Linear Systems (034032)

tutorial 10

1 Topics

Linear algebra revision, matrix functions, Cayley–Hamilton theorem, matrix calculus.

2 Background

2.1 Linear Algebra Revision

Let $A, B \in \mathbb{R}^{n\times n}$ be real valued square $n \times n$ matrices.

2.1.1 Eigenvalues and Characteristic Polynomial

The *characteristic polynomial* of A is:

$$\chi_A(\lambda) := \det(\lambda I - A) = \lambda^n + \chi_{n-1}\lambda^{n-1} + \cdots + \chi_1\lambda + \chi_0 = 0$$

The solutions to the polynomial $\lambda_i \in \mathbb{C}$ are called the *eigenvalues*. The set of all eigenvalues of matrix A is known as its *spectrum*, denoted $\text{spec}(A)$.

2.1.2 Eigenvalues and Eigenvector

A vector $0 \neq \eta_i \in \mathbb{C}^n$ is a *right eigenvector* of A associated with the eigenvalue λ_i if

$$(\lambda_i I - A)\eta_i = 0$$

and $0 \neq \tilde{\eta}_i \in \mathbb{C}^n$ is a *left eigenvector* associated with the eigenvalue λ_i if

$$\tilde{\eta}_i'(\lambda_i I - A) = 0$$

2.1.3 Multiplicity

The *algebraic multiplicity* of an eigenvalue λ_i is the number of times it appears in the characteristic polynomial. Its *geometric multiplicity* is defined as $n - \text{rank}(\lambda_i I - A)$, where n is the the dimension of matrix A and $\text{rank}(A)$ is the rank of A (the number of linearly independent rows or columns in it).

2.1.4 Similarity

We say that a matrix A is *similar* to a matrix B (and vice versa) if there exists a nonsingular transformation matrix T, such that

$$A = TBT^{-1}$$

2.1.5 Diagonalization

We say that a matrix A is *diagonalizable* if there exists a matrix T and a diagonal matrix Λ_A such that

$$A = T\Lambda_A T^{-1},$$

where the diagonal elements of Λ_A are eigenvalues of A and the columns of T are the right eigenvectors of A, i.e.

$$T = \begin{bmatrix}\eta_1 & \eta_2 & \cdots & \eta_n\end{bmatrix} \in \mathbb{C}^{n\times n}$$

or, conversely, we can write the transformation as

$$A = \tilde{T}^{-1}\Lambda_A \tilde{T},$$

where the rows of $\tilde{T}$ are the left eigenvectors of A (transposed of course),

$$\tilde{T} = \begin{bmatrix}\tilde{\eta}_1' \\ \tilde{\eta}_2' \\ \vdots \\ \tilde{\eta}_n'\end{bmatrix} \in \mathbb{C}^{n\times n}$$

If η_i and $\tilde{\eta}_i$ are normalized, then $\tilde{T} = T^{-1}$.

Theorem 1. *A matrix A is diagonalizable iff the geometric multiplicity of each its eigenvalue is equal to its algebraic multiplicity. Otherwise, there exists an eigenvalue $\lambda_i \in \mathrm{spec}(A)$ such that its geometric multiplicity is smaller than its algebraic multiplicity, and A is called* defective.

2.1.6 Real "Diagonalization" of a Matrix with Complex Eigenvalues

If there is a pair of complex eigenvalues $\lambda_i, \overline{\lambda_i} = \sigma \pm \omega \mathrm{j}$, then we also have two complex conjugate eigenvectors,

$$\eta_i, \overline{\eta_i} = \alpha \pm \mathrm{j}\beta$$

where $\alpha, \beta \in \mathbb{R}^n$ are real valued vectors.

We now show a special representation of the matrix called "real diagonalization of a matrix with complex eigenvalues." We first define two linear combinations of our eigenvectors,

$$\frac{\eta_i + \bar{\eta}_i}{2} = \alpha \quad \text{and} \quad \frac{\eta_i - \bar{\eta}_i}{2\mathrm{j}} = \beta$$

Defining

$$T = \begin{bmatrix}\alpha & \beta\end{bmatrix} \quad \text{and} \quad \hat{\Lambda}_A = \begin{bmatrix}\sigma & \omega \\ -\omega & \sigma\end{bmatrix}$$

we get that

$$A = T\hat{\Lambda}_A T^{-1}.$$

Note that when doing this form of "diagonalizing" of the entire matrix A, we get Λ_A with either real eigenvalues or blocks of 2×2 $\hat{\Lambda}_A$s on the main diagonal:

$$\Lambda_A = \begin{bmatrix}\lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & \sigma_2 & \omega_2 & 0 & \cdots \\ 0 & -\omega_2 & \sigma_2 & 0 & \cdots \\ 0 & 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

2.2 Matrix Functions

Let $f : \mathbb{R} \to \mathbb{R}$ be analytic, i.e. representable via its power series

$$f(x) = \sum_{j=0}^{\infty} f_j x^j = f_0 + f_1 x + f_2 x^2 + \cdots$$

Its matrix version $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is defined as

$$f(A) := \sum_{j=0}^{\infty} f_j A^j$$

for the very same scalar coefficients f_j. We can use two methods to calculate $f(A)$:

1. Via **diagonalization**: If A is diagonalizable, then

$$A = T\Lambda_A T^{-1} \quad\implies\quad f(A) = Tf(\Lambda_A)T^{-1} = T^{-1}\begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix} T$$

   If A has a pair of complex eigenvalues, we can use the “real diagonalization” form for that spesific block:

$$A = T\hat{\Lambda}_A T^{-1} \quad\implies\quad f(A) = Tf(\hat{\Lambda}_A)T^{-1}$$

2. Via **Cayley–Hamilton**: By Cayley–Hamilton arguments we know there is $\{g_i\}_{i=0}^{n}$ such that

$$f(A) = \sum_{j=0}^{\infty} f_j A^j = \sum_{i=0}^{n-1} g_i A^i$$

   In order to find the coefficients g_i, define $g(x) := \sum_{i=0}^{n-1} g_i x^i$ and, assuming that all the eigenvalues of A are simple, obtain that

$$\begin{bmatrix} g_0 & g_1 & \cdots & g_{n-1} \end{bmatrix} = \begin{bmatrix} f(\lambda_1) & f(\lambda_2) & \cdots & f(\lambda_n) \end{bmatrix} V^{-1}$$

   where V is the (invertible) Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

   If there is an eigenvalue λ_i of A, with the algebraic multiplicity $\mu_i > 1$, then we need to add extra conditions, viz.

$$\left.\frac{\mathrm{d}^j f(x)}{\mathrm{d}x^j}\right|_{x=\lambda_i} = \left.\frac{\mathrm{d}^j g(x)}{\mathrm{d}x^j}\right|_{x=\lambda_i}, \quad \forall j \in \mathbb{Z}_{1..\mu_i-1}$$

2.2.1 Matrix Exponential

The matrix exponential is a particular case of matrix functions, which is of a special interest. Let $f(x) = \exp(x) = \mathrm{e}^x$. Its power series is

$$\mathrm{e}^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

The form of the matrix version of this function that we are interested in is $\exp(At) = \mathrm{e}^{At}$, where $t \in \mathbb{R}$. The solution to the matrix function is as before. Specifically,

1. Via diagonalization,

$$\mathrm{e}^{At} = T\mathrm{e}^{\Lambda_A t}T^{-1} = T\begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\lambda_n t} \end{bmatrix} T^{-1}$$

   Or for the "real diagonalization of complex eigenvalues,"

$$\mathrm{e}^{At} = T\mathrm{e}^{\hat{A}t}T^{-1} = T\mathrm{e}^{\sigma t}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} T^{-1}$$

2. Via Cayley–Hamilton:

$$\begin{bmatrix} g_0 & g_1 & \cdots & g_{n-1} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & \mathrm{e}^{\lambda_2 t} & \cdots & \mathrm{e}^{\lambda_n t} \end{bmatrix} V^{-1}$$

   where V is the Vandermonde matrix.

2.3 Matrix Calculus

The derivative of a matrix $A(t)$ by a scalar $t \in \mathbb{R}$ is done component-wise:

$$\frac{\mathrm{d}}{\mathrm{d}t}(A(t)) = \begin{bmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \cdots & \dot{a}_{1n}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \cdots & \dot{a}_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \dot{a}_{n1}(t) & \dot{a}_{n2}(t) & \cdots & \dot{a}_{nn}(t) \end{bmatrix}$$

Some properties:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(A_1(t)A_2(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(A_1(t))A_2(t) + A_1(t)\frac{\mathrm{d}}{\mathrm{d}t}(A_2(t)) \\ &\frac{\mathrm{d}}{\mathrm{d}t}A^{-1}(t) = -A^{-1}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)A^{-1}(t) \\ &\frac{\mathrm{d}}{\mathrm{d}t}(At)^k = A^k(kt^{k-1}) \\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{At} = A\mathrm{e}^{At} = \mathrm{e}^{At}A \end{aligned}$$

The derivative of $f : \mathbb{R}^m \to \mathbb{R}^n$ by its vector argument $x \in \mathbb{R}^m$ is defined as

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

3 Problems

Question 1. Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$$

1. Find the diagonalizing transformation of A (in the real form if it exists).
2. Calculate the matrix exponent e^{At} using diagonalization.
3. Calculate the matrix exponent e^{At} using Cayley-Hamilton.

Solution.

1. First, calculate the characteristic polynomial:

$$\begin{aligned}
\chi_A(\lambda) = \det(\lambda I - A) &= \det\left(\begin{bmatrix} \lambda-4 & 0 & -1 \\ 1 & \lambda+6 & 2 \\ -5 & 0 & \lambda \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \lambda+6 & 1 & 2 \\ 0 & \lambda-4 & -1 \\ 0 & -5 & \lambda \end{bmatrix}\right) && \left(\begin{smallmatrix} R_1 \leftrightarrow R_2 \\ C_1 \leftrightarrow C_2 \end{smallmatrix}\right) \\
&= (\lambda+6)\det\left(\begin{bmatrix} \lambda-4 & -1 \\ -5 & \lambda \end{bmatrix}\right) = (\lambda+6)((\lambda-4)\lambda - 5) \\
&= (\lambda+6)(\lambda^2 - 4\lambda - 5) = (\lambda+6)(\lambda+1)(\lambda-5)
\end{aligned}$$

Therefore, the eigenvalues are

$$\lambda_1 = -6, \quad \lambda_2 = -1, \quad \lambda_3 = 5$$

Now we calculate the eigenvectors:
<u>For $\lambda_1 = -6$:</u>

$$\begin{aligned}
&(\lambda_1 I - A)\eta_1 = 0 \\
&\left(\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} \eta_{1,1} \\ \eta_{1,2} \\ \eta_{1,3} \end{bmatrix} = 0 \\
&\begin{bmatrix} -10 & 0 & -1 \\ 1 & 0 & 2 \\ -5 & 0 & -6 \end{bmatrix}\begin{bmatrix} \eta_{1,1} \\ \eta_{1,2} \\ \eta_{1,3} \end{bmatrix} = 0 \\
&\begin{bmatrix} -10\eta_{1,1} - \eta_{1,3} \\ \eta_{1,1} + 2\eta_{1,3} \\ -5\eta_{1,1} - 6\eta_{1,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

To solve the set of equations we demand: $\eta_{1,1} = \eta_{1,3} = 0$, and $\eta_{1,2}$ is a free variable. Thus, the eigenvector that corresponds to an eigenvalue of $\lambda_1 = -6$ is:

$$\eta_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = -1$:

$$(\lambda_2 I - A)\eta_2 = 0$$

$$\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} \eta_{2,1} \\ \eta_{2,2} \\ \eta_{2,3} \end{bmatrix} = 0$$

$$\begin{bmatrix} -5 & 0 & -1 \\ 1 & 5 & 2 \\ -5 & 0 & -1 \end{bmatrix} \begin{bmatrix} \eta_{2,1} \\ \eta_{2,2} \\ \eta_{2,3} \end{bmatrix} = 0 \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} -5 & 0 & -1 \\ 1 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{2,1} \\ \eta_{2,2} \\ \eta_{2,3} \end{bmatrix} = 0$$

$$\begin{bmatrix} -5\eta_{2,1} - \eta_{2,3} \\ \eta_{2,1} + 5\eta_{2,2} + 2\eta_{2,3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve the set of equations we demand: $\eta_{2,3} = -5\eta_{2,1}$. Choosing $\eta_{2,1} = -5$ we get $\eta_{2,3} = 25$ and $\eta_{2,2} = -9$:

$$\eta_2 = \begin{bmatrix} -5 \\ -9 \\ 25 \end{bmatrix}$$

For $\lambda_3 = 5$:

$$(\lambda_2 I - A)\eta_2 = 0$$

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 11 & 2 \\ -5 & 0 & -5 \end{bmatrix} \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = 0 \xrightarrow{R_3 = R_3 + 5R_1} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = 0$$

$$\begin{bmatrix} -\eta_{3,1} - \eta_{3,3} \\ \eta_{3,1} + 11\eta_{3,2} + 2\eta_{3,3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve the set of equations we choose: $\eta_{3,1} = \eta_{3,1} = 11$ and $\eta_{3,2} = -3$:

$$\eta_2 = \begin{bmatrix} 11 \\ -3 \\ 11 \end{bmatrix}$$

The diagonalizing transformation is therefore $\Lambda_A = T^{-1}AT$, where:

$$\Lambda_A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & -5 & 11 \\ 1 & -9 & -3 \\ 0 & 25 & 11 \end{bmatrix}$$

2. To calculate the matrix exponent we'll use the diagonal transformation:

$$\begin{aligned}
e^{At} &= T e^{\Lambda_A t} T^{-1} \\
&= \begin{bmatrix} 0 & -5 & 11 \\ 1 & -9 & -3 \\ 0 & 25 & 11 \end{bmatrix} \begin{bmatrix} e^{-6t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 0 & -5 & 11 \\ 1 & -9 & -3 \\ 0 & 25 & 11 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & -5 & 11 \\ 1 & -9 & -3 \\ 0 & 25 & 11 \end{bmatrix} \begin{bmatrix} e^{-6t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{5t} \end{bmatrix} \frac{1}{330} \begin{bmatrix} -24 & 330 & 114 \\ -11 & 0 & 11 \\ 25 & 0 & 5 \end{bmatrix} \\
&= \frac{1}{330} \begin{bmatrix} 55(e^{-t} + 5e^{5t}) & 0 & 55(e^{5t} - e^{-t}) \\ 3(33e^{-t} - 8e^{-6t} - 25e^{5t}) & 330e^{-6t} & 3(38e^{-6t} - 33e^{-t} - 5e^{5t}) \\ 275(e^{5t} - e^{-t}) & 0 & 55(5e^{-t} + e^{5t}) \end{bmatrix}
\end{aligned}$$

3. To calculate the matrix exponent using Cayley-Hamilton we first define:

$$e^{At} = \sum_{i=0}^{2} g_i A^i$$

Using the eigenvalues of A, we can find these coefficients:

$$\begin{aligned}
e^{\lambda_1 t} &= g_0 + g_1\lambda_1 + g_2\lambda_1^2 \\
e^{\lambda_2 t} &= g_0 + g_1\lambda_2 + g_2\lambda_2^2 \\
e^{\lambda_3 t} &= g_0 + g_1\lambda_3 + g_2\lambda_3^2
\end{aligned}$$

Or in matrix form:

$$\begin{bmatrix} e^{\lambda_1 t} & E^{\lambda_2 t} & E^{\lambda_3 t} \end{bmatrix} = \begin{bmatrix} g_0 & g_1 & g_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

Substituting the values of $\lambda_1, \lambda_2, \lambda_3$:

$$\begin{bmatrix} e^{-6t} & e^{-t} & e^{5t} \end{bmatrix} = \begin{bmatrix} g_0 & g_1 & g_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -6 & -1 & 5 \\ 36 & 1 & 25 \end{bmatrix}$$

thus,

$$\begin{aligned}
\begin{bmatrix} g_0 & g_1 & g_2 \end{bmatrix} &= \begin{bmatrix} e^{-6t} & e^{-t} & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -6 & -1 & 5 \\ 36 & 1 & 25 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} e^{-6t} & e^{-t} & e^{5t} \end{bmatrix} \frac{1}{330} \mathrm{Adj}\left(\begin{bmatrix} 1 & 1 & 1 \\ -6 & -1 & 5 \\ 36 & 1 & 25 \end{bmatrix} \right) \\
&= \begin{bmatrix} e^{-6t} & e^{-t} & e^{5t} \end{bmatrix} \frac{1}{330} \begin{bmatrix} -30 & 330 & 30 \\ -24 & -11 & 35 \\ 6 & -11 & 5 \end{bmatrix}' \\
&= \begin{bmatrix} e^{-6t} & e^{-t} & e^{5t} \end{bmatrix} \frac{1}{330} \begin{bmatrix} -30 & -24 & 6 \\ 330 & -11 & -11 \\ 30 & 35 & 5 \end{bmatrix}' \\
&= \frac{1}{330} \begin{bmatrix} -30e^{-6t} + 330e^{-t} + 30e^{5t} & -24e^{-6t} - 11e^{-t} + 35e^{5t} & 6e^{-6t} - 11e^{-t} + 5e^{5t} \end{bmatrix}
\end{aligned}$$

Substituting the coefficients back into the matrix function:

$$
\begin{aligned}
e^{At} &= g_0 I + g_1 A + g_2 A^2 \\
&= \frac{1}{330}\left(-30e^{-6t} + 330e^{-t} + 30e^{5t}\right)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \frac{1}{330}\left(-24e^{-6t} - 11e^{-t} + 35e^{5t}\right)\begin{bmatrix}4 & 0 & 1\\ -1 & -6 & -2\\ 5 & 0 & 0\end{bmatrix} + \\
&+ \frac{1}{330}\left(6e^{-6t} - 11e^{-t} + 5e^{5t}\right)\begin{bmatrix}4 & 0 & 1\\ -1 & -6 & -2\\ 5 & 0 & 0\end{bmatrix}^2 \\
&= \frac{1}{330}\left(-30e^{-6t} + 330e^{-t} + 30e^{5t}\right)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \frac{1}{330}\left(-24e^{-6t} - 11e^{-t} + 35e^{5t}\right)\begin{bmatrix}4 & 0 & 1\\ -1 & -6 & -2\\ 5 & 0 & 0\end{bmatrix} + \\
&+ \frac{1}{330}\left(6e^{-6t} - 11e^{-t} + 5e^{5t}\right)\begin{bmatrix}21 & 0 & 4\\ -8 & 36 & 11\\ 20 & 0 & 5\end{bmatrix} \\
&= \frac{1}{330}\begin{bmatrix}55(e^{-t} + 5e^{5t}) & 0 & 55(e^{5t} - e^{-t})\\ 3(33e^{-t} - 8e^{-6t} - 25e^{5t}) & 330e^{-6t} & 3(38e^{-6t} - 33e^{-t} - 5e^{5t})\\ 275(e^{5t} - e^{-t}) & 0 & 55(5e^{-t} + e^{5t})\end{bmatrix}
\end{aligned}
$$

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Question 2. Let A be the matrix:

$$\begin{bmatrix}-1 & 4\\ -1 & -1\end{bmatrix}$$

1. Find the diagonalizing transformation (in the real form if it exists).
2. Find the matrix exponential e^{At}.

Solution.

1. To find the diagonalizing transformation we'll first find the eigenvalues and eigenvectors. First we'll calculate the eigenvalues:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix}\lambda + 1 & -4\\ 1 & \lambda + 1\end{bmatrix}\right) = \lambda^2 + 2\lambda + 5 = 0$$

Solving the quadratic equations gives the complex eigenvalues:

$$\lambda_{12} = -1 \pm 2j = \sigma \pm j\omega$$

While the eigenvalues are complex, the calculation for the eigenvectors is still the same,

$$
\begin{aligned}
&(\lambda_{12} I - A)\eta_{12} = 0 \\
&\left(\begin{bmatrix}-1 \pm 2j & 0\\ 0 & -1 \pm 2j\end{bmatrix} - \begin{bmatrix}-1 & 4\\ -1 & -1\end{bmatrix}\right)\eta_{12} = 0 \\
&\begin{bmatrix}\pm 2j & -4\\ 1 & \pm 2j\end{bmatrix}\begin{bmatrix}\eta_{12,1}\\ \eta_{12,2}\end{bmatrix} = 0 \xrightarrow{R_2 = \pm 2j R_2} \begin{bmatrix}\pm 2j & -4\\ \pm 2j & -4\end{bmatrix}\begin{bmatrix}\eta_{12,1}\\ \eta_{12,2}\end{bmatrix} = 0 \\
&\begin{bmatrix}\pm 2j\eta_{12,1} - 4\eta_{12,2}\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}
$$

We choose $\eta_{12,2} = 1$ and get $\eta_{12,1} \mp 2\text{j}$, thus:

$$\eta_{1,(\lambda_1=-1+2\text{j})} = \begin{bmatrix} -2\text{j} \\ 1 \end{bmatrix}$$
$$\eta_{2,(\lambda_2=-1-2\text{j})} = \begin{bmatrix} 2\text{j} \\ 1 \end{bmatrix}$$

We can rewrite the eigenvectors in the following form:

$$\eta_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \text{j}$$
$$\eta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{j}$$

In general, in this course we can say that complex eigenvalues come in pairs because the matrix A (which will usually represent masses, forces, moments, etc.) will be real. So we can always write:

$$\eta_{12} = \alpha \pm \beta \text{j}$$

and in our case:

$$\eta_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -2 \\ 0 \end{bmatrix} \text{j}$$

In order for us to calculate using real values only, we define two new vectors:

$$\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \beta = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Using these two vectors we define the modal matrix:

$$T = \begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

and the real "diagonal" matrix is:

$$\Lambda_A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

2. The matrix exponential can be found using the "diagonalizing" transformation:

$$\begin{aligned}
\text{e}^{At} &= T\text{e}^{\Lambda_A}T^{-1} = \begin{bmatrix} \alpha & \beta \end{bmatrix} \text{e}^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \text{e}^{-t} \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \text{e}^{-t} \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \\
&= \text{e}^{-t} \begin{bmatrix} \cos(2t) & 2\sin(2t) \\ -0.5\sin(2t) & \cos(2t) \end{bmatrix}
\end{aligned}$$

∇

Question 3. Let A be the matrix:

$$\begin{bmatrix} 1 & 2 & -8 \\ 0 & -1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

1. Find the diagonalizing transformation (in the real form if it exists).
2. Find the matrix exponential e^{At}.

Solution.

1. We first find the eigenvalues and eigenvectors:

$$\det(\lambda I - A) = \begin{bmatrix} \lambda - 1 & -2 & 8 \\ 0 & \lambda + 1 & -4 \\ 0 & 1 & \lambda + 1 \end{bmatrix} = (\lambda - 1)(\lambda^2 + 2\lambda + 5) = 0$$

From Question 2 we know that there are two complex eigenvalues, so in total the eigenvalues are:

$$\lambda_1 = 1, \lambda_{23} = -1 \pm 2\mathrm{j}$$

We now calculate the eigenvectors:
For $\lambda_1 = 1$:

$$\begin{bmatrix} 0 & -2 & 8 \\ 0 & 2 & -4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \eta_{1,1} \\ \eta_{1,2} \\ \eta_{1,3} \end{bmatrix} = 0 \rightarrow \eta_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_2 = -1 + 2\mathrm{j}$:

$$\begin{bmatrix} -2 + 2\mathrm{j} & -2 & 8 \\ 0 & 2\mathrm{j} & -4 \\ 0 & 1 & 2\mathrm{j} \end{bmatrix} \begin{bmatrix} \eta_{2,1} \\ \eta_{2,2} \\ \eta_{2,3} \end{bmatrix} = 0 \rightarrow \eta_2 = \begin{bmatrix} 1 + 3\mathrm{j} \\ -2\mathrm{j} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \mathrm{j} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

Defining:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \mathrm{j} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \alpha + \mathrm{j}\beta$$

We know that η_3 is the complex conjugate of η_2. Constructing the transformation matrix T (notice the order of the columns) in the real form:

$$T = \begin{bmatrix} \eta_1 & \alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

and the "real diagonal" matrix is:

$$\Lambda_A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

2. We can now use the ”diagonalizing” transformation to calculate the matrix exponent:

$$\begin{aligned}
e^{At} &= T e^{\Lambda_A t} T^{-1} \\
&= \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ 0 & -e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \\
&= e^{-t} \begin{bmatrix} e^{2t} & 1.5e^{2t} - 0.5\sin(2t) - 1.5\cos(2t) & -e^{2t} - \cos(2t) - 3\sin(2t) \\ 0 & \cos(2t) & 2\sin(2t) \\ 0 & -0.5\sin(2t) & \cos(2t) \end{bmatrix}
\end{aligned}$$

▽

Question 4. Let $A \in \mathbb{R}^{n \times m}$ be a matrix and $x \in \mathbb{R}^m$ be a vector. Calculate the derivative of

$$y = Ax \in \mathbb{R}^n$$

by x.

Solution. Let us first express the vector y:

$$y = Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nm}x_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i}x_i \\ \sum_{i=1}^m a_{2i}x_i \\ \vdots \\ \sum_{i=1}^m a_{ni}x_i \end{bmatrix}$$

The derivative of y by x is done element-wise so:

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\sum_{i=1}^m a_{1i}x_i\right) & \frac{\partial}{\partial x_2}\left(\sum_{i=1}^m a_{1i}x_i\right) & \cdots \\ \frac{\partial}{\partial x_1}\left(\sum_{i=1}^m a_{2i}x_i\right) & \frac{\partial}{\partial x_2}\left(\sum_{i=1}^m a_{2i}x_i\right) & \cdots \\ \vdots & \vdots & \\ \frac{\partial}{\partial x_1}\left(\sum_{i=1}^m a_{ni}x_i\right) & \frac{\partial}{\partial x_2}\left(\sum_{i=1}^m a_{ni}x_i\right) & \cdots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = A$$

▽

4 Homework problems

Question 5. Consider:

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

1. Find the diagonalizing transformation of A (in the real form if it exists).
2. Calculate the matrix exponent using Cayley-Hamilton.

Solution.

1. The characteristic polynomial:

$$\chi_A(\lambda) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+1 & -2 & -2 \\ -2 & \lambda+1 & -2 \\ -2 & -2 & \lambda+1 \end{bmatrix}\right) =$$

$$\xrightarrow{R_3=R_3-R_2} \det\left(\begin{bmatrix} \lambda+1 & -2 & -2 \\ -2 & \lambda+1 & -2 \\ 0 & -\lambda-3 & \lambda+3 \end{bmatrix}\right) =$$

$$\xrightarrow{C_2=C_2+C_3} \det\left(\begin{bmatrix} \lambda+1 & -4 & -2 \\ -2 & \lambda-1 & -2 \\ 0 & 0 & \lambda+3 \end{bmatrix}\right) =$$

$$= (\lambda+3)\det\left(\begin{bmatrix} \lambda+1 & -4 \\ -2 & \lambda-1 \end{bmatrix}\right) = (\lambda+3)((\lambda+1)(\lambda-1)-8) = \cdots = (\lambda+3)^2(\lambda-3)$$

The eigenvalues are therefore:

$$\lambda_{12} = -3, \lambda_3 = 3$$

where λ_{12} has an algebraic multiplicity of 2 and λ_3 has an algebraic multiplicity of 1. Now we calculate the eigenvectors:

For $\lambda_{12} = -3$:

$$(\lambda_{12} I - A)\eta_{12} = 0$$

$$\left(\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\right)\begin{bmatrix} \eta_{12,1} \\ \eta_{12,2} \\ \eta_{12,3} \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} \eta_{12,1} \\ \eta_{12,2} \\ \eta_{12,3} \end{bmatrix} = 0$$

Notice that:

$$\text{rank}\left(\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}\right) = 1$$

So the geometric multiplicity of λ_{12} is also 2, thus A is not defective and is diagonalizable.

After some row reduction, we get the below system of equations:

$$\begin{bmatrix} \eta_{12,1} + \eta_{12,2} + \eta_{12,3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Choosing $\eta_{12,2} = s$ and $\eta_{12,3} = t$ we get:

$$\eta_{12,1} = -s - t$$

thus,

$$\eta_{12} = \begin{bmatrix} \eta_{12,1} \\ \eta_{12,2} \\ \eta_{12,3} \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$$

Since the above equation is valid for all s and t, and so for $s = t = 1$ in particular, we write:

$$\eta_{12} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

From here we have two eigenvectors corresponding to the eigenvalue $\lambda_{12} = -3$:

$$\eta_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \eta_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

<u>For $\lambda_3 = 3$</u>:

$$(\lambda_3 I - A)\eta_3 = 0$$

$$\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \right) \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = 0$$

$$\xrightarrow[R_1=R_1+2R_2]{R_3=R_3-R_2} \begin{bmatrix} 0 & 6 & -6 \\ -2 & 4 & -2 \\ 0 & -6 & 6 \end{bmatrix} \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = 0$$

$$\xrightarrow[R_2=R_2-R_1/3]{R_3=R_3+R_1} \begin{bmatrix} 0 & 6 & -6 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = 0$$

$$\begin{bmatrix} \eta_{3,2} - \eta_{3,3} \\ \eta_{3,1} - \eta_{3,2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the eigenvector corresponding to eigenvalue $\lambda_3 = 3$ is:

$$\eta_3 = \begin{bmatrix} \eta_{3,1} \\ \eta_{3,2} \\ \eta_{3,3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We can now diagonalize the matrix using the transformation $\Lambda_A = T^{-1}AT$, where:

$$\Lambda_A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$T = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

2. We now use Cayley-Hamilton to calculate the matrix exponent. Since the multiplicity of $\lambda_{12} = -3$ is 2 we need to use the extension of Cayley-Hamilton. We are looking for coefficients g_0, g_1, g_2 that fulfill:

$$\exp(At) = \sum_{i=0}^{2} g_i A^i$$

Using the eigenvalues of A, we can find these coefficients:

$$\begin{aligned} e^{\lambda_{12}t} &= g_0 + g_1\lambda_{12} + g_2\lambda_{12}^2 \\ \lambda_{12}e^{\lambda_{12}t} &= g_1 + 2g_2\lambda_{12} \\ e^{\lambda_3 t} &= g_0 + g_1\lambda_3 + g_2\lambda_3^2 \end{aligned}$$

Or in matrix form:

$$\begin{bmatrix} e^{\lambda_{12}t} & \lambda_{12}E^{\lambda_{12}t} & E^{\lambda_3 t} \end{bmatrix} = \begin{bmatrix} g_0 & g_1 & g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ \lambda_{12} & 1 & \lambda_3 \\ \lambda_{12}^2 & 2\lambda_{12} & \lambda_3^2 \end{bmatrix}$$

Where due to the fact that λ_{12} has an algebraic multiplicity of 2, we also had to use derivatives of the CH equations. Substituting the values of λ_{12}, λ_3:

$$\begin{bmatrix} e^{-3t} & -3e^{-3t} & e^{3t} \end{bmatrix} = \begin{bmatrix} g_0 & g_1 & g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & 3 \\ 9 & -6 & 9 \end{bmatrix}$$

thus,

$$\begin{aligned} \begin{bmatrix} g_0 & g_1 & g_2 \end{bmatrix} &= \begin{bmatrix} e^{-3t} & -3e^{-3t} & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & 3 \\ 9 & -6 & 9 \end{bmatrix}^{-1} = \\ &= \begin{bmatrix} e^{-3t} & -3e^{-3t} & e^{3t} \end{bmatrix} \begin{bmatrix} 0.75 & -1/6 & -1/36 \\ 1.5 & 0 & -1/6 \\ 0.25 & 1/6 & 1/36 \end{bmatrix} = \\ &= \begin{bmatrix} -(15/4)e^{-3t} + (1/4)e^{3t} & (-1/6)e^{-3t} + (1/6)e^{3t} & (17/36)e^{-3t} + (1/36)e^{3t} \end{bmatrix} \end{aligned}$$

Substituting back into the matrix exponential:

$$\begin{aligned} \exp(At) &= g_0 I + g_1 A + g_2 A^2 = \\ &= (-(15/4)e^{-3t} + (1/4)e^{3t})I + ((-1/6)e^{-3t} + (1/6)e^{3t})A + ((17/36)e^{-3t} + (1/36)e^{3t})A^2 = \\ &= \cdots = \begin{bmatrix} 2e^{-3t}/3 + e^{3t}/3 & e^{3t}/3 - e^{-3t}/3 & e^{3t}/3 - e^{-3t}/3 \\ e^{3t}/3 - e^{-3t}/3 & 2e^{-3t}/3 + e^{3t}/3 & e^{3t}/3 - e^{-3t}/3 \\ e^{3t}/3 - e^{-3t}/3 & e^{3t}/3 - e^{-3t}/3 & (2e^{-3t}/3 + e^{3t}/3 \end{bmatrix} \end{aligned}$$

▽

Question 6. Find the derivative of the vector $f \in \mathbb{R}^3$ as a function of $x \in \mathbb{R}^2$:

$$\begin{bmatrix} x_1^3 - 2x_2 + 5\sin(x2) \\ x_2 e^{x_1} - 3x_2 \\ \frac{x_1}{1-x_2-x_2^2} \end{bmatrix}$$

Solution. The derivative of $f \in \mathbb{R}^3$ by $x \in \mathbb{R}^2$ is:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix}$$

Calculating each scalar derivative individually we get:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial(x_1^3 - 2x_2 + 5\sin(x2))}{\partial x_1} = 3x_1^2$$
$$\frac{\partial f_1}{\partial x_2} = \frac{\partial(x_1^3 - 2x_2 + 5\sin(x2))}{\partial x_2} = -2 + 5\cos(x_2)$$
$$\frac{\partial f_2}{\partial x_1} = \frac{\partial(x_2 e^{x_1} - 3x_2)}{\partial x_1} = x_2 e^{x_1}$$
$$\frac{\partial f_2}{\partial x_2} = \frac{\partial(x_2 e^{x_1} - 3x_2)}{\partial x_2} = e^{x_1} - 3$$
$$\frac{\partial f_3}{\partial x_1} = \frac{\partial\left(\frac{x_1}{1-x_2-x_2^2}\right)}{\partial x_1} = \frac{1}{1 - x_2 - x_2^2}$$
$$\frac{\partial f_3}{\partial x_2} = \frac{\partial\left(\frac{x_1}{1-x_2-x_2^2}\right)}{\partial x_2} = \frac{1 + 2x_2}{(1 - x_2 - x_2^2)^2}$$

Plugging into the matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 3x_1^2 & -2 + 5\cos(x_2) \\ x_2 e^{x_1} & e^{x_1} - 3 \\ \frac{1}{1-x_2-x_2^2} & \frac{1+2x_2}{(1-x_2-x_2^2)^2} \end{bmatrix}$$

Notice that when differentiating a vector of size 3×1 by a vector of size 2×1, we get a matrix of size 3×2. ▽

Question 7. Find the derivative of matrix $A \in \mathbb{R}^{3\times3}$ by scalar $t \in \mathbb{R}$:

$$A = \begin{bmatrix} t & \sin t & \frac{1}{t^2} e^{-t} \\ \ln t & \frac{cos(t)}{1+sin t} & (2t+1)^4(3t-5)^2 \\ \sqrt{3t + t^2} & 3t^2 - 2t + 1 & \ln(e^t + 1)^2 \end{bmatrix}$$

Solution. The derivative of A by t is done element-wise:

$$\frac{\mathrm{d}}{\mathrm{d}t}t = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\sin t) = \cos t$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t^2}\mathrm{e}^{-t}\right) = -\frac{1}{t^3}\mathrm{e}^{-t} - \frac{1}{t^2}\mathrm{e}^{-t} = -\frac{1}{t^3}\mathrm{e}^{-t}(1+t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln t) = \frac{1}{t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\cos t}{1+\sin t}\right) = \frac{(-\sin t)(1+\sin t) - \cos^2 t}{(1+\sin t)^2} = \frac{-\sin t - \sin^2 t - \cos^2 t}{(1+\sin t)^2} = -\frac{1+\sin}{(1+\sin t)^2} = -\frac{1}{(1+\sin t)}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left((2t+1)^4(3t-5)^2\right) = 8(2t+1)^3(3t-5)^2 + 6(2t+1)^4(3t-5) =$$
$$= (2t+1)^3(3t-5)\left[8(3t-5) + 6(2t+1)\right] = (2t+1)^3(3t-5)\left[24t - 40 + 12t + 6\right] =$$
$$= 2(2t+1)^3(3t-5)(18t-17)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{3t+t^2}\right) = \frac{3+2t}{2\sqrt{3t+t^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(3t^2 - 2t + 1) = 6t - 2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln^2(\mathrm{e}^t + 1)) = 2\ln(\mathrm{e}^t + 1)\mathrm{e}^t$$

So the matrix derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}t}A = \begin{bmatrix} 1 & \cos t & -\frac{1}{t^3}\mathrm{e}^{-t}(1+t) \\ \frac{1}{t} & -\frac{1}{(1+\sin t)} & 2(2t+1)^3(3t-5)(18t-17) \\ \frac{3+2t}{2\sqrt{3t+t^2}} & 6t-2 & 2\ln(\mathrm{e}^t+1)\mathrm{e}^t \end{bmatrix}$$

∇

Question 8. Let $A(t) \in \mathbb{R}^{n\times p}$, $B(t) \in \mathbb{R}^{p\times m}$ and $C \in \mathbb{R}^{n\times n}$ be matrices that depend on scalar $t \in \mathbb{R}$. Prove the following statements:

1. $\frac{\mathrm{d}}{\mathrm{d}t}(A(t)B(t)) = (\frac{\mathrm{d}}{\mathrm{d}t}A(t))B(t) + A(t)(\frac{\mathrm{d}}{\mathrm{d}t}B(t))$.
2. $\frac{\mathrm{d}}{\mathrm{d}t}(C^{-1}(t)) = -C^{-1}(\frac{\mathrm{d}}{\mathrm{d}t}C(t))C^{-1}$.

Solution.

1. We first express the matrices $A(t)B(t)$:

$$A(t)B(t) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} =$$
$$= \begin{bmatrix} \sum_{i=1}^{p} a_{1i}b_{i1} & \dots & \sum_{i=1}^{p} a_{1i}b_{im} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{p} a_{ni}b_{i1} & \dots & \sum_{i=1}^{p} a_{ni}b_{im} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

Differentiating element-wise by t gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}(A(t)B(t)) = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{p} a_{1i}b_{i1}\right) & \dots & \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{p} a_{1i}b_{im}\right) \\ \vdots & \ddots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{p} a_{ni}b_{i1}\right) & \dots & \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{p} a_{ni}b_{im}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{1i})b_{i1} + a_{1i}(\frac{\mathrm{d}}{\mathrm{d}t}b_{i1})\right) & \dots & \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{1i})b_{im} + a_{1i}(\frac{\mathrm{d}}{\mathrm{d}t}b_{im})\right) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{ni})b_{i1} + a_{ni}(\frac{\mathrm{d}}{\mathrm{d}t}b_{i1})\right) & \dots & \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{ni})b_{im} + a_{ni}(\frac{\mathrm{d}}{\mathrm{d}t}b_{im})\right) \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{1i})b_{i1}\right) & \dots & \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{1i})b_{im}\right) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{ni})b_{i1}\right) & \dots & \sum_{i=1}^{p}\left((\frac{\mathrm{d}}{\mathrm{d}t}a_{ni})b_{im}\right) \end{bmatrix}}_{\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)B(t)} + \underbrace{\begin{bmatrix} \sum_{i=1}^{p}\left(a_{1i}(\frac{\mathrm{d}}{\mathrm{d}t}b_{i1})\right) & \dots & \sum_{i=1}^{p}\left(a_{1i}(\frac{\mathrm{d}}{\mathrm{d}t}b_{im})\right) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{p}\left(a_{ni}(\frac{\mathrm{d}}{\mathrm{d}t}b_{i1})\right) & \dots & \sum_{i=1}^{p}\left(a_{ni}(\frac{\mathrm{d}}{\mathrm{d}t}b_{im})\right) \end{bmatrix}}_{A(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}B(t)\right)} =$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)B(t) + A(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}B(t)\right)$$

Q.E.D.

2. Using the definition of matrix inversion:

$$C^{-1}C = I$$

Deriving both sides of the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(C^{-1}C) = \frac{\mathrm{d}}{\mathrm{d}t}I = 0_{n\times n}$$

From section 1 of this question:

$$\frac{\mathrm{d}}{\mathrm{d}t}(C^{-1}C) = \frac{\mathrm{d}}{\mathrm{d}t}(C^{-1})C + C^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(C)$$

therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(C^{-1})C + C^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(C) = 0_{n\times n}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(C^{-1})C = -C^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(C)$$

Multiplying both sides of the equation by C^{-1} from the right gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}(C^{-1}) = -C^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(C)C^{-1}$$

Q.E.D.

▽