Linear Systems (034032)

TUTORIAL 11

1 Topics

Solving state equations, canonical realizations, and linearization.

2 Background

2.1 State space representation

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$, the state space representation of a system of a continuous-time system $G : u \mapsto y$ is given by

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} .$$

One can show that the impulse response (i.e. $u = \delta$) of this system is given by

$$g(t) = Ce^{At}B\mathbb{1}(t) + D\delta(t).$$

With the impulse response known, we can get the response for any input by using the convolution property of LTI systems.

$$y = g * u$$

2.2 State space to transfer function

Going from a state space representation to a transfer function representation is very simple.

$$G(s) = C(sI - A)^{-1}B + D$$

One can show that $G(s)$ is a rational function of s. Moreover, the poles of $G(s)$ are among the eigenvalues of A.

2.3 Similar realizations

Let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix. Define $\tilde{x} := Tx$, where x is the state of the system. Then, the following realizations are called similar.

$$(TAT^{-1}, TB, CT^{-1}, D) \text{ and } (A, B, C, D)$$

These similar realizations have the same I/O relations (impulse response and transfer function).

2.4 Transfer function to state space

2.4.1 "Physical" realization

If

$$G(s) = \frac{b}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0},$$

then its possible state-space realization is

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{cccc:c} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -a_0 & -a_1 & \dots & -a_{n-1} & b \\ \hdashline 1 & 0 & \dots & 0 & 0 \end{array}\right].$$

2.4.2 Canonical realization: companion form

Let $G(s)$ be strictly proper, i.e.

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

The state-space realization discussed above, known as the companion form, is

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{c:c} A_{\text{cf}} & B_{\text{cf}} \\ \hdashline C_{\text{cf}} & D_{\text{cf}} \end{array}\right] := \left[\begin{array}{cccc:c} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -a_0 & -a_1 & \dots & -a_{n-1} & 1 \\ \hdashline b_0 & b_1 & \dots & b_{n-1} & 0 \end{array}\right]$$

Note that this is very similar to the "physical" realization in that the matrix A is the same.

2.4.3 Canonical realization: observer form

Let $G(s)$ be strictly proper, i.e.

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

Its state-space realization in observer form has

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{c:c} A_{\text{of}} & B_{\text{of}} \\ \hdashline C_{\text{of}} & D_{\text{of}} \end{array}\right] := \left[\begin{array}{cccc:c} -a_{n-1} & 1 & \dots & 0 & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \dots & 1 & b_1 \\ -a_0 & 0 & \dots & 0 & b_0 \\ \hdashline 1 & 0 & \dots & 0 & 0 \end{array}\right].$$

2.4.4 Canonical realization: bi-proper case

Let

$$G(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

for $b_n \neq 0$. The trick is to rewrite it as

$$\frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} = b_n + \underbrace{\frac{(b_{n-1} - b_n a_{n-1})s^{n-1} + \cdots + b_0 - b_n a_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}}_{\tilde{G}(s)}.$$

Hence, canonical realizations of $G(s)$ are those of the (strictly proper) $\tilde{G}(s)$ complemented by $D = b_n$.

2.5 Linearization

A class of continuous-time nonlinear systems $G : u \mapsto y$ can be described by

$$G : \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

for some functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, such that

- both they (i.e. f and h) and their derivatives in x and u are continuous.

An equilibrium of the system is any pair $(x_{\text{eq}}, u_{\text{eq}}) \in \mathbb{R}^n \times \mathbb{R}$ for which the system is at rest, i.e. for which $\dot{x} = f(x, u)$ can be solved for

$$\dot{x} = 0.$$

Hence, an equilibrium should satisfy the algebraic equation

$$f(x_{\text{eq}}, u_{\text{eq}}) = 0.$$

We can then define deviations from equilibrium as

$$\begin{aligned} x_\delta(t) &:= x(t) - x_{\text{eq}} \\ u_\delta(t) &:= u(t) - u_{\text{eq}} \\ y_\delta(t) &:= y(t) - h(x_{\text{eq}}, u_{\text{eq}}). \end{aligned}$$

The linearization of G around an equilibrium $(x_{\text{eq}}, u_{\text{eq}})$ is then given by the linear system

$$G_\delta : \begin{cases} \dot{x}_\delta(t) = Ax_\delta(t) + Bu_\delta(t) \\ y_\delta(t) = Cx_\delta(t) + Du_\delta(t) \end{cases}$$

where

$$A := \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} \in \mathbb{R}^{n\times n}, \qquad B := \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} \in \mathbb{R}^{n},$$

$$C := \frac{\partial h(x,u)}{\partial x}\Big|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} \in \mathbb{R}^{1\times n}, \qquad D := \frac{\partial h(x,u)}{\partial u}\Big|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} \in \mathbb{R}.$$

2.6 Matlab commands

Some Matlab commands to create transfer functions and their step response:

- `G = ss(A,B,C,D);` generates a system object in a state-space form from realization parameters.
- `Gss = ss(G);` generates a `ss` form of a system `G`
  e.g., `Gss = ss(tf([1],[1 2 3]));` generates a state-space realization of $G(s) = \dfrac{1}{s^2 + 2s + 3}$.
- `Gtilde = ss2ss(G,T);` performs the similarity transformation $\tilde{x} = Tx$ on the state vector x of a state-space model `G`.

3 Problems

Question 1. Given the following second-order differential equation.

$$\ddot{y}(t) + y(t) = u(t)$$

1. Find a “physical” state space realization.
2. Use this state space model to calculate the impulse response of the system.

Question 2. Consider the strictly proper transfer function

$$G(z) = \frac{z^2 + 2z + 1}{z^3 + 6z^2 + 11z + 6}$$

(yes, this is the transfer function of a discrete system).

1. Find the state space realization in companion form.
2. Find the state space realization in observer form.

Question 3. Consider the following bi-proper transfer function.

$$G(s) = \frac{s^2 + 2s + 1}{s^2 + 5s + 6}$$

1. Find the state space realization in companion form.
2. Use the following transformation matrix to get a similar realization.

$$T = \begin{bmatrix} -5 & -3 \\ -7 & -5 \end{bmatrix}$$

   What does this similar realization correspond to?

3. Check that $G(s)$ calculated from the similar realization is the same as the given transfer function.

Question 4. Consider the system shown in Fig. 1. We will take the following parameters.

$$a = b = 1\ \text{m}, \quad g = 10\ \text{m}\,\text{s}^{-2}$$
$$k_1 = k_2 = 6\ \text{N}\,\text{m}^{-1}$$
$$c_1 = 7\ \text{N}\,\text{s}\,\text{m}^{-1}, \quad c_2 = 8\ \text{N}\,\text{s}\,\text{m}^{-1}$$
$$m_1 = 1\ \text{kg}, \quad m_2 = 2\ \text{kg}$$

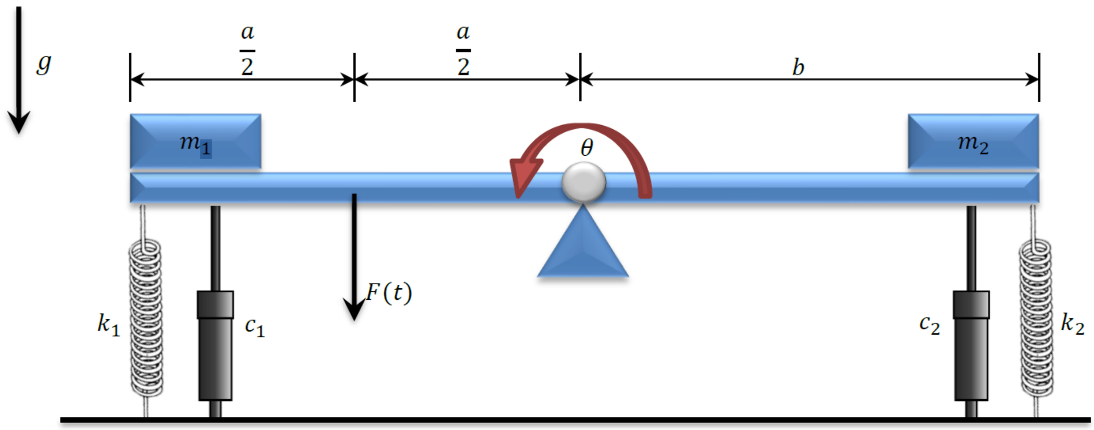

Fig. 1: Seesaw system.

Assuming that the springs and dampers only elongate vertically, it can be shown that the dynamics are given by the following second order differential equation.

$$\ddot{\theta} = \frac{1}{6} F \cos\theta - 2\sin 2\theta - 5\dot{\theta}\cos^2\theta - \frac{10}{3}\cos\theta$$

1. Rewrite the dynamics in the following form.

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases},$$

with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, $u = F$, and $y = \theta$.

2. Find u_{eq} such that $x_{\text{eq}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point.

3. Linearize the system around this equilibrium point.

4 Homework problems

Question 5. Consider the following third-order differential equation.

$$\dddot{y}(t) + 4\ddot{y}(t) + 6\dot{y}(t) + y(t) = \dddot{u}(t) + 3\ddot{u}(t) + 3\dot{u}(t) + u(t)$$

1. Calculate the transfer function $G(s)$ of the system.
2. Find the state space realization in observer form.
3. Check that $G(s)$ calculated from the realization in observer form is the same as the given transfer function.

Question 6. Consider the tank system shown in Fig. 2.

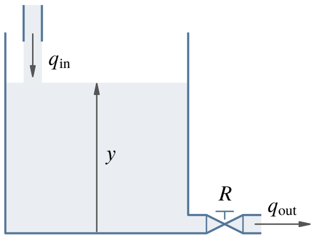

Fig. 2: Tank system.

The state of the system is given by the height of the liquid level y. The flow rate of the entering liquid is q_{in} and the flow rate of the exiting liquid is q_{out}. The dynamics are given by the following equation.

$$q_{out}(t) = R\sqrt{y(t)}$$
$$\dot{y}(t) = \frac{1}{S}\left(q_{in}(t) - q_{out}(t)\right),$$

with S being the cross-sectional area of the tank and R being the resistance coefficient of the outlet.

1. Rewrite the dynamics in the following form.

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases},$$

   with $x = y$, $u = q_{in}$.

2. Find all the equilibrium points (x_{eq}, u_{eq}) of the system.
3. Linearize the system around the equilibrium point corresponding to $q_{in} = 1$.
4. Find the transfer function of the linearized system.
5. Given that $q_{in}(t) = 1 + a\sin(\omega t)$, approximate $y(t)$ using the linearized system.

Question 7. Consider the magnetic levitation system shown in Fig. 3.

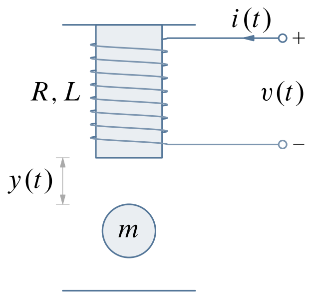

Fig. 3: Magnetic levitation system

The electric current i running through a coil, having resistance R and inductance L, creates a magnetic field, which attracts an iron ball of mass m. The electromagnetic force applied by the magnetic field to the ball is

$$F_{\mathrm{em}}(t) = \alpha \frac{i^2(t)}{y^2(t)},$$

where y given the position of the ball, and $\alpha > 0$ is constant. The ball is also subject to gravity, and the force of gravity is given by

$$F_{\mathrm{g}}(t) = mg.$$

The dynamics of the electric RL circuit are

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big(Li(t)\big) + Ri(t) = v(t).$$

1. Rewrite the dynamics in the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

   with $x = \begin{bmatrix} y \\ \dot{y} \\ i \end{bmatrix}$, $u = v$, and $y = y$.

2. Find the equilibrium points of the system.

3. Linearize the system around the equilibrium points.