D$$

One can show that $G(s)$ is a rational function of s. Moreover, the poles of $G(s)$ are among the eigenvalues of A.

2.3 Similar realizations

Let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix. Define $\tilde{x} := Tx$, where x is the state of the system. Then, the following realizations are called similar.

$$(TAT^{-1}, TB, CT^{-1}, D) \text{ and } (A, B, C, D)$$

These similar realizations have the same I/O relations (impulse response and transfer function).

2.4 Transfer function to state space

2.4.1 "Physical" realization

If

$$G(s) = \frac{b}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0},$$

then its possible state-space realization is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ -a_0 & -a_1 & \ldots & -a_{n-1} & b \\ \hline 1 & 0 & \ldots & 0 & 0 \end{array}\right].$$

2.4.2 Canonical realization: companion form

Let $G(s)$ be strictly proper, i.e.

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

The state-space realization discussed above, known as the companion form, is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A_{\mathrm{cf}} & B_{\mathrm{cf}} \\ \hline C_{\mathrm{cf}} & D_{\mathrm{cf}} \end{array}\right] := \left[\begin{array}{cccc|c} 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ -a_0 & -a_1 & \ldots & -a_{n-1} & 1 \\ \hline b_0 & b_1 & \ldots & b_{n-1} & 0 \end{array}\right]$$

Note that this is very similar to the "physical" realization in that the matrix A is the same.

2.4.3 Canonical realization: observer form

Let $G(s)$ be strictly proper, i.e.

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

Its state-space realization in observer form has

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A_{\mathrm{of}} & B_{\mathrm{of}} \\ \hline C_{\mathrm{of}} & D_{\mathrm{of}} \end{array}\right] := \left[\begin{array}{cccc|c} -a_{n-1} & 1 & \ldots & 0 & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \ldots & 1 & b_1 \\ -a_0 & 0 & \ldots & 0 & b_0 \\ \hline 1 & 0 & \ldots & 0 & 0 \end{array}\right].$$

2.4.4 Canonical realization: bi-proper case

Let

$$G(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

for $b_n \neq 0$. The trick is to rewrite it as

$$\frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} = b_n + \underbrace{\frac{(b_{n-1} - b_n a_{n-1})s^{n-1} + \cdots + b_0 - b_n a_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}}_{\tilde{G}(s)}.$$

Hence, canonical realizations of $G(s)$ are those of the (strictly proper) $\tilde{G}(s)$ complemented by $D = b_n$.

2.5 Linearization

A class of continuous-time nonlinear systems $G : u \mapsto y$ can be described by

$$G : \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

for some functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, such that

- both they (i.e. f and h) and their derivatives in x and u are continuous.

An equilibrium of the system is any pair $(x_{\text{eq}}, u_{\text{eq}}) \in \mathbb{R}^n \times \mathbb{R}$ for which the system is at rest, i.e. for which $\dot{x} = f(x, u)$ can be solved for

$$\dot{x} = 0.$$

Hence, an equilibrium should satisfy the algebraic equation

$$f(x_{\text{eq}}, u_{\text{eq}}) = 0.$$

We can then define deviations from equilibrium as

$$\begin{aligned} x_\delta(t) &:= x(t) - x_{\text{eq}} \\ u_\delta(t) &:= u(t) - u_{\text{eq}} \\ y_\delta(t) &:= y(t) - h(x_{\text{eq}}, u_{\text{eq}}). \end{aligned}$$

The linearization of G around an equilibrium $(x_{\text{eq}}, u_{\text{eq}})$ is then given by the linear system

$$G_\delta : \begin{cases} \dot{x}_\delta(t) = A x_\delta(t) + B u_\delta(t) \\ y_\delta(t) = C x_\delta(t) + D u_\delta(t) \end{cases}$$

where

$$A := \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \in \mathbb{R}^{n \times n}, \qquad B := \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \in \mathbb{R}^{n},$$

$$C := \frac{\partial h(x,u)}{\partial x}\Big|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \in \mathbb{R}^{1 \times n}, \qquad D := \frac{\partial h(x,u)}{\partial u}\Big|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \in \mathbb{R}.$$

2.6 Matlab commands

Some Matlab commands to create transfer functions and their step response:

- `G = ss(A,B,C,D);` generates a system object in a state-space form from realization parameters.
- `Gss = ss(G);` generates a `ss` form of a system `G`
  e.g., `Gss = ss(tf([1],[1 2 3]));` generates a state-space realization of $G(s) = \dfrac{1}{s^2 + 2s + 3}$.
- `Gtilde = ss2ss(G,T);` performs the similarity transformation $\tilde{x} = Tx$ on the state vector x of a state-space model `G`.

3 Problems

Question 1. Given the following second-order differential equation.

$$\ddot{y}(t) + y(t) = u(t)$$

1. Find a "physical" state space realization.
2. Use this state space model to calculate the impulse response of the system.

Solution.

1. We can define the state.

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \dot{y}(t) \end{aligned}$$

   Then, we get the following state space equations.

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

   In short, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $D = 0$.

2. We can use the following result.

$$g(t) = D\delta(t) + C\mathrm{e}^{At}B\mathbb{1}(t)$$

   Plugging in our values for (A, B, C, D), we get

$$g(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathrm{e}^{At} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{1}(t).$$

   You have seen in the lectures that

$$\exp\left(\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t \right) = \mathrm{e}^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

   Therefore, in our case ($\sigma = 0$ and $\omega = 1$), we get

$$\mathrm{e}^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

   Therefore, we get the following impulse response.

$$g(t) = \sin(t)\mathbb{1}(t)$$

$\triangledown$

Question 2. Consider the strictly proper transfer function

$$G(z) = \frac{z^2 + 2z + 1}{z^3 + 6z^2 + 11z + 6}$$

(yes, this is the transfer function of a discrete system).

1. Find the state space realization in companion form.
2. Find the state space realization in observer form.

Solution. We must compare our transfer function to the standard form.

$$G(z) = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

Thus, $b_3 = 0$, $b_2 = 1$, $b_1 = 2$, $b_0 = 1$, $a_2 = 6$, $a_1 = 11$, and $a_0 = 6$. Note that $b_3 = 0$ because our transfer function is strictly proper. Therefore, the canonical forms simplify considerably.

1. For the companion canonical form, we have the following matrices.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \\ D &= 0 \end{aligned}$$

2. For the observer canonical form, we have the following matrices.

$$\begin{aligned} A &= \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ D &= 0 \end{aligned}$$

As a sanity check, you can calculate $G(z) = C(zI - A)^{-1}B + D$ for both realizations and see that you get the same transfer function. ▽

Question 3. Consider the following bi-proper transfer function.

$$G(s) = \frac{s^2 + 2s + 1}{s^2 + 5s + 6}$$

1. Find the state space realization in companion form.

2. Use the following transformation matrix to get a similar realization.

$$T = \begin{bmatrix} -5 & -3 \\ -7 & -5 \end{bmatrix}$$

What does this similar realization correspond to?

3. Check that $G(s)$ calculated from the similar realization is the same as the given transfer function.

Solution. We must first separate our transfer function into a constant and a strictly proper transfer function.

$$G(s) = \frac{s^2 + 2s + 1}{s^2 + 5s + 6} = 1 + \frac{-3s - 5}{s^2 + 5s + 6}$$

Given that $G(s) = C(sI - A)^{-1}B + D$, we can see that $D = 1$. The matrices A, B, C can be obtained by using the canonical form for strictly proper transfer function. Again, we must compare our strictly proper transfer function to the standard form.

$$\frac{-3s - 5}{s^2 + 5s + 6} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

Thus, $b_1 = -3$, $b_0 = -5$, $a_1 = 5$, and $a_0 = 6$.

1. For the companion canonical form, we have the following matrices.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} b_0 & b_1 \end{bmatrix} = \begin{bmatrix} -5 & -3 \end{bmatrix} \\ D &= 1 \end{aligned}$$

2. The inverse of the transformation matrix is given by

$$T^{-1} = \frac{1}{4}\begin{bmatrix} -5 & 3 \\ 7 & -5 \end{bmatrix}.$$

For the similar state space realization, we have the following matrices.

$$\begin{aligned} \tilde{A} &= TAT^{-1} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \\ \tilde{B} &= TB = \begin{bmatrix} -3 \\ -5 \end{bmatrix} \\ \tilde{C} &= CT^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \tilde{D} &= D = 1 \end{aligned}$$

This realization corresponds to the observer canonical form.

3. We can get the transfer function with the following formula.

$$G(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & -1 \\ 6 & s \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ -5 \end{bmatrix} + 1$$

For the matrix inverse, we have

$$\begin{bmatrix} s+5 & -1 \\ 6 & s \end{bmatrix}^{-1} = \frac{1}{s^2+5s+6}\begin{bmatrix} s & 1 \\ -6 & s+5 \end{bmatrix}$$

Thus, we get the following transfer function.

$$G(s) = \frac{1}{s^2+5s+6}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ -6 & s+5 \end{bmatrix}\begin{bmatrix} -3 \\ -5 \end{bmatrix} + 1 = \frac{-3s-5}{s^2+5s+6} + 1$$

This is exactly the same transfer function we started with.

▽

Question 4. Consider the system shown in Fig. 1. We will take the following parameters.

$$\begin{aligned} a = b = 1\ \text{m}, \quad g = 10\ \text{m}\,\text{s}^{-2} \\ k_1 = k_2 = 6\ \text{N}\,\text{m}^{-1} \\ c_1 = 7\ \text{N}\,\text{s}\,\text{m}^{-1}, \quad c_2 = 8\ \text{N}\,\text{s}\,\text{m}^{-1} \\ m_1 = 1\ \text{kg}, \quad m_2 = 2\ \text{kg} \end{aligned}$$

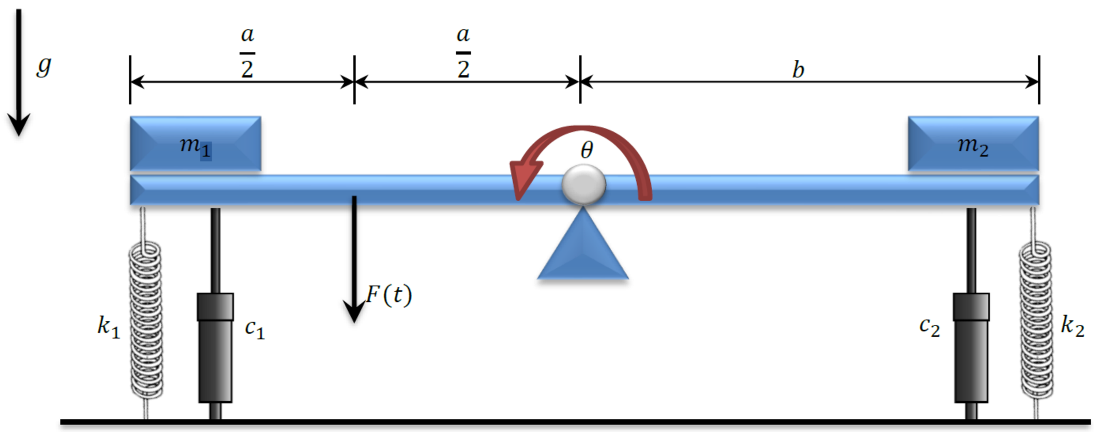

Fig. 1: Seesaw system.

Assuming that the springs and dampers only elongate vertically, it can be shown that the dynamics are given by the following second order differential equation.

$$\ddot{\theta} = \frac{1}{6}F\cos\theta - 2\sin 2\theta - 5\dot{\theta}\cos^2\theta - \frac{10}{3}\cos\theta$$

1. Rewrite the dynamics in the following form.

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases},$$

with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, $u = F$, and $y = \theta$.

2. Find u_{eq} such that $x_{\text{eq}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point.

3. Linearize the system around this equilibrium point.

Solution.

1. The dynamics can be rewritten as follows.

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} x_2(t) \\ \frac{1}{6}u(t)\cos x_1(t) - 2\sin 2x_1(t) - 5x_2(t)\cos^2 x_1(t) - \frac{10}{3}\cos x_1(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

2. To find our equilibrium point, we must solve the equation $f(x_{\text{eq}}, u_{\text{eq}}) = 0$. This gives us the following.

$$\begin{cases} x_{2,\text{eq}} = 0 \\ \frac{1}{6}u_{\text{eq}}\cos x_{1,\text{eq}} - 2\sin 2x_{1,\text{eq}} - 5x_{2,\text{eq}}\cos^2 x_{1,\text{eq}} - \frac{10}{3}\cos x_{1,\text{eq}} = 0 \end{cases}$$

We can immediately see that $x_{2,\text{eq}} = 0$, independent of F. We can thus rewrite the second equation as follows.

$$\frac{1}{6}u_{\text{eq}}\cos x_{1,\text{eq}} - 2\sin 2x_{1,\text{eq}} - \frac{10}{3}\cos x_{1,\text{eq}} = 0$$

There are many pairs $(x_{1,\text{eq}}, u_{\text{eq}})$ that satisfy this equation. However, we are only interested in the pair that satisfies $x_{1,\text{eq}} = 0$. This gives us the following.

$$\frac{1}{6}u_{\text{eq}} - \frac{10}{3} = 0 \quad \Rightarrow \quad \boxed{u_{\text{eq}} = 20 \text{ [N]}}$$

In conclusion, we have that $(x_{\text{eq}}, u_{\text{eq}}) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 20\right)$ is an equilibrium point of our system.

3. We must first define deviations from equilibrium.

$$\begin{cases} x_\delta = x - x_{\text{eq}} = x \\ u_\delta = u - u_{\text{eq}} = u - 20 \\ y_\delta = y - h(x_{\text{eq}}, u_{\text{eq}}) = y - x_{1,\text{eq}} = y \end{cases}$$

Then, we can calculate the matrices (A, B, C, D) as follows.

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \left.\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} \\ &= \left.\begin{bmatrix} 0 & 1 \\ -\frac{1}{6}\sin x_1 - 4\cos 2x_1 + 10x_2\cos x_1 \sin x_1 + \frac{10}{3}\sin x_1 & -5\cos^2 x_1 \end{bmatrix}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} \\ &= \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \\ B &= \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \left.\begin{bmatrix} 0 \\ \frac{1}{6}\cos x_1 \end{bmatrix}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \begin{bmatrix} 0 \\ \frac{1}{6} \end{bmatrix} \\ C &= \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \left.\begin{bmatrix} 1 & 0 \end{bmatrix}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ D &= \left.\frac{\partial h}{\partial u}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = 0\Big|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = 0 \end{aligned}$$

Thus, the state space realization of the linearized system is given by the following.

$$G_\delta : \begin{cases} \dot{x}_\delta(t) = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} x_\delta(t) + \begin{bmatrix} 0 \\ \frac{1}{6} \end{bmatrix} u_\delta(t) \\ y_\delta(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_\delta(t) \end{cases}$$

▽

4 Homework problems

Question 5. Consider the following third-order differential equation.

$$\dddot{y}(t) + 4\ddot{y}(t) + 6\dot{y}(t) + y(t) = \dddot{u}(t) + 3\ddot{u}(t) + 3\dot{u}(t) + u(t)$$

1. Calculate the transfer function $G(s)$ of the system.
2. Find the state space realization in observer form.
3. Check that $G(s)$ calculated from the realization in observer form is the same as the given transfer function.

Solution.

1. The transfer function is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^3 + 3s^2 + 3s + 1}{s^3 + 4s^2 + 6s + 1}$$

2. We must first separate our transfer function into a constant and a strictly proper transfer function.

$$G(s) = \frac{s^3 + 3s^2 + 3s + 1}{s^3 + 4s^2 + 6s + 1} = 1 + \frac{-s^2 - 3s}{s^3 + 4s^2 + 6s + 1}$$

Given that $G(s) = C(sI - A)^{-1}B + D$, we can see that $D = 1$. The matrices A, B, C can be obtained by using the canonical form for strictly proper transfer function. We must compare our strictly proper transfer function to the standard form.

$$\frac{-s^2 - 3s}{s^3 + 4s^2 + 6s + 1} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Thus, $b_2 = -1$, $b_1 = -3$, $b_0 = 0$, $a_2 = 4$, $a_1 = 6$, $a_0 = 1$. Now, we get the following matrices for the state space realization in observer form.

$$A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ -6 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$
$$D = 1$$

3. We can get the transfer function with the following formula.

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s+4 & -1 & 0 \\ 6 & s & -1 \\ 1 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} + 1 = \frac{s^3 + 3s^2 + 3s + 1}{s^3 + 4s^2 + 6s + 1}$$

This is exactly the same transfer function we started with.

▽

Question 6. Consider the tank system shown in Fig. 2.

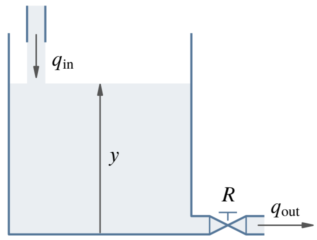

Fig. 2: Tank system.

The state of the system is given by the height of the liquid level y. The flow rate of the entering liquid is q_{in} and the flow rate of the exiting liquid is q_{out}. The dynamics are given by the following equation.

$$q_{out}(t) = R\sqrt{y(t)}$$
$$\dot{y}(t) = \frac{1}{S}\left(q_{in}(t) - q_{out}(t)\right),$$

with S being the cross-sectional area of the tank and R being the resistance coefficient of the outlet.

1. Rewrite the dynamics in the following form.

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases},$$

with $x = y$, $u = q_{in}$.

2. Find all the equilibrium points (x_{eq}, u_{eq}) of the system.

3. Linearize the system around the equilibrium point corresponding to $q_{in} = 1$.

4. Find the transfer function of the linearized system.

5. Given that $q_{in}(t) = 1 + a\sin(\omega t)$, approximate $y(t)$ using the linearized system.

Solution.

1. The dynamics are given by the following equation.

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) = \frac{1}{S}\left(u(t) - R\sqrt{x(t)}\right) \\ y(t) = h(x(t), u(t)) = x(t) \end{cases}$$

2. The equilibrium points are characterized by the following equation.

$$f(x_{eq}, u_{eq}) = \frac{1}{S}\left(u_{eq} - R\sqrt{x_{eq}}\right) = 0$$

Thus, we have the following equilibrium points.

$$(x_{eq}, u_{eq}) = \left(\frac{u_{eq}^2}{R^2}, u_{eq}\right), \quad u_{eq} > 0$$

Note that $(x_{\text{eq}}, u_{\text{eq}}) = (0, 0)$ is not a valid equilibrium point, because $\sqrt{x}$ is not continuous at $x = 0$. Moreover, the right derivative of $\sqrt{x}$ at $x = 0$ is infinite.

$$\lim_{x \to 0^+} \frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x}) = \lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty$$

3. For $q_{\text{in}} = 1$, we have the following equilibrium point.

$$(x_{\text{eq}}, u_{\text{eq}}) = \left(\frac{1}{R^2}, 1\right)$$

We now define deviations from equilibrium as follows.

$$\begin{aligned} x_\delta(t) &= x(t) - x_{\text{eq}} = x(t) - \frac{1}{R^2} \\ u_\delta(t) &= u(t) - u_{\text{eq}} = u(t) - 1 \\ y_\delta(t) &= y(t) - h(x_{\text{eq}}, u_{\text{eq}}) = y(t) - \frac{1}{R^2} \end{aligned}$$

We can now find the matrices A, B, C and D of the linearized system.

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \left.\frac{\partial}{\partial x}\left(\frac{1}{S}\left(u - R\sqrt{x}\right)\right)\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \left.\frac{1}{S}\left(-\frac{R}{2\sqrt{x}}\right)\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = -\frac{R}{2S\sqrt{x_{\text{eq}}}} = -\frac{R^2}{2S} \\ B &= \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \left.\frac{\partial}{\partial u}\left(\frac{1}{S}\left(u - R\sqrt{x}\right)\right)\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \left.\frac{1}{S}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \frac{1}{S} \\ C &= \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \left.\frac{\partial}{\partial x}(x)\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = 1|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = 1 \\ D &= \left.\frac{\partial h}{\partial u}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \left.\frac{\partial}{\partial u}(x)\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = 0|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = 0 \end{aligned}$$

Note that $A = \infty$ for $x_{\text{eq}} = 0$, which shows again that $(x_{\text{eq}}, u_{\text{eq}}) = (0, 0)$ is not a valid equilibrium point. The linearized system is thus given by the following equations.

$$G_\delta : \begin{cases} \dot{x}_\delta(t) = -\frac{R^2}{2S} x_\delta(t) + \frac{1}{S} u_\delta(t) \\ y_\delta(t) = x_\delta(t) \end{cases}$$

4. The transfer function of the linearized system is given by the following equation.

$$G(s) = C(sI - A)^{-1}B + D = \frac{1/S}{s + \frac{R^2}{2S}}$$

5. For the linearized system, we must first find $u\delta$.

$$u_\delta(t) = u(t) - u_{\text{eq}} = q_{\text{in}}(t) - 1 = a\sin(\omega t)$$

We know that the response to a sinusoidal input is a sinusoidal output with the same frequency but with a different amplitude and phase.

$$y_\delta(t) = a|G(\mathrm{j}\omega)|\sin(\omega t + \arg G(\mathrm{j}\omega))$$

Finally, to get $y_{\text{approx}}(t)$, we must remember that $y_\delta = y - y_{\text{eq}}$.

$$y_{\text{approx}}(t) = y_{\text{eq}} + y_\delta(t) = \frac{1}{R^2} + a|G(\mathrm{j}\omega)|\sin(\omega t + \arg G(\mathrm{j}\omega))$$

Note that this is just an approximation. The response of the nonlinear system will not be exactly sinusoidal. The lower the amplitude a of the input, the better the approximation will be.

▽

Question 7. Consider the magnetic levitation system shown in Fig. 3.

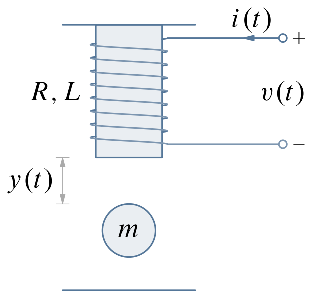

Fig. 3: Magnetic levitation system

The electric current i running through a coil, having resistance R and inductance L, creates a magnetic field, which attracts an iron ball of mass m. The electromagnetic force applied by the magnetic field to the ball is

$$F_{\text{em}}(t) = \alpha \frac{i^2(t)}{y^2(t)},$$

where y given the position of the ball, and $\alpha > 0$ is constant. The ball is also subject to gravity, and the force of gravity is given by

$$F_{\text{g}}(t) = mg.$$

The dynamics of the electric RL circuit are

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big(Li(t)\big) + Ri(t) = v(t).$$

1. Rewrite the dynamics in the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

with $x = \begin{bmatrix} y \\ \dot{y} \\ i \end{bmatrix}$, $u = v$, and $y = y$.

2. Find the equilibrium points of the system.

3. Linearize the system around the equilibrium points.

Solution.

1. The dynamics of the system are given by the following equations.

$$\begin{cases} m\ddot{y}(t) = -\alpha \frac{i^2(t)}{y^2(t)} + mg \\ \frac{\mathrm{d}}{\mathrm{d}t}\big(Li(t)\big) + Ri(t) = v(t) \end{cases}$$

   Given that $x = \begin{bmatrix} y \\ \dot{y} \\ i \end{bmatrix}$, $u = v$, and $y = y$, we can rewrite the dynamics in the following form.

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ g - \frac{\alpha}{m} x_3^2(t)/x_1^2(t) \\ -\frac{R}{L}x_3(t) + \frac{1}{L}u(t) \end{bmatrix} \\ y(t) = x_1(t) \end{cases}$$

   This is indeed of the form $\dot{x} = f(x,u)$ and $y = h(x,u)$.

2. The equilibrium points of the system is given by $f(x_{\mathrm{eq}}, u_{\mathrm{eq}}) = 0$. In our case this reads

$$\begin{cases} x_{2,\mathrm{eq}} = 0 \\ g - \frac{\alpha}{m} x_{3,\mathrm{eq}}^2 / x_{1,\mathrm{eq}}^2 = 0 \\ -\frac{R}{L} x_{3,\mathrm{eq}} + \frac{1}{L} u_{\mathrm{eq}} = 0 \end{cases}$$

   Noting $x_{1,\mathrm{eq}} = y_0 \geq 0$, we get the following equilibrium points:

$$(x_{\mathrm{eq}}, u_{\mathrm{eq}}) = \left(\begin{bmatrix} y_0 \\ 0 \\ \sqrt{\frac{mg}{\alpha}} y_0 \end{bmatrix}, R\sqrt{\frac{mg}{\alpha}} y_0 \right)$$

3. We now define deviations from the equilibrium points,

$$x_\delta(t) = x(t) - \begin{bmatrix} y_0 \\ 0 \\ \sqrt{\frac{mg}{\alpha}} y_0 \end{bmatrix}, \quad u_\delta(t) = u(t) - R\sqrt{\frac{mg}{\alpha}} y_0, \quad \text{and} \quad y_\delta(t) = y(t) - y_0.$$

   and the realization (A, B, C, D) of the linearized system,

$$A = \frac{\partial f}{\partial x}\bigg|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha x_3^2/(m x_1^3) & 0 & -2\alpha x_3/(m x_1^2) \\ 0 & 0 & -R/L \end{bmatrix}\Bigg|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = \begin{bmatrix} 0 & 1 & 0 \\ 2g/y_0 & 0 & -2\sqrt{g\alpha/m}/y_0 \\ 0 & 0 & -R/L \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\bigg|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix}\Bigg|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix}$$

$$C = \frac{\partial h}{\partial x}\bigg|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\Big|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$D = \frac{\partial h}{\partial u}\bigg|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = 0\Big|_{\substack{x = x_{\mathrm{eq}} \\ u = u_{\mathrm{eq}}}} = 0$$

Therefore, the state space representation of the linearized system is

$$G_\delta : \begin{cases} \dot{x}_\delta(t) = \begin{bmatrix} 0 & 1 & 0 \\ 2g/y_0 & 0 & -2\sqrt{g\alpha/m}/y_0 \\ 0 & 0 & -R/L \end{bmatrix} x_\delta(t) + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} u_\delta(t) \\ y_\delta(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_\delta(t) \end{cases}$$

That's all ... ▽