<div align="center">

LINEAR SYSTEMS (034032)

TUTORIAL 9

</div>

1 Topics

Frequency response of an LTI system, Bode and polar diagrams, filters.

2 Background

2.1 Frequency Response

Theorem 1. *Let $G : u \mapsto y$ be a stable continuous-time LTI system. Its response to the sinusoidal test input u such that*

$$u(t) = a \sin(\omega t + \phi)\mathbb{1}(t)$$

in steady state is also sinusoidal. Specifically

$$y_{\text{ss}}(t) = a|G(\mathrm{j}\omega)| \sin(\omega t + \phi + \arg G(\mathrm{j}\omega)),$$

where $|G(\mathrm{j}\omega)|$ and $\arg(G(\mathrm{j}\omega))$ are the gain (magnitude) and phase of the frequency response $G(\mathrm{j}\omega)$ of G, respectively, which can be calculated as

$$|G(\mathrm{j}\omega)| = \sqrt{x^2 + y^2} \quad and \quad \arg(G(\mathrm{j}\omega)) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \arctan(\frac{y}{x}) + \pi & x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & x < 0, y < 0 \\ +\frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ undefined & x = 0, y = 0 \end{cases}$$

where $x := \operatorname{Re} G(\mathrm{j}\omega)$ and $y := \operatorname{Im} G(\mathrm{j}\omega)$.

2.2 Bode Diagram

The Bode diagram is a way of visualizing $|G(\mathrm{j}\omega)|$ and $\arg(G(\mathrm{j}\omega))$. In order to draw the Bode diagram by hand we actually draw the asymptotic diagram.

For example, Fig. 1 shows the Bode diagram of the transfer function:
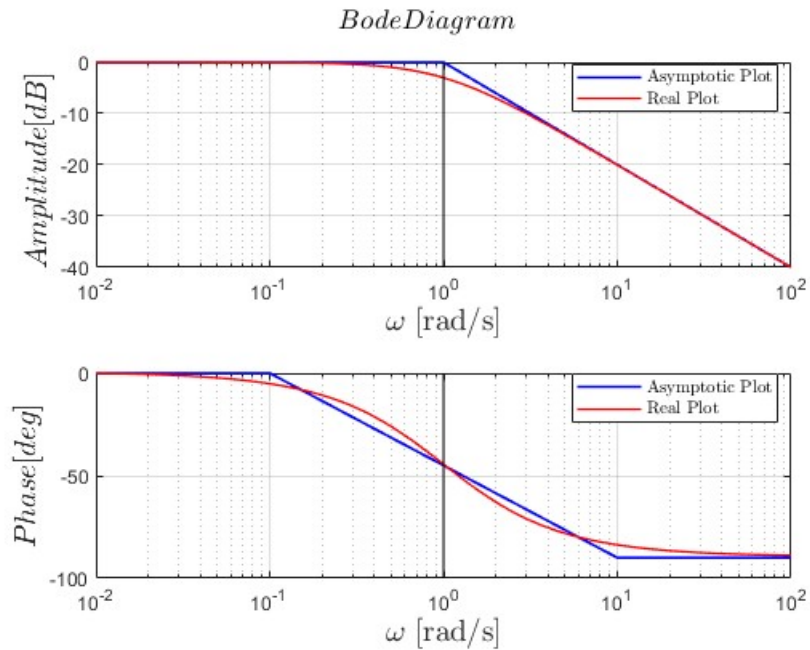
$$G(s) = \frac{1}{s + 1}$$

Fig. 1: Real and Asymptotic Bode Diagram

2.2.1 Bode Scales

The Bode diagram's horizontal axis is the frequency ω in logarithmic scale. The magnitude part is represented in dB and the phase part in deg. dB is the unit decibel and is defined as:

$$k_{(\text{dB})} = 20 \log k$$
$$k = 10^{k_{(\text{dB})}/20}$$

We also define dec, decade, which is the distance of 10 units (10 scales on the horizontal axis).

2.2.2 Steps to Create an Asymptotic Diagram

1. Decomposing the system into the product of sub-systems.

$$G(s) = G_1(s) \cdot G_2(s) \cdot \ldots \cdot G_m(s)$$

where each one of these subsystems $i \in \{1, m\}$ are first and second order systems of the form:

$$G_i(s) = \frac{1}{s}, s, \frac{1}{as + b}, as + b, \frac{1}{as^2 + bs + c}, as^2 + bs + c$$

2. Every first order system of the form $\frac{1}{as+b}$ we convert to

$$\frac{k_{\text{st}}}{\tau s + 2} \tag{1}$$

and every first order system of the form $as + b$ we convert to

$$k_{\text{st}}(\tau s + 1)$$

In the same way, we convert second order systems of the form $\frac{1}{as^2+bs+c}$ to

$$\frac{k_{st}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k_{st}}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

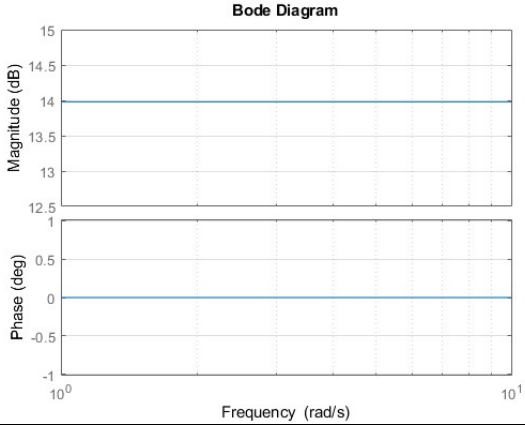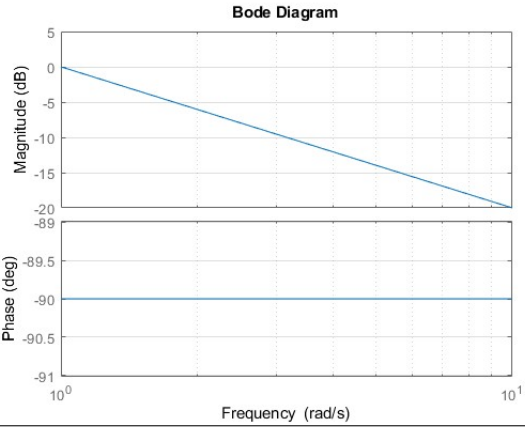and second order systems of the form $as^2 + bs + c$ to

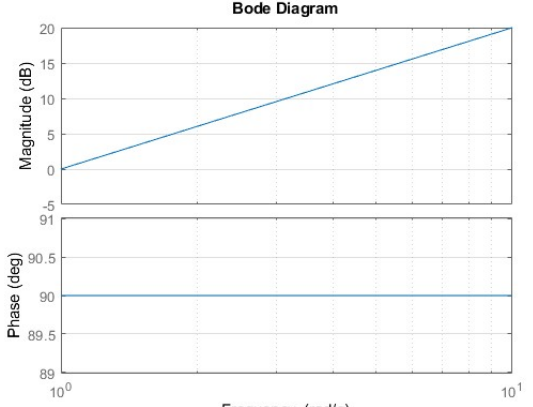$$\frac{k_{st}(s^2 + 2\zeta\omega_n s + \omega_n^2)}{\omega_n^2} = (s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1$$
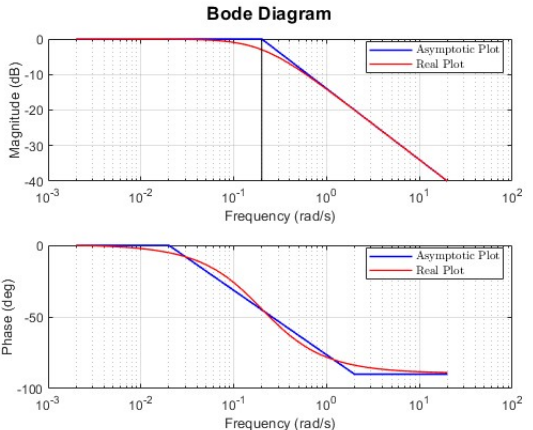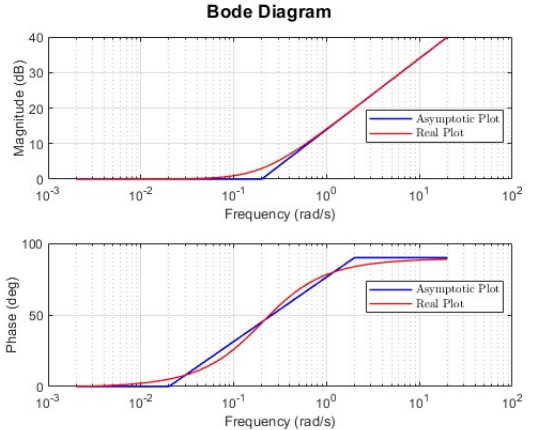
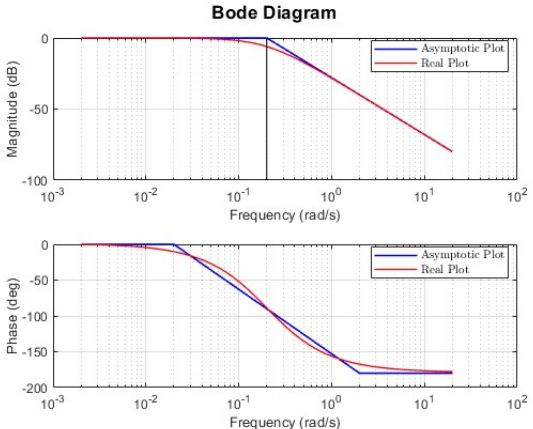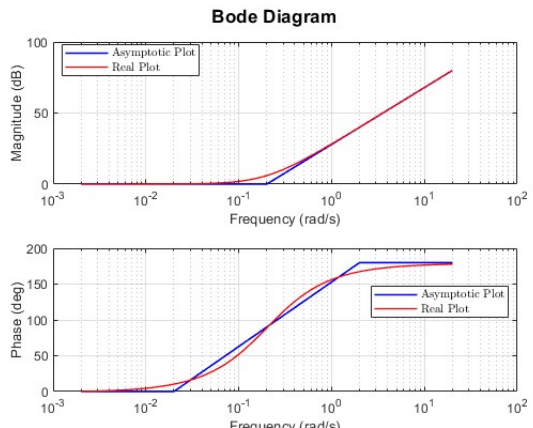3. We unite all the static gains by multiplying all k_{st} elements, getting:
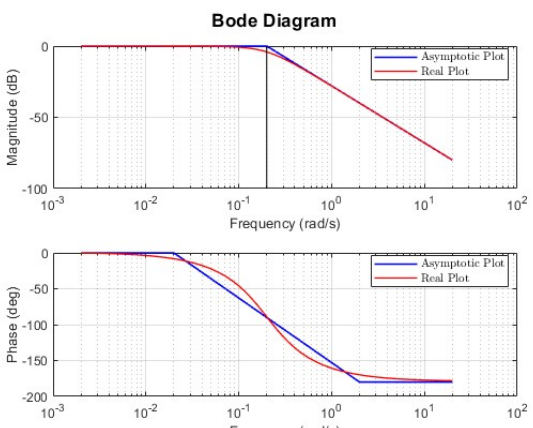
$$G(s) = k \cdot \mathcal{G}_1(s) \cdot \mathcal{G}_2(s) \cdot \ldots \cdot \mathcal{G}_m(s)$$
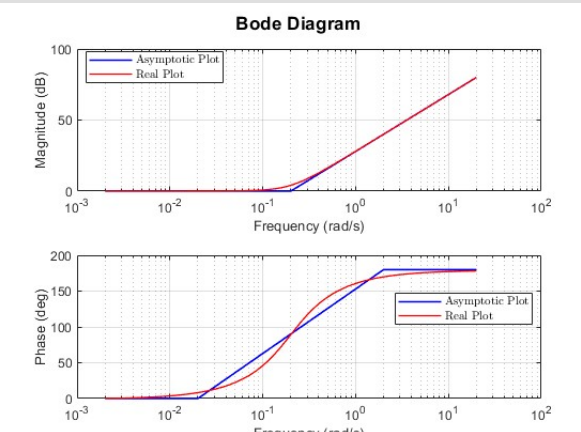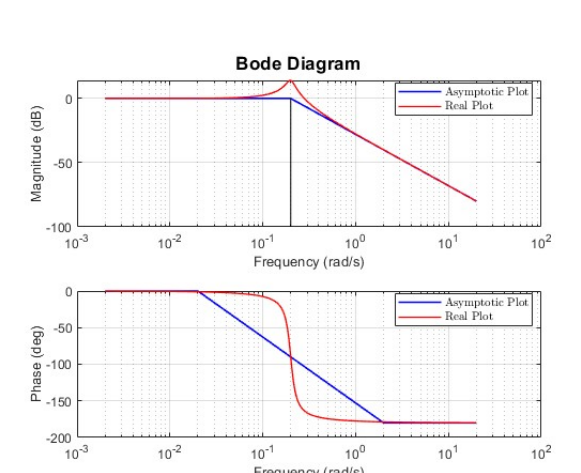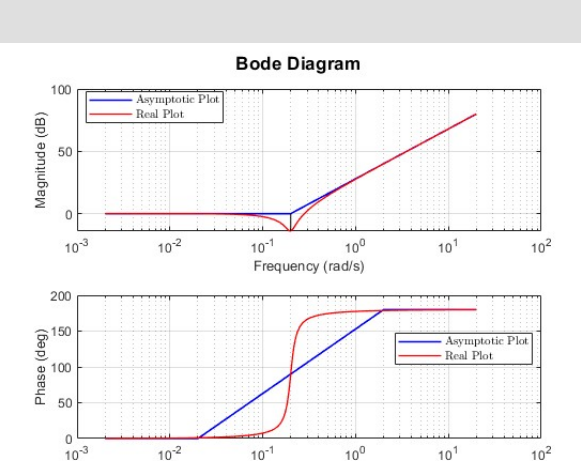
where $\mathcal{G}_i(s)$ are the standard transfer functions.

4. Using the table below, we draw the asymptotic Bode diagram of the system as a combination of the Bodes of the standard systems.

Function	Bode Graph	Asymptotic Behaviour
k		The magnitude is a straight horizontal line with a constant gain of $k_{(dB)} = 20\log k$. The phase is $0°$.
$\dfrac{1}{s}$		The magnitude is a straight line that crosses the horizontal axis at $1[\frac{rad}{sec}]$ and has a slope of $-20\left[\frac{dB}{dec}\right]$. The phase is constant at $-90°$.

Function	Bode Graph	Comment
s		The magnitude is a straight line that crosses the horizontal axis at $1[\frac{rad}{sec}]$ and has a slope of $20\left[\frac{dB}{dec}\right]$. The phase is constant at $90°$.
$\dfrac{1}{\tau s + 1}$		The magnitude is constant at $0\,[dB]$ until the corner frequency of $\omega_p = \frac{1}{\tau}$, after which it is a straight line with slope $-20\left[\frac{dB}{dec}\right]$. The phase is constant at $0°$ until $0.1\omega_p$ after which it is a straight line that crosses $-45°$ at frequency ω_p, and at $10\omega_p$ again constant at $-90°$.
$\tau s + 1$		The magnitude is constant at $0[dB]$ until the corner frequency of $\omega_p = \frac{1}{\tau}$, after which it is a straight line with slope $20\left[\frac{dB}{dec}\right]$. The phase is constant at $0°$ until $0.1\omega_p$ after which it is a straight line that crosses $45°$ at frequency ω_p, and at $10\omega_p$ again constant at $90°$.

Function	Bode Graph	Comment
$\dfrac{1}{(\tau s + 1)^n}$		The magnitude is constant at $0\,[\mathrm{dB}]$ until the corner frequency of $\omega_\mathrm{p} = \frac{1}{\tau}$, after which it is a straight line with slope $-20 \cdot n \left[\frac{\mathrm{dB}}{\mathrm{dec}}\right]$. The phase is constant at $0°$ until $0.1\omega_\mathrm{p}$ after which it is a straight line that crosses $-45n°$ at frequency ω_p, and at $10\omega_\mathrm{p}$ again constant at $-90n°$.
$(\tau s + 1)^n$		The magnitude is constant at $0[\mathrm{dB}]$ until the corner frequency of $\omega_\mathrm{p} = \frac{1}{\tau}$, after which it is a straight line with slope $20 \cdot n \left[\frac{\mathrm{dB}}{\mathrm{dec}}\right]$. The phase is constant at $0°$ until $0.1\omega_\mathrm{p}$ after which it is a straight line that crosses $45n°$ at frequency ω_p, and at $10\omega_\mathrm{p}$ again constant at $90n°$.
$\dfrac{\omega_\mathrm{n}^2}{s^2 + 2\zeta\omega_\mathrm{n}s + \omega_\mathrm{n}^2}$ $\zeta \geq \frac{1}{\sqrt{2}}$		The magnitude is constant at $0[\mathrm{dB}]$ until the natural frequency of ω_n, after which it is a straight line with slope $-40 \left[\frac{\mathrm{dB}}{\mathrm{dec}}\right]$. The phase is constant at $0°$ until $0.1\omega_\mathrm{n}$ after which it is a straight line that crosses $-90°$ at frequency ω_n, and at $10\omega_\mathrm{n}$ again constant at $-180°$.

Function	Bode Graph	Comment
$\dfrac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}$ $\zeta \geq \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $40\left[\frac{dB}{dec}\right]$. The phase is constant at 0° until $0.1\omega_n$ after which it is a straight line that crosses 90° at frequency ω_n, and at $10\omega_n$ again constant at 180°.
$\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $-40\left[\frac{dB}{dec}\right]$. The real system has a resonance peak at frequency $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$ The phase is constant at 0° until $0.1\omega_n$ after which it is a straight line that crosses $-90°$ at frequency ω_n, and at $10\omega_n$ again constant at $-180°$.
$\dfrac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $40\left[\frac{dB}{dec}\right]$. The real system has a resonance dip at frequency $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$ The phase is constant at 0° until $0.1\omega_n$ after which it is a straight line that crosses 90° at frequency ω_n, and at $10\omega_n$ again constant at 180°.

Function	Bode Graph	Comment
$\left(\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)'$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $-40 \cdot n \left[\frac{dB}{dec}\right]$. The real system has a resonance peak at frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ The phase is constant at $0°$ until $0.1\omega_n$ after which it is a straight line that crosses $-90n°$ at frequency ω_n, and at $10\omega_n$ again constant at $-180n°$.
$\left(\dfrac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}\right)'$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $40 \cdot n \left[\frac{dB}{dec}\right]$. The real system has a resonance dip at frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ The phase is constant at $0°$ until $0.1\omega_n$ after which it is a straight line that crosses $90n°$ at frequency ω_n, and at $10\omega_n$ again constant at $180n°$.

For more examples see the lecture slides.

2.2.3 General Guidelines for Asymptotic Bode

- Each pole adds $-20\left[\frac{dB}{dec}\right]$ to the magnitude's slope.
- Each zero adds $+20\left[\frac{dB}{dec}\right]$ to the magnitude's slope.
- Every pole in $\mathbb{C}\backslash\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ adds a phase lag of $-90°$.
- Every pole in $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ adds a phase lead of $+90°$.
- Every zero in $\mathbb{C}\backslash\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ adds a phase lead of $+90°$.
- Every zero in $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ adds a phase lag of $-90°$.

2.3 Polar Diagram

The polar diagram is another way to represemt the frequency response of the system. Similarly to the Bode diagram, the polar diagram shows $G(j\omega)$. But, unlike the Bode diagram which is comprised of two parts: the magnitude as a function of frequency $|G(j\omega)|$ and the phase as a function of frequncy $\arg(G(j\omega))$, the polar diagram is comprised of only one graph where we can see the real and imaginary parts $\Re\{G(j\omega)\}$, $\Im\{G(j\omega)\}$ as a function of the frequency which **isn't** shown directly on the graph.

For example, Fig. 2 shows the polar diagram of the transfer function:
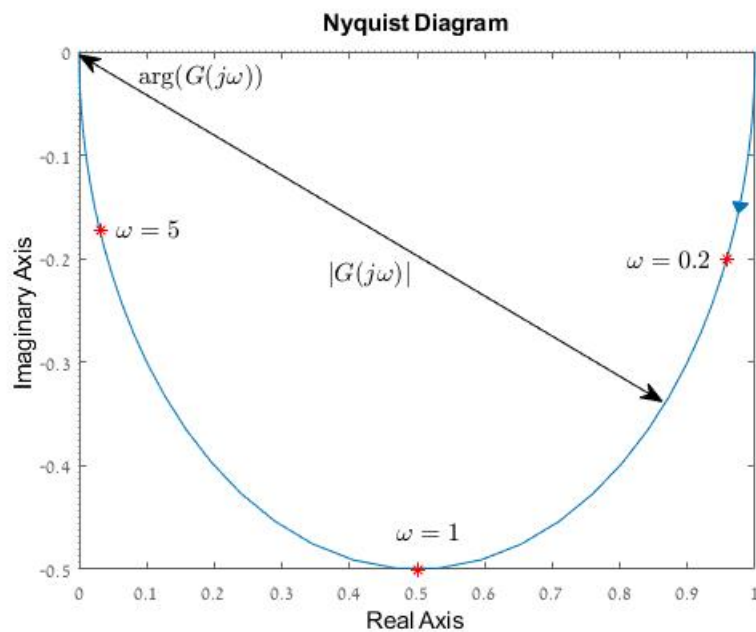
$$G(s) = \frac{1}{s+1}$$

Fig. 2: Polar Diagram

The blue arrow denotes the direction in which the frequency increases. The red points denote the frequency response at chosen frequencies.

2.3.1 The Magnitude and Phase in the Polar Diagram

Similarly to the Bode diagram, we can extract the magnitude and phase of the system for a given frequency from the polar diagram. But, here we do not know the actual frequency. The magnitude at a given point $|G(j\omega)|$ is the distance of that point from the origin. The phase $\arg(G(j\omega))$ is the angle between the line connecting that point to the origin, and the positive direction of the real axis.

When looking back at the Bode diagram of the same system in Fig. 1, we can see that the magnitude decreases monotonically. This can also be seen in the polar diagram as the distance from the origin decreases until it reaches 0. The system's phase also decreases which can again be seen in the polar diagram.

2.4 Filters

Using the frequency response we can design filters to shape the spectra of signals. 4 categories of filters are generally used:

1. Low-pass Filters:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \leq \omega_b$$

2. High-pass Filter:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \geq \omega_c$$

3. Band-pass Filter:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \in [\omega_1, \omega_2]$$

4. Band-stop Filter:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \notin (\omega_1, \omega_2)$$

2.4.1 Butterworth Filter

A filter based on the Butterwoth polynomials:
$$B_n(s) = \begin{cases} \prod_{i=1}^{n/2}(s^2 + 2\zeta_i s + 1) & \text{if } n \text{ is even} \\ (s + 1)\prod_{i=1}^{(n-1)/2}(s^2 + 2\zeta_i s + 1) & \text{if } n \text{ is odd} \end{cases}$$

where,
$$\zeta_i := \sin\left(\frac{2i - 1}{2n}\pi\right) \in (0, 1)$$

The low-pass Butterworth filter is:
$$F(s) = \frac{1}{B_n(s/\omega_b)} \implies |F(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_b)^{2n}}}$$
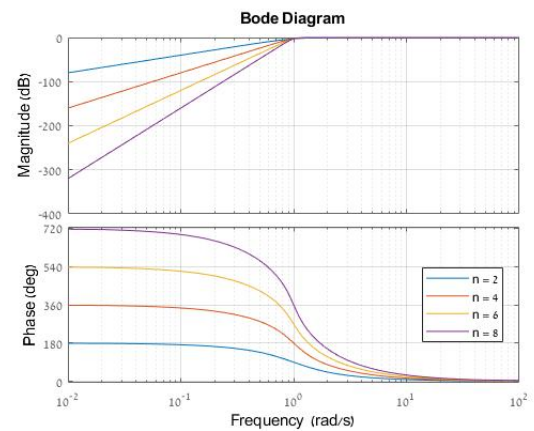
The high-pass Butterworth filter is:
$$F(s) = \frac{s/\omega_c}{B_n(s/\omega_c)} \implies |F(j\omega)| = \frac{(\omega/\omega_c)^n}{\sqrt{1 + (\omega/\omega_c)^{2n}}}$$

Their Bode diagrams are shown in figures 3(a) and 3(b).
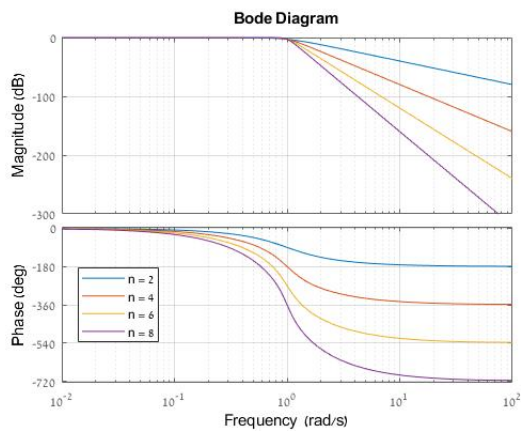
(a) Low-pass Butterworth Filter, $\omega_b = 1$

(b) High-pass Butterworth Filter, $\omega_c = 1$

Fig. 3: Butterworth Filters Bode Diagrams

2.4.2 Notch Filter

A narrow stopband filter of the form:

$$F(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \implies |F(s)| = \sqrt{\frac{(\omega^2 - \omega_0^2)^2}{(\omega^2 - \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$

It has a stopband of (ω_1, ω_2) with:

$$\omega_1 := \left(\sqrt{1 + \zeta^2} - \zeta\right)\omega_0$$
$$\omega_2 := \left(\sqrt{1 + \zeta^2} + \zeta\right)\omega_0 = 1/\omega_1$$

The Bode diagram is shown in Fig. 4
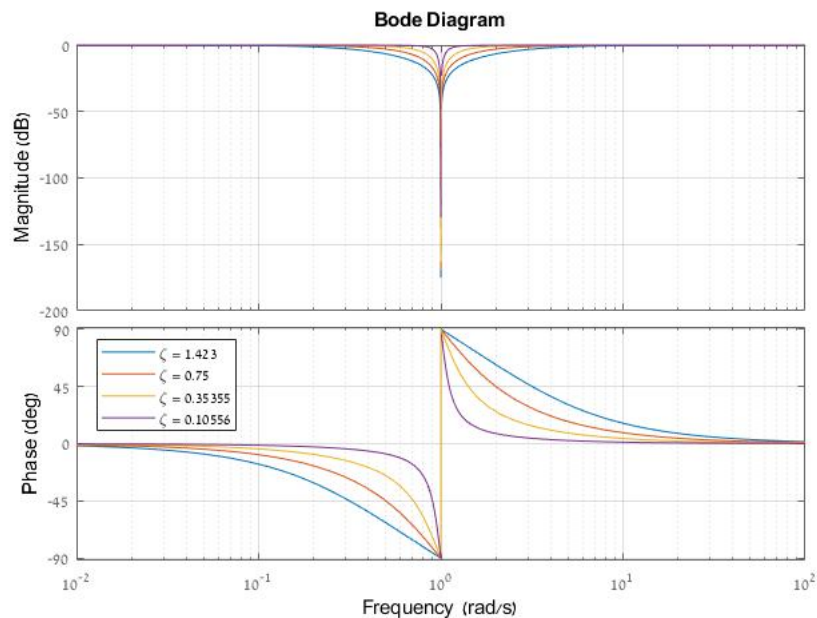
Fig. 4: Notch Filter Bode Diagram

2.5 Matlab Commands

Some useful Matlab functions:

1. `mag2db` and `db2mag` convert the magnitude to and from dB.

2. `bode(G)` and `bodeplot(G)` plot the bode diagram of the system G. `bodeplot(G)` has more plot options.

3. `nyquist(G)` and `nyquistplot(G)` plot a diagram of the system G in polar coordinates (called a Nyquist diagram). To create a plain polar plot ($\omega > 0$), use:
   `setoptions(nyquistplot(G),'ShowFullContour','off')`.

4. `[num,den] = butter(n,wb,'s');` generates the numerator and denominator of the low-pass Butterworth filter of order `n` with the bandwidth `wn`.

3 Problems

Question 1. Draw the asymptotic Bode magnitude plots of the transfer function

$$G(s) = \frac{k}{(\tau_1 s + 1)(\tau_2 s + 1)},$$

where $\tau_1 > 0$ and $\tau_2 > 0$.

Solution. Let us factor $G(s)$ as

$$G(s) = k \cdot \frac{1}{\tau_1 s + 1} \cdot \frac{1}{\tau_2 s + 1} =: G_0(s)G_1(s)G_2(s),$$

The transfer function $G_0(s) = k$ is static, whose magnitude bode diagram is the straight horizontal line at the level $20 \log k$ (remember, the Bode plot is in dB), see Fig. 5(a). The two other transfer functions are

(a) $G(s) = k$ (b) $G(s) = 1/(\tau s + 1)$ (c) $G(s) = G_0(s)G_1(s)G_2(s)$
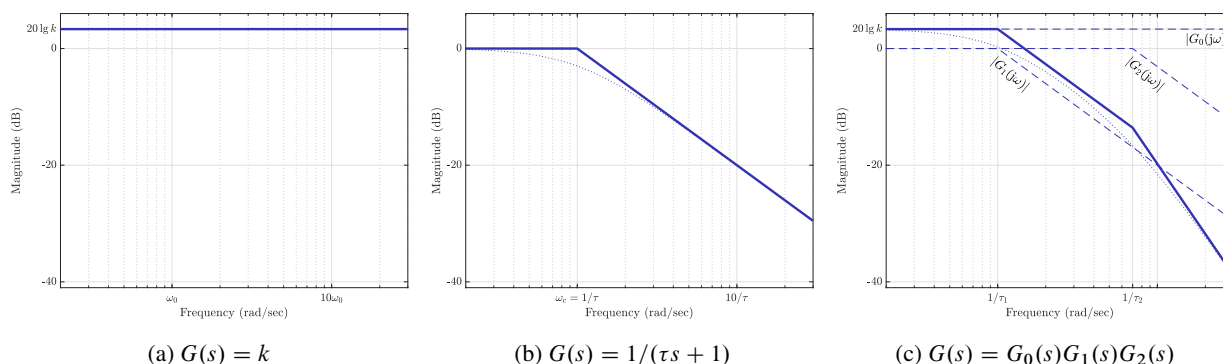
Fig. 5: Asymptotic Bode magnitude diagrams (here $k > 1$); dotted lines correspond to actual Bode plots

first-order transfer functions with the unit static gain of the form $1/(\tau s + 1)$. The asymptotic magnitude Bode plot of these kinds of transfer functions comprises two straight lines: a horizontal one at 0 dB in the low-frequency range, up to the cutoff frequency $\omega_c = 1/\tau$, and a straight line starting at ω_c and decaying with the slope of -20 deg/dec (sometimes referred to as having a *rolloff* of 1), see Fig. 5(b). Now, we know (see lecture 9) that the Bode magnitude plot of the cascade of systems is the superposition of their individual Bode magnitude plots. We then end up with the diagram presented by the solid line in Fig. 5(c).   ▽

Question 2. Draw the Bode and polar plots for the following transfer functions:

1. $G_1(s) = \dfrac{1}{(\tau s + 1)^2}$ for $\tau > 0$;

2. $G_2(s) = \dfrac{k}{s(\tau s + 1)}$ for $\tau > 0$ and $k > 0$;

3. $G_3(s) = \dfrac{\tau_2 s + 1}{\tau_1 s + 1}$ for $\tau_1 = \frac{1}{3}$ and $\tau_2 = \frac{5}{3}$ and then for $\tau_1 = \frac{5}{3}$ and $\tau_2 = \frac{1}{3}$.

Solution. We shall follow the following procedure: first, draw asymptotic Bode plots, then actual Bode plots (via "rounding corners"), then present frequency responses at several frequencies on the polar plot plane (from Bode), and then actual polar plots via connecting those points. The magnitude and the phase at the chosen points can be evaluated via the analytic expressions for the corresponding frequency responses.

1. This $G_1(s)$ can be presented as the cascade of two identical systems with the transfer function $1/(\tau s + 1)$. The asymptotic plots for $G(j\omega)$ can be presented, following the steps of Question 1, as in Fig. 6(a). The actual Bode plots are then shown by the solid lines in Fig. 6(b). Analytic expression for the frequency response is

$$G_1(j\omega) = \frac{1}{(j\omega\tau + 1)^2} = \left(\frac{1}{\sqrt{\tau^2\omega^2 + 1}}\right)^2 e^{j2\arg(1/(j\omega\tau+1))} = \frac{1}{\tau^2\omega^2 + 1}e^{-j2\arctan(\tau\omega)}.$$

Let us pick the following frequency points:

ω	0	$\omega_1 = 0.1/\tau$	$\omega_2 = 1/\tau$	$\omega_3 = 3/\tau$	∞
$\lvert G_1(j\omega)\rvert$	1	≈ 0.99	0.5	0.1	0
$\arg(G_1(j\omega))$	$0°$	$\approx -11°$	$-90°$	$\approx -143°$	$-180°$

The frequency responses at the frequencies ω_i, $i = 1, 2, 3$, are marked by large dots in Fig. 6(b). The corresponding points at the complex plane, which is the plane of the polar plot, are presented in Fig. 6(c). Connecting these dots we end up with the polar plot in Fig. 6(d), where the arrow shows the direction of the plot as ω increases. Note that as $\omega \uparrow \infty$, the plot approaches the origin along the negative real axis, because the argument of $G_1(j\omega)$ approaches $-180°$ then.

2. The steps here are similar to those taken in the previous system. The asymptotic and actual Bode diagrams are then presented in Figs. 7(a) and 7(c), respectively. The frequency response

$$G_2(j\omega) = \frac{k}{j\omega(j\tau\omega + 1)} = \frac{k}{\omega\sqrt{\tau^2\omega^2 + 1}}e^{-j(\pi/2+\arctan(\tau\omega))},$$

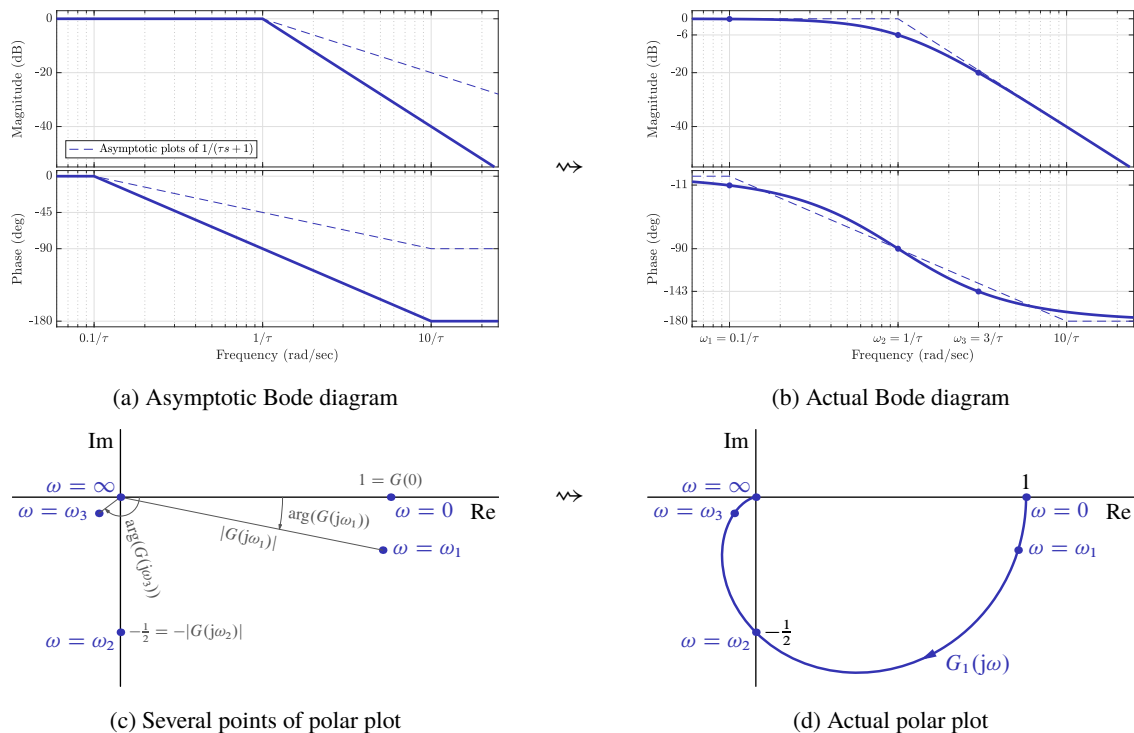from which the frequency responses at the chosen points are

(a) Asymptotic Bode diagram

(b) Actual Bode diagram

(c) Several points of polar plot

(d) Actual polar plot

Fig. 6: Frequency response plots of $G_1(s)$

ω	0	$\omega_1 = 0.6/\tau$	$\omega_2 = 1/\tau$	$\omega_3 = 3/\tau$	∞
$\lvert G_2(j\omega)\rvert$	∞	$\approx 1.43 k\tau$	$k\tau/\sqrt{2}$	$\approx 0.11 k\tau$	0
$\arg(G_2(j\omega))$	$-90°$	$\approx -121°$	$-135°$	$\approx -153°$	$-180°$

The only nontrivial difference is that due to the presence of an integrator in $G_2(s)$, $\lvert G_2(0)\rvert = \infty$. To understand the behavior of the polar graph at small frequencies, rewrite

$$G_2(j\omega) = \frac{k}{j\omega(j\tau\omega + 1)} = -\frac{k\tau}{\tau^2\omega^2 + 1} - j\frac{k}{\omega(\tau^2\omega^2 + 1)}.$$

It is now seen that while the imaginary part goes to $-\infty$, the real part approaches a finite value, $-k\tau$ (in fact, the real part belongs to $(-k\tau, 0)$ for all ω). This yields the polar plot in Figs. 7(d).

3. This transfer function can be presented as

$$G_3(s) = \frac{\tau_2 s + 1}{\tau_1 s + 1} = (\tau_2 s + 1) \cdot \frac{1}{\tau_1 s + 1},$$

which is the cascade of a first-order system and the inverse of another first-order system. The asymptotic plots of the former are as the dashed lines in Fig. 6(a) and those of the latter—are the same plots modulo the opposite signs (inversion of a transfer function means sign inversion on Bode). The form of the convolution of such plots depends on the relation between τ_1 and τ_2.

(a) Asymptotic Bode diagram

(b) Actual Bode diagram

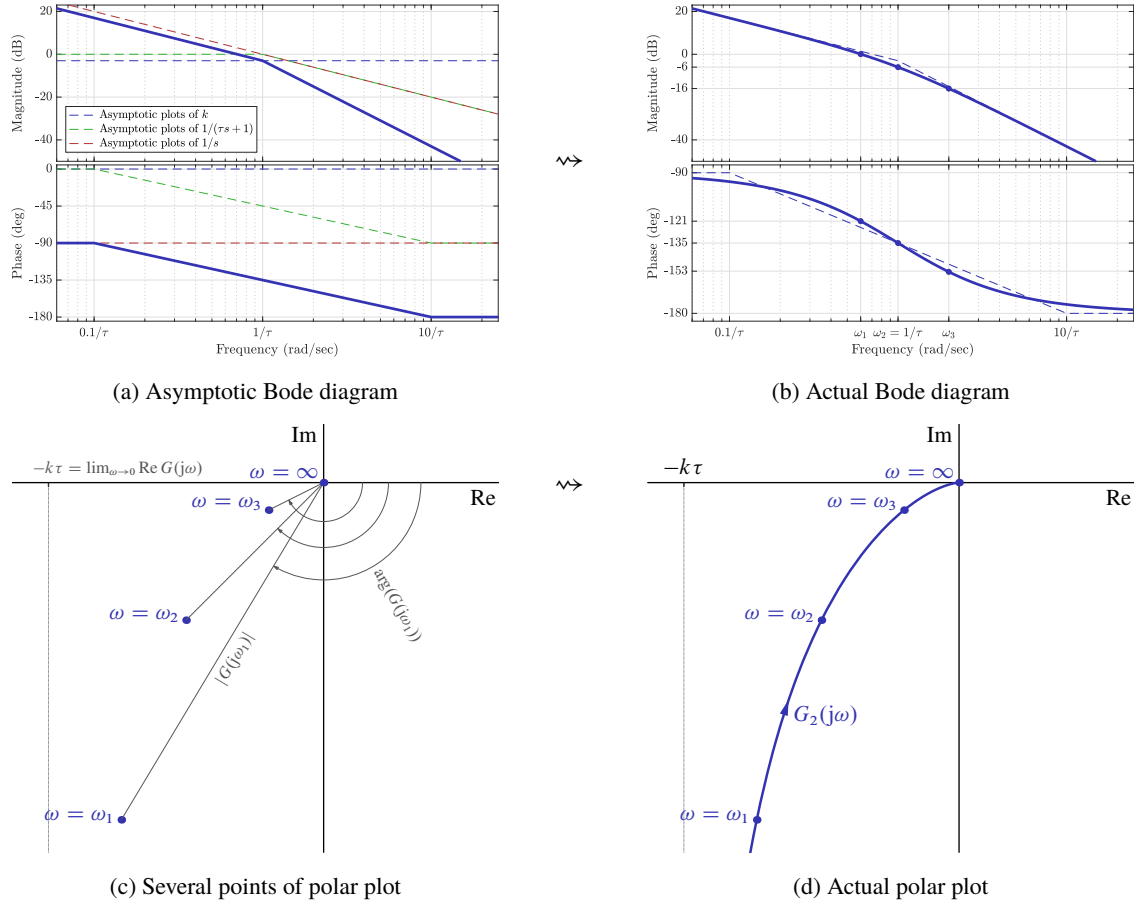(c) Several points of polar plot

(d) Actual polar plot

Fig. 7: Frequency response plots of $G_2(s)$ (here $k = 1/\sqrt{2}$ and $\tau = 1$)

- If $\frac{1}{3} = \tau_1 < \tau_2 = \frac{5}{3}$, the effect of the zero precedes that of the pole (as ω increases). Hence, the magnitude starts at $0\,\mathrm{bB}$ (this is the static gain), then gets up at $\omega_{c2} := 1/\tau_2 = 0.6$ and then becomes flat again at $\omega_{c1} := 1/\tau_1 = 3$. This is what we can see in Fig. 8(a). The actual Bode diagram is presented in Fig. 8(b). To construct the polar plot, pick the following frequencies:

ω	0	$\omega_1 = 0.1342$	$\omega_2 = 1.342$	$\omega_3 = 13.42$	∞
$\lvert G_3(\mathrm{j}\omega) \rvert$	1	≈ 1.02	≈ 2.2361	≈ 4.88	5
$\arg(G_3(\mathrm{j}\omega))$	$0°$	$\approx 10.04°$	≈ 41.81	≈ 10.04	$0°$

where the values can be obtained from the frequency response

$$G_3(\mathrm{j}\omega) = \frac{\mathrm{j}\tau_2\omega + 1}{\mathrm{j}\tau_1\omega + 1} = \sqrt{\frac{\tau_2^2\omega + 1}{\tau_1^2\omega + 1}}\, e^{\mathrm{j}(\arctan(\tau_2\omega) - \arctan(\tau_1\omega))}$$

(with $\frac{1}{3} = \tau_1 > \tau_2 = \frac{5}{3}$, although it is also true for $\tau_1 > \tau_2$). The polar plot is then as shown in Fig. 8(d). Note that $\arg(G_3(\mathrm{j}\omega_1)) = \arg(G_3(\mathrm{j}\omega_3))$, so the corresponding points lie on the same radial line in the complex plane.

- If $\frac{5}{3} = \tau_1 > \tau_2 = \frac{1}{3}$, the effect of the pole precedes that of the zero (as ω increases). Hence, the magnitude starts at $0\,\mathrm{bB}$ (this is the static gain), then gets down at $\omega_{c1} := 1/\tau_1 = 0.6$ and then becomes flat again at $\omega_{c2} := 1/\tau_2 = 3$. This is what we can see in Fig. 9(a). The actual Bode diagram is presented in Fig. 9(b). To construct the polar plot, pick the same frequencies as in the previous case. We then have:

ω	0	$\omega_1 = 0.1342$	$\omega_2 = 1.342$	$\omega_3 = 13.42$	∞
$\lvert G_3(\mathrm{j}\omega) \rvert$	1	≈ 0.98	≈ 0.45	≈ 0.2	0.2
$\arg(G_3(\mathrm{j}\omega))$	$0°$	$\approx -10.04°$	≈ -41.81	≈ -10.04	$0°$

The polar plot is then as shown in Fig. 9(d).

(a) Asymptotic Bode diagram

(b) Actual Bode diagram
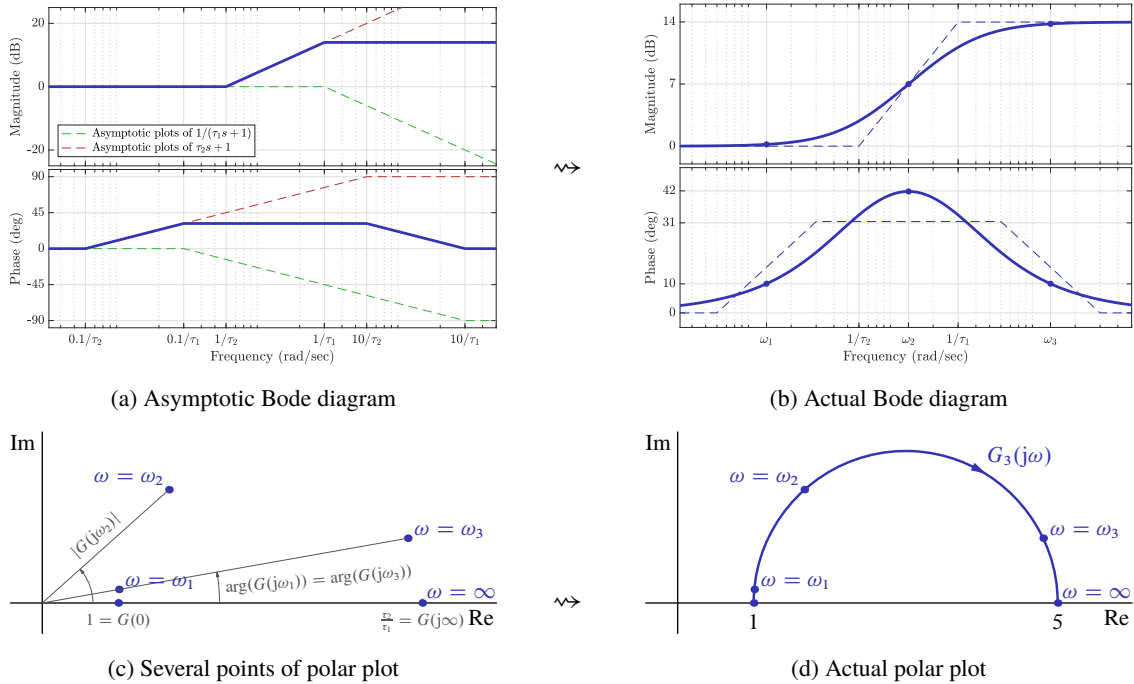
(c) Several points of polar plot

(d) Actual polar plot

Fig. 8: Frequency response plots of $G_3(s)$ for $\frac{1}{3} = \tau_1 < \tau_2 = \frac{5}{3}$

As a matter of fact, it can be verified that for all τ_1 and τ_2, the real and imaginary parts of $G_3(j\omega)$,

$$\text{Re}\, G_3(j\omega) = \frac{\tau_1\tau_2\omega^2 + 1}{\tau_1^2\omega^2 + 1} \quad \text{and} \quad \text{Im}\, G_3(j\omega) = \frac{(\tau_2 - \tau_1)\omega}{\tau_1^2\omega^2 + 1},$$

verify

$$\left(\text{Re}\, G_3(j\omega) - \frac{\tau_1 + \tau_2}{2\tau_1}\right)^2 + \left(\text{Im}\, G_3(j\omega)\right)^2 = \left(\frac{\tau_1 - \tau_2}{2\tau_1}\right)^2.$$

This implies that whenever $\tau_1 \neq \tau_2$, the polar plot of $G_3(j\omega)$ is a semi-circle centered at $\frac{1}{2}(1 + \frac{\tau_2}{\tau_1})$ and having the radius $\frac{1}{2}|1 - \frac{\tau_2}{\tau_1}|$. This is also true for the particular case when $\tau_2 = 0$, which is the standard first-order transfer function.

$\triangledown$

Question 3. A signal u passes a stable system $F(s)$, whose frequency response is presented by its polar plot in Fig. 10. The magnitude $|F(j\omega)|$ is a monotonically decreasing function of ω. Denote by y the resulting output signal, i.e. $y = F(s)u$.

1. Find $y(t)$ for $u(t) = 2\sin(t)$.

2. Find $y(t)$ for $u(t) = \sin(\frac{2}{3}t + 2) + 3\sin(\frac{1}{10}t + 0.356)$.

3. In what frequency range harmonic u's are attenuated by at least a factor of 5?

4. What information about the bandwidth of $F(s)$ can be extracted from Fig. 10?

Solution. All reasonings below are based on the Frequency Response Theorem.

1. With this input, $y(t) = 2|F(j)| \sin(t + \arg(F(j))) = 2 \times 0.5 \sin(t - \pi) = -\sin(t)$.

(a) Asymptotic Bode diagram

(b) Actual Bode diagram

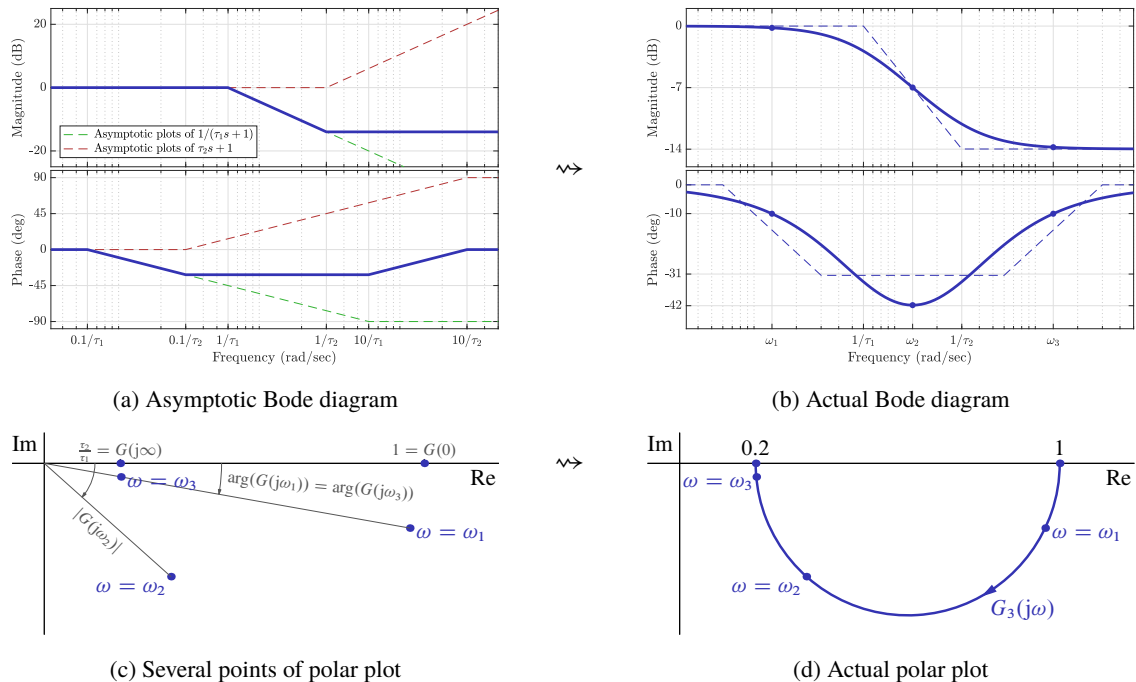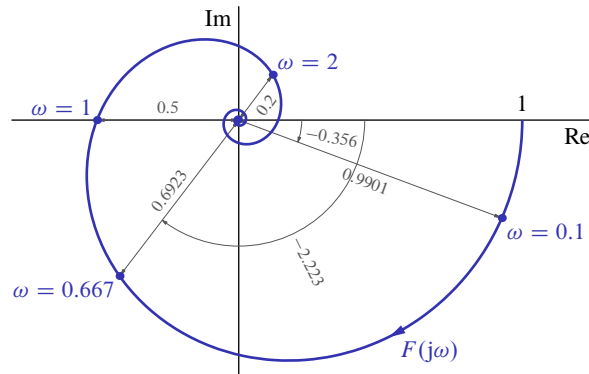(c) Several points of polar plot

(d) Actual polar plot

Fig. 9: Frequency response plots of $G_3(s)$ for $\frac{5}{3} = \tau_1 > \tau_2 = \frac{1}{3}$

Fig. 10: Polar plot of $F(j\omega)$ in Question 3

2. In this case,

$$
\begin{aligned}
y(t) &= |F(j\tfrac{2}{3})| \sin(\tfrac{2}{3}t + 2 + \arg(F(j\tfrac{2}{3}))) + 3|F(j\tfrac{1}{10})| \sin(\tfrac{1}{10}t + 0.356 + \arg(F(j\tfrac{1}{10}))) \\
&= 0.6923 \sin(\tfrac{2}{3}t + 2 - 2.223) + 3 \times 0.9901 \sin(\tfrac{1}{10}t + 0.356 - 0.356) \\
&= 0.6923 \sin(\tfrac{2}{3}t - 0.223) + 2.9703 \sin(\tfrac{1}{10}t).
\end{aligned}
$$

3. A harmonic $u(t) = c e^{j\omega t}$ is attenuated by $F(s)$ by at least a factor of 5 iff $|F(j\omega)| \le \tfrac{1}{5}$. From the plot in Fig. 10 and the fact that the frequency response gain is monotonically decreasing, we have that the required range is $\omega \in [2, \infty)$.

4. Because $|F(j\omega)|$ is monotonically decreasing and $F(0) = 1 > 1/\sqrt{2}$, this $F(s)$ is a low-pass filter. Its bandwidth is the smallest frequency where $|F(j\omega_b)| = 1/\sqrt{2} \approx 0.7071$. But the polar plot does not contain frequency information (apart from what is marked). All we can say is that $\omega_b \in (\tfrac{1}{10}, \tfrac{2}{3})$, as $|F(j\tfrac{1}{10})| = 0.9901 > 1/\sqrt{2}$ and $|F(j\tfrac{2}{3})| = 0.6923 < 1/\sqrt{2}$ and there are no frequency points in between. Because $|F(j\tfrac{2}{3})|$ is very close to $1/\sqrt{2}$, we may be tempted to say that ω_b is very close to $\tfrac{2}{3}$. But this statement might be risky, as we cannot see the slope of $|F(j\omega)|$ at the polar plot.

$\triangledown$

Question 4. Three sensors, $H_1(s)$, $H_2(s)$, and $H_3(s)$, were tested on the same signal:
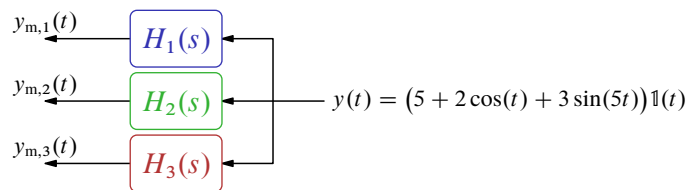
The results (measurements) were saved, see parts of them, in the time interval $t \in [20, 30]$, in Fig. 11(a). Unfortunately, the information about what sensor each measurement belongs to got lost. Fortunately, we still have frequency response plots of each sensor, see Fig. 11(b). Use it to reconstruct the lost information.

Solution. All measurements are already in steady state. By the Frequency Response Theorem, the steady-state response of the nth measurement is

$$
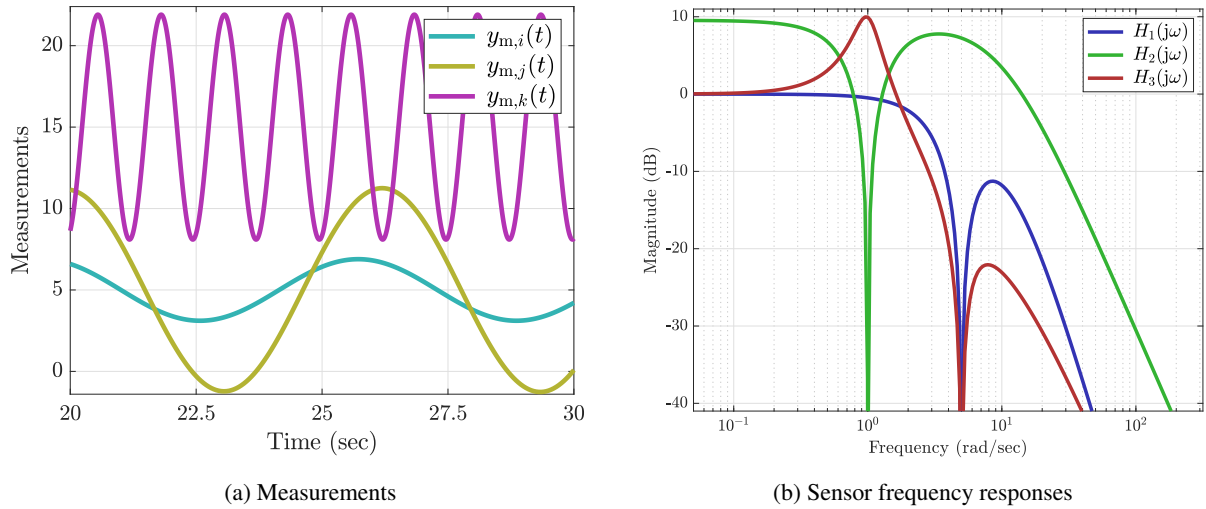y_{m,n}(t) = 5H_n(0) + 2|H_n(j)| \cos(t + \arg(H_n(j))) + 3|H_n(j5)| \sin(5t + \arg(H_n(j5))).
$$

(a) Measurements

(b) Sensor frequency responses

Fig. 11: Plots for Question 4

The magenta plot in Fig. 11(a) contains 2 harmonics: an offset (i.e. $\omega = 0$) and $\omega = 5$ (its period is $\approx 1.25\,\text{sec}$). In other words, the harmonic at $\omega = 1$ is filtered out by the sensor. The only frequency response in Fig. 11(b) having zero gain at that frequency is drawn with the green line, which belongs to $H_2(s)$. Hence,

$$y_{\mathrm{m},k} = y_{\mathrm{m},2} \quad (\text{i.e. } k = 2).$$

As a matter of fact, the match between $y_{\mathrm{m},k}$ and $H_2(s)$ can also be seen via the highest offset of the former, which corresponds to the highest static gain of $H_2(s)$ among the sensors.

Now, both lime and cyan plots in Fig. 11(a) have the harmonic at $\omega = 5$ filtered out in them, consistently with the zero gains of $H_1(j\omega)$ and $H_3(j\omega)$ at $\omega = 5$ in Fig. 11(b). Measurements $y_{\mathrm{m},i}$ and $y_{\mathrm{m},j}$ have identical offsets (consistently with the identical static gains of $H_1(s)$ and $H_3(s)$), but different amplitudes of harmonics with $\omega = 1$ in them. This difference must be manifested in different gains of the frequency responses of the remaining sensors at $\omega = 1$. That is indeed what we can see in Fig. 11(b) (compare the blue and red lines there). Because $|H_1(\mathrm{j})| < |H_3(\mathrm{j})|$, we end up with

$$y_{\mathrm{m},i} = y_{\mathrm{m},1} \quad (\text{i.e. } i = 1) \qquad \text{and} \qquad y_{\mathrm{m},j} = y_{\mathrm{m},3} \quad (\text{i.e. } j = 3).$$

$\triangledown$

Question 5. During an experiment we measured the below signal.

$$y(t) = \big(5 + 3\cos(2\pi \cdot t) + 0.1\cos(2\pi \cdot 60t)\big)\mathbb{1}(t)$$

But, this signal has some noise due to the electrical line (a sinusoidal noise at $50\,\text{Hz}$) and some noise due to the environment as a sinusoidal signal with frequency $600\,\text{Hz}$. The signal with the noise is as described below and is shown in the time domain in Fig. 12.

$$y(t) = \big(5 + 3\cos(2\pi \cdot t) + 0.1\cos(2\pi \cdot 60t) + \sin(2\pi \cdot 50t) + \sin(2\pi \cdot 600t)\big)\mathbb{1}(t)$$

We wish to filter out the noise by using notch filters and Butterworth filters. The noise must have an attenuation to at most 1%. The signal itself is allowed to have an attenuation to at least 89%. Design the notch filters with the largest stop-band possible and the Butterworth filters with the smallest order that meet the requirements.
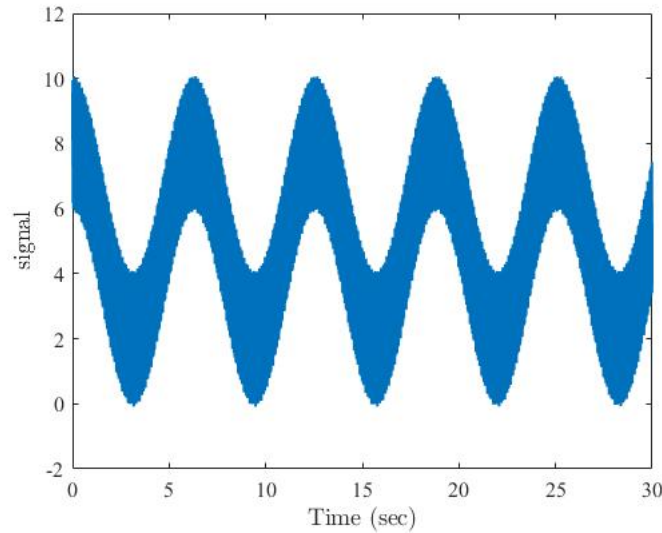
Fig. 12: The Noisy Signal in Question 5

Solution. For the 50 Hz noise, since it's close and to the left of the 60 Hz signal, we'll design a notch filter. To do so we use the notch filter's transfer function:

$$F_{\text{notch}}(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Demanding:

$$\omega_0 = 2\pi \cdot 50 = 100\pi$$

We do not want to attenuate the original signal at 60 Hz more than 89%. To do so and since we need to design 2 filters, we require an attenuation of the signal of each filter of at most $\sqrt{0.89} = 0.94$. We need to design the right edge frequency so that the filter's magnitude at 60 Hz is $0.94 \approx -0.54$[dB]. So we demand that at $60\,\text{Hz} = 120\pi \left[\frac{\text{rad}}{\text{sec}}\right]$ frequency the filter magnitude is 0.94 which determines the ζ parameter:

$$|F_{\text{notch}}(j\omega)| = \sqrt{\frac{(\omega^2 - \omega_0^2)^2}{(\omega^2 - \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$

$$|F_{\text{notch}}(j120\pi)| = \sqrt{\frac{((120\pi)^2 - (100\pi)^2)^2}{((120\pi)^2 - (100\pi)^2)^2 + 4\zeta^2(100\pi)^2(120\pi)^2}} = 0.94 \implies \zeta = 0.0645$$

We therefor design the notch filter as:

$$F_{\text{notch}}(s) = \frac{s^2 + (100\pi)^2}{s^2 + 2 \cdot 0.0645 \cdot 100\pi \cdot s + (100\pi)^2}$$

The filter's Bode diagram is shown in Fig. 13 (with the frequency in Hz for convenience). We can see that the magnitude at 60 Hz is -0.51[dB] ≈ 0.94. So at 60 Hz the signal passes. (While our standard does define a magnitude of -3[dB] as passing, since at that magnitude the signal's power is halved we actually want a higher magnitude at the interested frequencies which is why the question demands 89% attenuation total instead). Also, at this frequency there is a phase lead of $\approx 20°$.
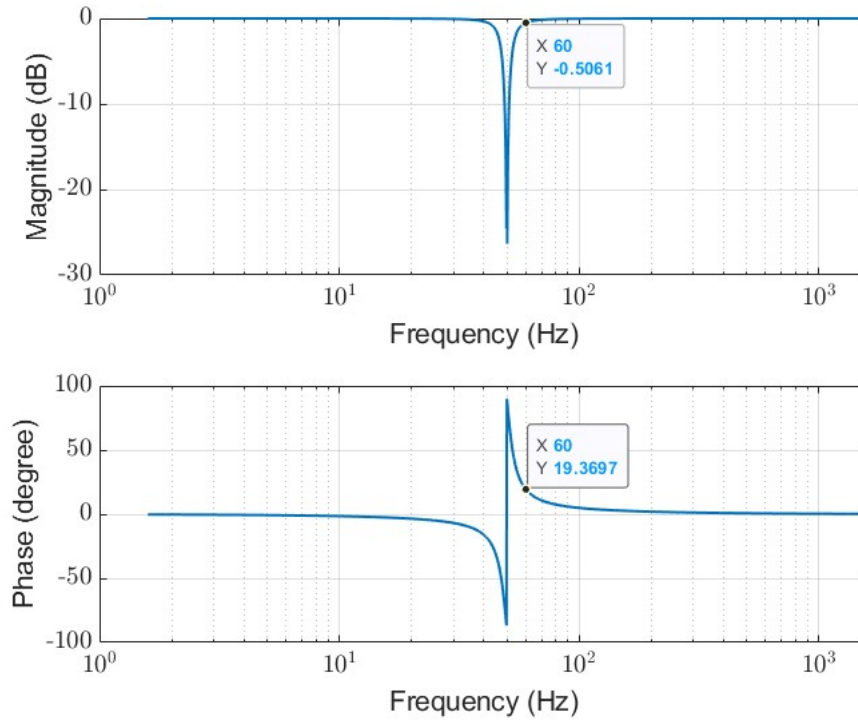
Fig. 13: Bode of the Notch Filter Designed in Question 5

For the noise at $600\,\text{Hz}$, since here we know there is no signal above it we can design a low pass Butterworth filter.

$$F_{\text{butter}}(s) = \frac{1}{B_n(s/\omega_b)}$$

For an attenuation of 1%, this means that the filter's magnitude should be at most $|F_{\text{butter}}(j\omega)| = 0.01 = -40[\text{dB}]$ at frequency $\omega = 2\pi \cdot 600 = 1200\pi\left[\frac{\text{rad}}{\text{sec}}\right]$. Again, we do not want attenuation of more than 0.89 at our signal frequency of 60 Hz total, which means an attenuation of no more than 0.94 for this filter. To find the proper bandwidth frequency and order, we do some trial and error. We first demand that the bandwidth frequency will be $\omega_b = 120\pi\left[\frac{\text{rad}}{\text{sec}}\right]$ (60 Hz). To find the order of the Butterworth polynomial, we first find the slope needed for the line of the asymptotic Bode. Since we are looking at the asymptotic graph this is the slope of the line that runs from $0[\text{dB}]$ to $-40[\text{dB}]$. Remember, since the Bode's horizontal axis is in logarithmic scale the equation for the slope is:

$$a = \frac{M_{0,\text{dB}} - M_{b,\text{dB}}}{\log \omega_0 - \log \omega_b} = \frac{M_{0,\text{dB}} - M_{b,\text{dB}}}{\log(\omega_0/\omega_b)} = \frac{-40 - 0}{\log((1200\pi)/(120\pi))} = -40\left[\frac{\text{dB}}{\text{dec}}\right]$$

For an nth order Butterworth filter the slope is $-20 \cdot n\left[\frac{\text{dB}}{\text{dec}}\right]$, thus we need an order of:

$$n = \lceil\frac{a}{-20}\rceil = 2$$

Thus, we have a Butterworth low-pass filter of:

$$F_{\text{butter}}(s) = \frac{1}{B_2(s/(120\pi))}$$

where,

$$B_2(s) = \prod_{i=1}^{1}(s^2 + 2\zeta_i s + 1)$$

and ζ_i is calculated as explained in the background section. The Bode diagram of the filter is shown in Fig. 14 (with the frequency in Hz for convenience). Notice here that the magnitude at frequency 60 Hz is -3[dB].
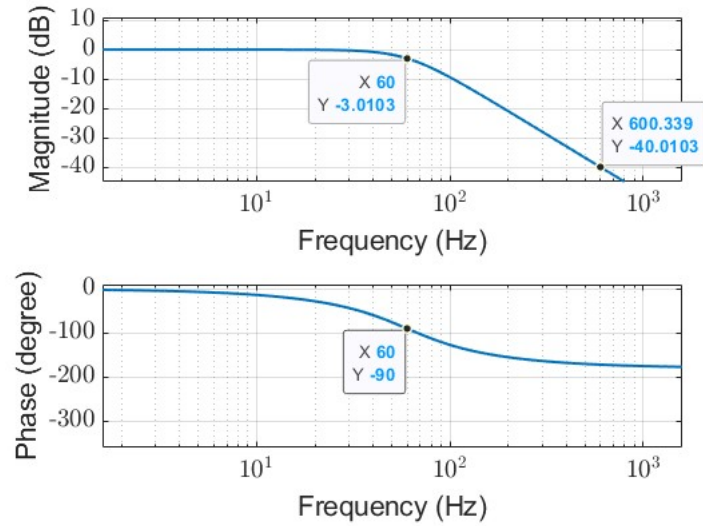

Fig. 14: Bode of the First Butterworth Filter Designed in Question 5

We need a higher gain at this frequency so through some trial and error we choose a bandwidth frequency of $\omega_b = 200\pi \left[\frac{\text{rad}}{\text{sec}}\right]$ (100 Hz). We now do the same calculation of the slope and order:

$$a = \frac{-40 - 0}{\log((1200\pi)/(200\pi))} = -51.5730 \left[\frac{\text{dB}}{\text{dec}}\right]$$

$$n = \lceil \frac{a}{-20} \rceil = 3$$

Recalculating the Butterworth filter we get:

$$F_{\text{butter}}(s) = \frac{1}{B_3(s/(200\pi))}$$

With the 3rd order Butterworth polynomial:

$$B_3(s) = (s + 1) \prod_{i=1}^{1}(s^2 + 2\zeta_i s + 1)$$

The Bode diagram of the filter is shown in Fig. 15 (again with Hz for convenience). Here, the -3[dB] frequency (the bandwidth frequency) is 100 Hz, while the signal frequency of 60 Hz has a gain of -0.2[dB] $\approx$ 0.98 as required. The noise frequency of 600 Hz is attenuated with a gain of -46.7[dB] again as required. Also, we can see that there is a phase lag of $-74°$ at the signal frequency of 60 Hz.
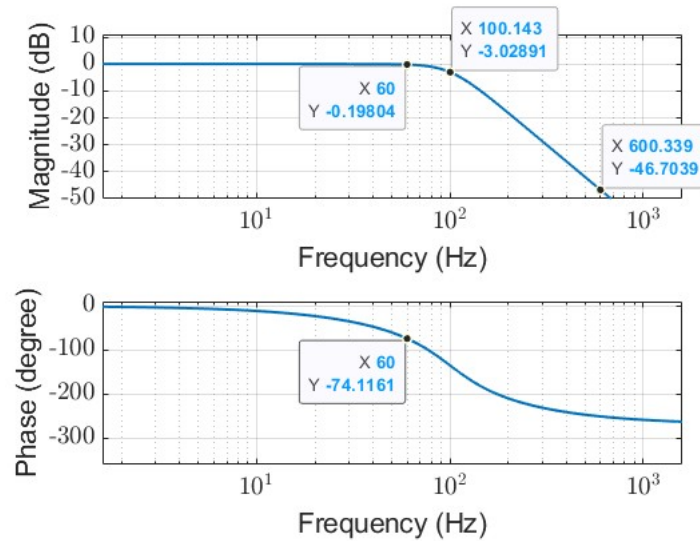
Fig. 15: Bode of the Second Butterworth Filter Designed in Question 5

In summary, taking into account both the Butterworth and notch filters, the 60 Hz signal has an attenuation of $0.94 \cdot 0.98 = 0.92$ as required. The total phase lag is $-74° + 20° = -54°$.

Using the notch and Butterworth filters, we can denoise the signal as shown in figures 16. While for the frequency of $1 \left[\frac{\text{rad}}{\text{sec}} \right]$ as well as the static gain, there is no change in amplitude nor phase, for the frequency of 60 Hz we see a phase lag of $-54°$, shown in Fig. 16(b), as well as a small change in magnitude. The phase will need to be fixed through some other means. $\triangledown$
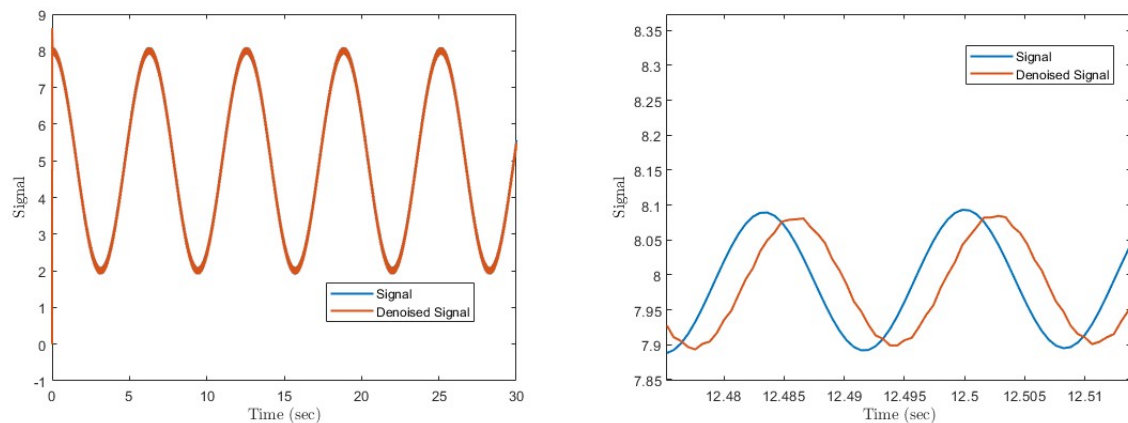
(a) Denoised Signal v.s. Original Signal

(b) Close up of Denoised Signal v.s. Original Signal

Fig. 16: Plots for Question 5

4   Homework Problems

Question 6. Given is a system represented by an ODE:

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = \dot{u}(t) + 2u(t)$$

and the input:

$$u(t) = (\delta(t) + (t-4) + 2\sin 2t)\mathbb{1}$$

Find the system response in steady state to the input u.

Solution. We perform the Laplace transform on the ODE:

$$s^2 Y(s) + 2sY(s) + Y(s) = sU(s) + 2U(s)$$

thus, the transfer function of the system is:

$$G_1(s) = \frac{s+2}{s^2 + 2s + 1}$$

Notice that the system is stable because its poles are both in $s = -1$. Since the system is linear the output will be a superposition of the 3 responses: the impulse response, the step response and the sinusoidal response.

$$y(t) = y_\delta(t) + y_{step}(t) + y_{sin}(t)$$

- The system is stable so in steady state the impulse response decays to zero:

$$y_{\delta,ss}(t) = 0$$

- For the same reason, the step response converges to the static gain:

$$y_{step,ss}(t) = G_1(0) = k_{st} = 2$$

- Due to the frequency response theorem, the response to a sinusoidal input will converge to a sinusoidal signal, where:

$$G_1(j\omega) = \frac{2 + j\omega}{j^2\omega^2 + 2j\omega + 1} = \frac{2 + j\omega}{1 - \omega^2 + 2\omega j}$$

$$|G_1(j\omega)| = \frac{|2 + j\omega|}{|1 - \omega^2 + 2\omega j|} = \frac{\sqrt{4 + \omega^2}}{\sqrt{(1-\omega^2)^2 + 4\omega^2}} = \sqrt{\frac{4 + \omega^2}{(1-\omega^2)^2 + 4\omega^2}}$$

$$\arg G_1(j\omega) = \arctan\left(\frac{\omega}{2}\right) - \arctan\left(\frac{2\omega}{1 - \omega^2}\right)$$

In our case, $\omega = 2$. Substituting into G_1 we get:

$$|G_1(j2)| = \frac{2\sqrt{2}}{5}$$

$$\arg G_1(j2) = -1.429 \,[\text{rad}]$$

So the sinusoidal response in steady state is:

$$y_{sin,ss}(t) = 2 \cdot \frac{2\sqrt{2}}{5}\sin(2t - 1.429)$$

The superposition of all these responses gives the steady state response of the system:

$$y_{ss}(t) = 2 + 2 \cdot \frac{2\sqrt{2}}{5}\sin(2t - 1.429)$$
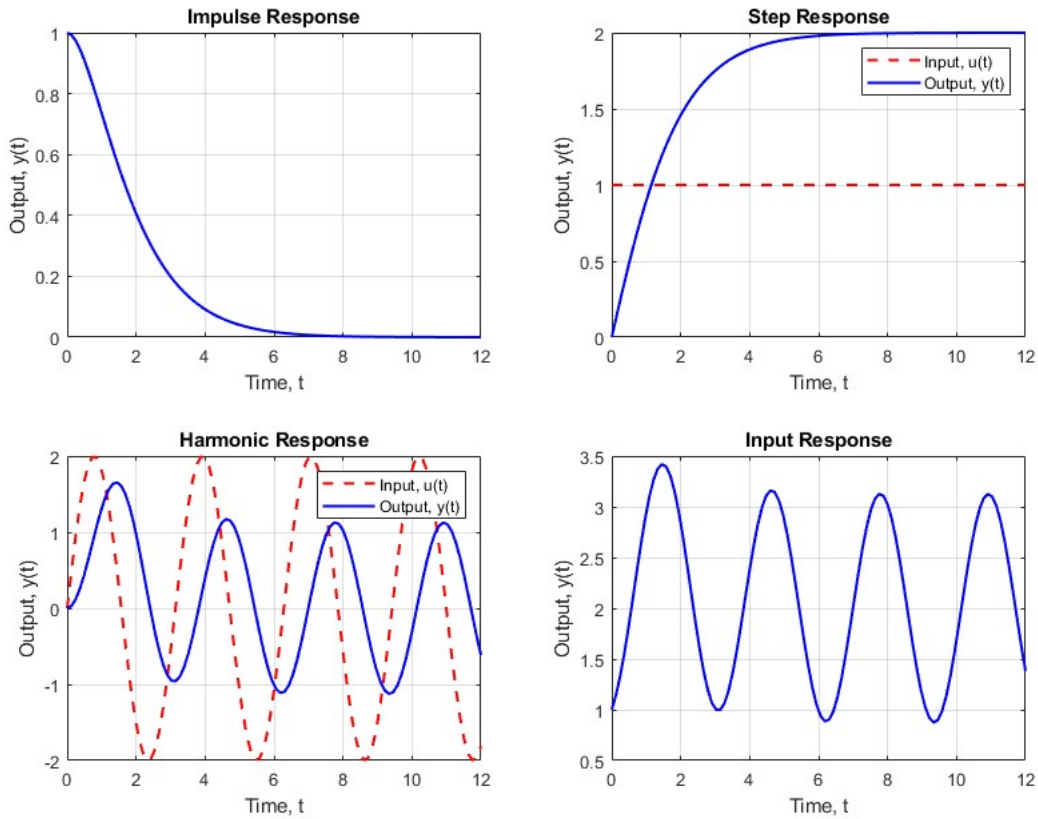
We show the solution graphically in Fig. 17   ▽

Fig. 17: Plots for Question 6

Question 7. Given is the below transfer function:

$$G(s) = \frac{256s}{(s + 2)(s + 8)^2}$$

1. Plot the asymptotic magnitude Bode diagram.

2. What is the deviation of the asymptotic diagram from the real frequency response at $\omega = 8 \left[\frac{\text{rad}}{\text{sec}}\right]$, and what is the phase delay at that frequency?

Solution.

1. We first unpack the system into its basic 1- and 2-order subsystems:

$$G(s) = 256 \cdot s \cdot \frac{1}{s + 2} \cdot \frac{1}{(s + 8)^2}$$

We now transform each of the subsystems into their standard form:

$$G(s) = 256 \cdot s \cdot \frac{1/2}{s/2 + 1} \cdot \frac{1/64}{(s/8 + 1)^2}$$

Combining the static gains:

$$G(s) = 2 \cdot s \cdot \frac{1}{s/2 + 1} \cdot \frac{1}{(s/8 + 1)^2}$$

We have 4 subsystems:

$$G_1(s) = 2$$
$$G_2(s) = s$$
$$G_3(s) = \frac{1}{s/2 + 1}$$
$$G_4(s) = \frac{1}{(s/8 + 1)^2}$$

We can now analyze each of the subsystems separately:

- The first system $G_1(s)$ is a static gain:

$$|G_1(j\omega)| = 2$$

  with magnitude $M_{1,\text{dB}} = 20 \log 2 \approx 6$. The asymptotic Bode diagram is shown in Fig. 18(a).

- The second system $G_2(s) = s$ is a differentiator:

$$|G_2(j\omega)| = \omega$$

  thus the magnitude in dB is $M_{2,\text{dB}} = 20 \log \omega$ and we get a straight line with slope $20 \left[\frac{\text{dB}}{\text{dec}}\right]$. The asymptotic Bode diagram is shown in Fig. 18(b).

- The third system $G_3(s)$ is a Low Pass Filter (LPF):

$$|G_3(j\omega)| = \frac{1}{\sqrt{\tau^2 \omega^2 + 1}} = \frac{1}{\sqrt{0.25\omega^2 + 1}}$$

  so the magnitude in dB will be

$$M_{3,\text{dB}} = 20 \log \frac{1}{0.25\omega^2 + 1} = 20 \log(0.25\omega^2 + 1)^{-0.5} = -10 \log(0.25\omega^2 + 1)$$

  It has a slope of $-20 \left[\frac{\text{dB}}{\text{dec}}\right]$ after the corner frequency of $\omega_\text{p} = 2 \left[\frac{\text{dB}}{\text{dec}}\right]$. The asymptotic Bode diagram is shown in Fig. 18(c).

- The fourth system $G_4(s)$ is also a LPF but of order 2. Similarly we get a gain of:
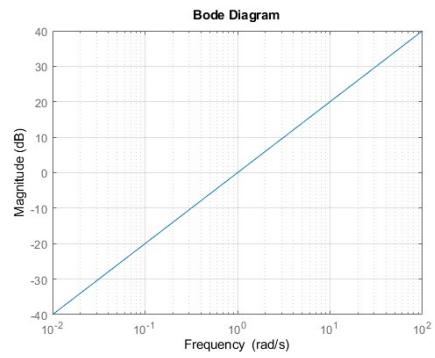
$$M_{4,\text{dB}} = -20 \log(0.015\omega^2 + 1)$$

  The slope of the system after the corner frequency of $\omega_\text{p} = 8 \left[\frac{\text{rad}}{\text{sec}}\right]$, is $-40 \left[\frac{\text{dB}}{\text{dec}}\right]$. The asymptotic Bode diagram is shown in Fig. 18(d).

We can now cascade all of the asymptotic Bodes and get the asymptotic Bode of the original system, as in Fig. 19. In summary we get a slope of:
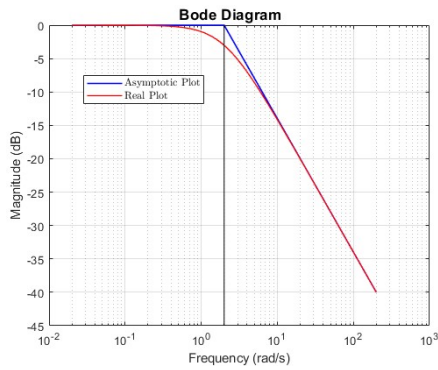
$$\text{slope} \left[\frac{\text{dB}}{\text{dec}}\right] = \begin{cases} 20 & \text{if } 0 \leq \omega \leq 2 \\ 0 & \text{if } 2 \leq \omega \leq 8 \\ -40 & \text{if } \omega \geq 8 \end{cases}$$
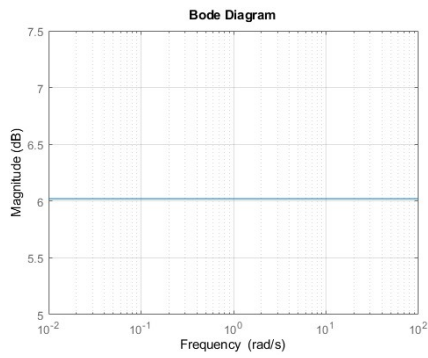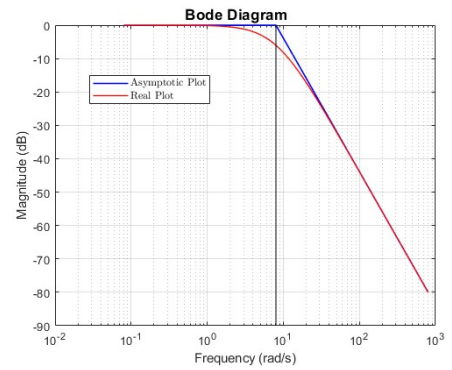
(a) Asymptotic Bode diagram of $G_1(s)$

(b) Asymptotic Bode diagram of $G_2(s)$

(c) Asymptotic (and real) Bode diagram of $G_3(s)$

(d) Asymptotic (and real) Bode diagram of $G_3(s)$
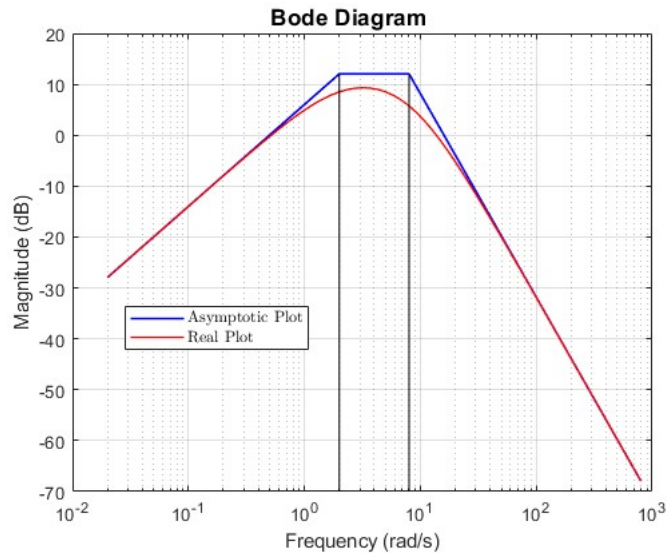
Fig. 18: Plots for question 7

Fig. 19: Asymptotic (and real) Bode plot of the system in question 7

2. From the asymptotic analysis, we know that the static gain is $M_{1,\mathrm{dB}} \approx 6\,[\mathrm{dB}]$. Also, the second subsystem gives a gain of 2 at frequency $\omega = 2\left[\frac{\mathrm{rad}}{\mathrm{sec}}\right]$. So the gain in dB is $M_{2,\mathrm{dB}} = 20\log 2 \approx 6\,[\mathrm{dB}]$. Thus, at a frequency of $\omega = 2\left[\frac{\mathrm{rad}}{\mathrm{sec}}\right]$ we have a magnitude of $12\,[\mathrm{dB}]$. But, we also saw that the

magnitude stays constant between $2\left[\frac{rad}{sec}\right]$ and $8\left[\frac{rad}{sec}\right]$ so the magnitude at frequency $\omega = 8\left[\frac{rad}{sec}\right]$ is also $12\,[dB]$.

We can now calculate the magnitude directly at $\omega = 8\left[\frac{rad}{sec}\right]$:

$$M_{(dB)}(\omega) = 6 + 20\log\omega - 10\log(0.25\omega^2 + 1) - 20log(0.015\omega^2 + 1)$$
$$M_{(dB)}(\omega = 8) = 5.74[dB]$$

So the error is:
$$error = 12 - 5.74 = 6.26[dB]$$

and the phase lag is:

$$G(j\omega) = 2 \cdot j\omega \cdot \frac{1}{j\omega/2 + 1} \cdot \frac{1}{(j\omega/8 + 1)^2}$$

$$G(j8) = 2 \cdot j8 \cdot \frac{1}{j4 + 1} \cdot \frac{1}{(j + 1)^2}$$

$$\arg G(j8) = \arctan(0/2) + \arctan(8/0) - \arctan(4/1) - 2\arctan(1/1) = 0° + 90° - 76° - 2 \cdot 45° = 76°$$
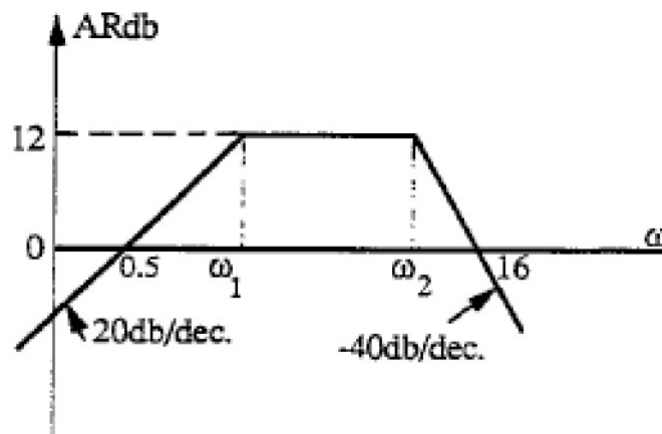
∇

Fig. 20: Asymptotic Bode plot of the system in question 8

Question 8. Given in Fig. 20 is the asymptotic Bode diagram of a system.

1. Find the order of the system.

2. Find ω_1, ω_2 and the transfer function of the system.

Solution.

1. We'll find the general shape of the system. In this system there is a static gain that doesn't affect its order. We shall mark it as:
$$G_1(s) = k$$

We can see that in the frequency range of $0 \leq \omega \leq \omega_1$ there is an incline with slope $20\left[\frac{dB}{dec}\right]$. This means there is a differentiator in the system. We write:
$$G_2(s) = s$$

In the frequency range of $\omega_1 \leq \omega \leq \omega_2$ the slope is 0, therefore there is a system with corner frequency ω_1 and a slope of $-20 \left[\frac{dB}{dec}\right]$ after the corner frequency that cancels the previous slope. We write:

$$G_3(s) = \frac{1}{s/\omega_1 + 1}$$

In the last frequency range of $\omega_2 \leq \omega$ we can see a slope of $-40 \left[\frac{dB}{dec}\right]$, so there must be a system with corner frequency ω_2 and a slope $-40 \left[\frac{dB}{dec}\right]$ after that.

$$G_4(s) = \frac{1}{(s/\omega_2 + 1)^2}$$

Note: We can't always definitively recreate the transfer function by analyzing the asymptotic Bode. For example, we could have assumed for subsystem 4:

$$G_4(s) = \frac{\omega_2^2}{s^2 + 2\zeta\omega_2 s + \omega_2^2}$$

and gotten the same asymptotic Bode. For example, for these two systems:

$$P_1(s) = \frac{ks}{(s/\omega_1 + 1)(s/\omega_2 + 1)^2}$$

$$P_2(s) = \frac{ks\omega_2^2}{(s/\omega_1 + 1)\left(s^2 + 2\zeta\omega_2 s + \omega_2^2\right)}$$

$P_1(s)$ is actually the system from question 7 and its Bode diagram is shown in Fig. 19, and $P_2(s)$'s Bode diagram is shown in Fig. 21
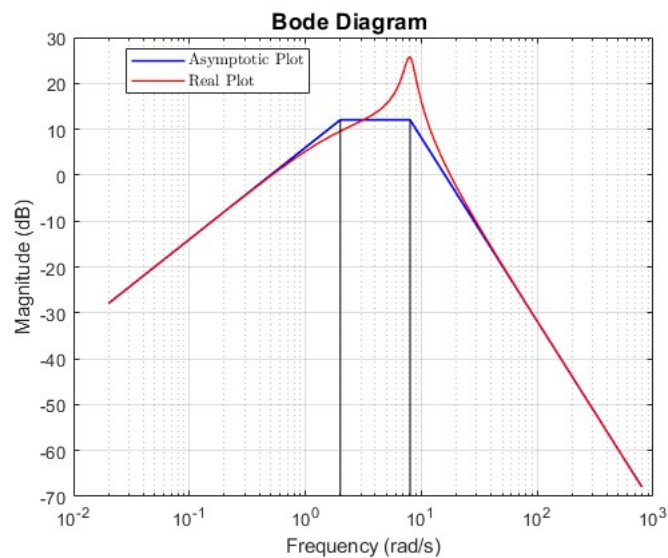

Fig. 21: Bode plot of $P_2(s)$ in question 8

Notice that the asymptotic Bode diagram of both systems is the same but the real diagrams are different.

In this question we'll assume:

$$G(s) = \frac{k_s s}{(s/\omega_1 + 1)(s/\omega_2 + 1)^2}$$

Either way, we can see that the system is a third order system.

2. It is known that given a linear function, its slope is:

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

For a Bode diagram, there is almost no difference. Given a straight line on the Bode diagram with slope $a\left[\frac{\text{dB}}{\text{dec}}\right]$, and keeping in mind that the horizontal axis is logarithmic, we get:

$$a\left[\tfrac{\text{dB}}{\text{dec}}\right] = \frac{M_2[\text{dB}] - M_1[\text{dB}]}{\log \omega_2 - \log \omega_1} = \frac{M_2[\text{dB}] - M_1[\text{dB}]}{\log(\omega_2/\omega_1)}$$

Substituting the magnitudes:

$$20 = \frac{12 - 0}{\log(\omega_1/0.5)}$$
$$\log(\omega_1/0.5) = 0.6$$
$$\omega_1 = 0.5 \cdot 10^{0.6} \approx 2\left[\tfrac{\text{rad}}{\text{sec}}\right]$$

$$-40 = \frac{12 - 0}{\log(\omega_2/16)}$$
$$\log(\omega_2/16) = -0.3$$
$$\omega_2 = 16 \cdot 10^{-0.3} \approx 8\left[\tfrac{\text{rad}}{\text{sec}}\right]$$

and the transfer function is:

$$G(s) = \frac{2s}{(s/2 + 1)(s/8 + 1)^2}$$

∇