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LINEAR SYSTEMS (034032)

TUTORIAL 9

</div>

1 Topics

Frequency response of an LTI system, Bode and polar diagrams, filters.

2 Background

2.1 Frequency Response

Theorem 1. *Let $G : u \mapsto y$ be a stable continuous-time LTI system. Its response to the sinusoidal test input u such that*

$$u(t) = a \sin(\omega t + \phi)\mathbb{1}(t)$$

in steady state is also sinusoidal. Specifically

$$y_{\text{ss}}(t) = a|G(\text{j}\omega)| \sin(\omega t + \phi + \arg G(\text{j}\omega)),$$

where $|G(\text{j}\omega)|$ and $\arg(G(\text{j}\omega))$ are the gain (magnitude) and phase of the frequency response $G(\text{j}\omega)$ of G, respectively, which can be calculated as

$$|G(\text{j}\omega)| = \sqrt{x^2 + y^2} \quad and \quad \arg(G(\text{j}\omega)) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \arctan(\frac{y}{x}) + \pi & x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & x < 0, y < 0 \\ +\frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ undefined & x = 0, y = 0 \end{cases}$$

where $x := \operatorname{Re} G(\text{j}\omega)$ and $y := \operatorname{Im} G(\text{j}\omega)$.

2.2 Bode Diagram

The Bode diagram is a way of visualizing $|G(\text{j}\omega)|$ and $\arg(G(\text{j}\omega))$. In order to draw the Bode diagram by hand we actually draw the asymptotic diagram.

For example, Fig. 1 shows the Bode diagram of the transfer function:
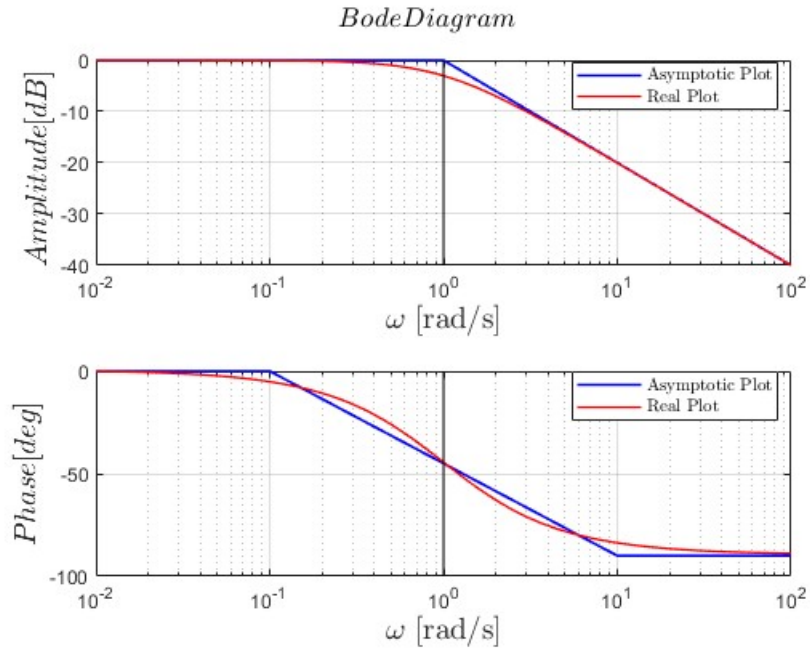
$$G(s) = \frac{1}{s + 1}$$

Fig. 1: Real and Asymptotic Bode Diagram

2.2.1 Bode Scales

The Bode diagram's horizontal axis is the frequency ω in logarithmic scale. The magnitude part is represented in dB and the phase part in deg. dB is the unit decibel and is defined as:

$$k_{\text{(dB)}} = 20 \log k$$
$$k = 10^{k_{\text{(dB)}}/20}$$

We also define dec, decade, which is the distance of 10 units (10 scales on the horizontal axis).

2.2.2 Steps to Create an Asymptotic Diagram

1. Decomposing the system into the product of sub-systems.

$$G(s) = G_1(s) \cdot G_2(s) \cdot \ldots \cdot G_m(s)$$

where each one of these subsystems $i \in \{1, m\}$ are first and second order systems of the form:

$$G_i(s) = \frac{1}{s}, s, \frac{1}{as+b}, as+b, \frac{1}{as^2+bs+c}, as^2+bs+c$$

2. Every first order system of the form $\frac{1}{as+b}$ we convert to

$$\frac{k_{\text{st}}}{\tau s + 2} \tag{1}$$

and every first order system of the form $as + b$ we convert to

$$k_{\text{st}}(\tau s + 1)$$

In the same way, we convert second order systems of the form $\frac{1}{as^2+bs+c}$ to

$$\frac{k_{st}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k_{st}}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

and second order systems of the form $as^2 + bs + c$ to

$$\frac{k_{st}(s^2 + 2\zeta\omega_n s + \omega_n^2)}{\omega_n^2} = (s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1$$
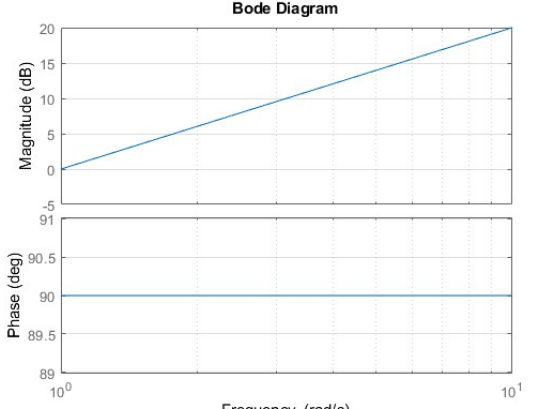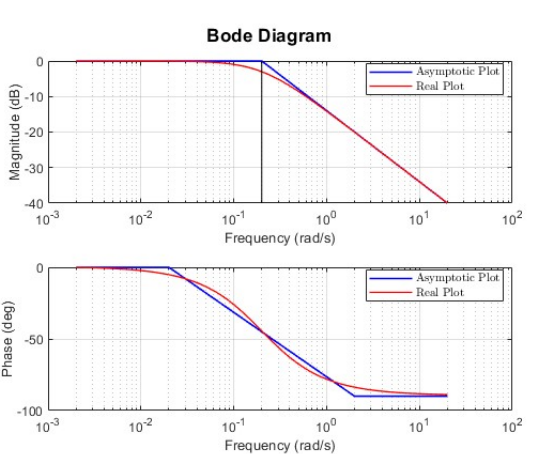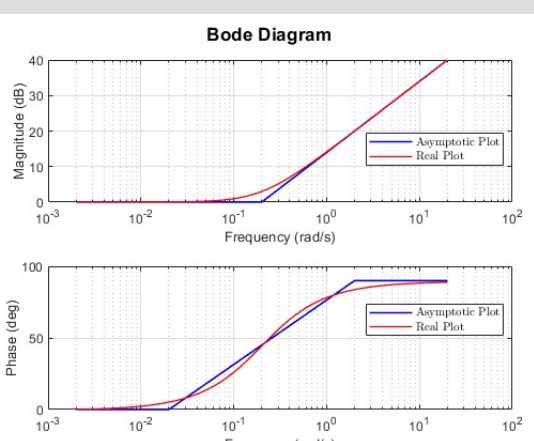
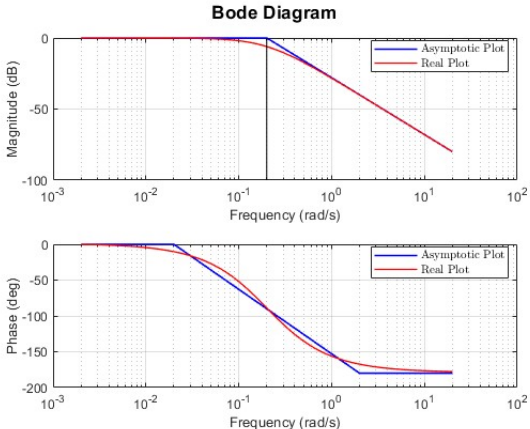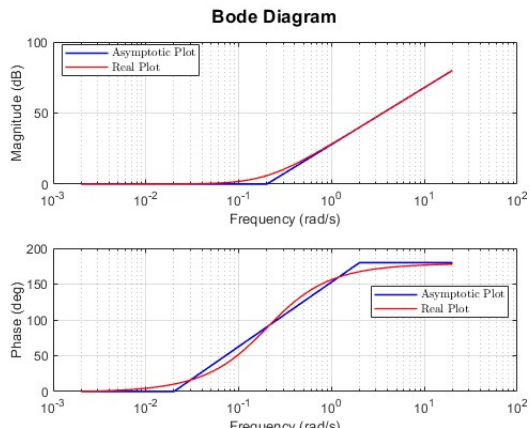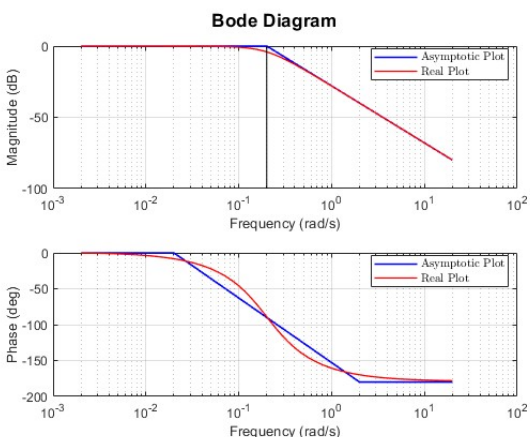3. We unite all the static gains by multiplying all k_{st} elements, getting:

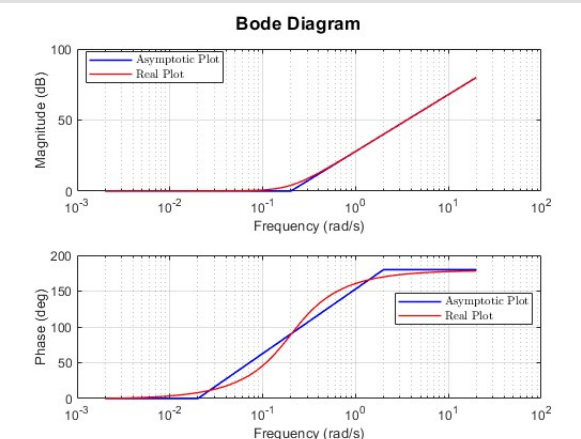$$G(s) = k \cdot \mathcal{G}_1(s) \cdot \mathcal{G}_2(s) \cdot \ldots \cdot \mathcal{G}_m(s)$$
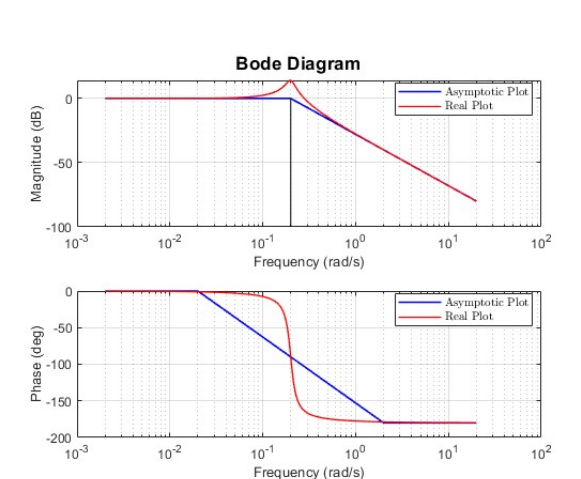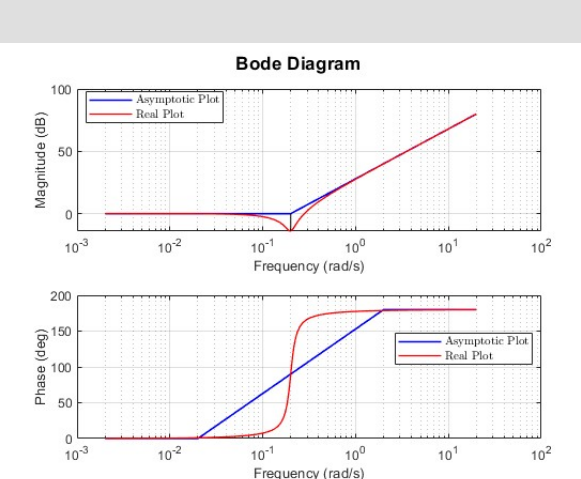
where $\mathcal{G}_i(s)$ are the standard transfer functions.

4. Using the table below, we draw the asymptotic Bode diagram of the system as a combination of the Bodes of the standard systems.

Function	Bode Graph	Asymptotic Behaviour
k		The magnitude is a straight horizontal line with a constant gain of $k_{(dB)} = 20\log k$. The phase is $0°$.
$\dfrac{1}{s}$		The magnitude is a straight line that crosses the horizontal axis at $1[\frac{rad}{sec}]$ and has a slope of $-20\left[\frac{dB}{dec}\right]$. The phase is constant at $-90°$.

Function	Bode Graph	Comment
s		The magnitude is a straight line that crosses the horizontal axis at $1[\frac{rad}{sec}]$ and has a slope of $20\left[\frac{dB}{dec}\right]$. The phase is constant at $90°$.
$\dfrac{1}{\tau s + 1}$		The magnitude is constant at $0\,[dB]$ until the corner frequency of $\omega_p = \frac{1}{\tau}$, after which it is a straight line with slope $-20\left[\frac{dB}{dec}\right]$. The phase is constant at $0°$ until $0.1\omega_p$ after which it is a straight line that crosses $-45°$ at frequency ω_p, and at $10\omega_p$ again constant at $-90°$.
$\tau s + 1$		The magnitude is constant at $0[dB]$ until the corner frequency of $\omega_p = \frac{1}{\tau}$, after which it is a straight line with slope $20\left[\frac{dB}{dec}\right]$. The phase is constant at $0°$ until $0.1\omega_p$ after which it is a straight line that crosses $45°$ at frequency ω_p, and at $10\omega_p$ again constant at $90°$.

Function	Bode Graph	Comment
$\dfrac{1}{(\tau s + 1)^n}$		The magnitude is constant at $0\,[\text{dB}]$ until the corner frequency of $\omega_{\text{p}} = \frac{1}{\tau}$, after which it is a straight line with slope $-20 \cdot n \left[\frac{\text{dB}}{\text{dec}} \right]$. The phase is constant at $0°$ until $0.1\omega_{\text{p}}$ after which it is a straight line that crosses $-45n°$ at frequency ω_{p}, and at $10\omega_{\text{p}}$ again constant at $-90n°$.
$(\tau s + 1)^n$		The magnitude is constant at $0[\text{dB}]$ until the corner frequency of $\omega_{\text{p}} = \frac{1}{\tau}$, after which it is a straight line with slope $20 \cdot n \left[\frac{\text{dB}}{\text{dec}} \right]$. The phase is constant at $0°$ until $0.1\omega_{\text{p}}$ after which it is a straight line that crosses $45n°$ at frequency ω_{p}, and at $10\omega_{\text{p}}$ again constant at $90n°$.
$\dfrac{\omega_{\text{n}}^2}{s^2 + 2\zeta\omega_{\text{n}}s + \omega_{\text{n}}^2}$ $\zeta \geq \frac{1}{\sqrt{2}}$		The magnitude is constant at $0[\text{dB}]$ until the natural frequency of ω_{n}, after which it is a straight line with slope $-40 \left[\frac{\text{dB}}{\text{dec}} \right]$. The phase is constant at $0°$ until $0.1\omega_{\text{n}}$ after which it is a straight line that crosses $-90°$ at frequency ω_{n}, and at $10\omega_{\text{n}}$ again constant at $-180°$.

Function	Bode Graph	Comment
$\dfrac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}$ $\zeta \geq \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $40\left[\frac{dB}{dec}\right]$. The phase is constant at 0° until $0.1\omega_n$ after which it is a straight line that crosses 90° at frequency ω_n, and at $10\omega_n$ again constant at 180°.
$\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $-40\left[\frac{dB}{dec}\right]$. The real system has a resonance peak at frequency $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$ The phase is constant at 0° until $0.1\omega_n$ after which it is a straight line that crosses $-90°$ at frequency ω_n, and at $10\omega_n$ again constant at $-180°$.
$\dfrac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at 0[dB] until the natural frequency of ω_n, after which it is a straight line with slope $40\left[\frac{dB}{dec}\right]$. The real system has a resonance dip at frequency $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$ The phase is constant at 0° until $0.1\omega_n$ after which it is a straight line that crosses 90° at frequency ω_n, and at $10\omega_n$ again constant at 180°.

Function	Bode Graph	Comment
$\left(\dfrac{\omega_{\mathrm{n}}^2}{s^2 + 2\zeta\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2}\right)'$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at $0[\mathrm{dB}]$ until the natural frequency of ω_{n}, after which it is a straight line with slope $-40 \cdot n \left[\frac{\mathrm{dB}}{\mathrm{dec}}\right]$. The real system has a resonance peak at frequency $\omega_{\mathrm{r}} = \omega_{\mathrm{n}}\sqrt{1 - 2\zeta^2}$ The phase is constant at $0°$ until $0.1\omega_{\mathrm{n}}$ after which it is a straight line that crosses $-90n°$ at frequency ω_{n}, and at $10\omega_{\mathrm{n}}$ again constant at $-180n°$.
$\left(\dfrac{s^2 + 2\zeta\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2}{\omega_{\mathrm{n}}^2}\right)'$ $\zeta < \dfrac{1}{\sqrt{2}}$		The magnitude is constant at $0[\mathrm{dB}]$ until the natural frequency of ω_{n}, after which it is a straight line with slope $40 \cdot n \left[\frac{\mathrm{dB}}{\mathrm{dec}}\right]$. The real system has a resonance dip at frequency $\omega_{\mathrm{r}} = \omega_{\mathrm{n}}\sqrt{1 - 2\zeta^2}$ The phase is constant at $0°$ until $0.1\omega_{\mathrm{n}}$ after which it is a straight line that crosses $90n°$ at frequency ω_{n}, and at $10\omega_{\mathrm{n}}$ again constant at $180n°$.

For more examples see the lecture slides.

2.2.3 General Guidelines for Asymptotic Bode

- Each pole adds $-20 \left[\frac{\mathrm{dB}}{\mathrm{dec}}\right]$ to the magnitude's slope.

- Each zero adds $+20 \left[\frac{\mathrm{dB}}{\mathrm{dec}}\right]$ to the magnitude's slope.

- Every pole in $\mathbb{C}\backslash\mathbb{C}_0 = \{s \in \mathbb{C} \mid \mathrm{Re}\, s \leq 0\}$ adds a phase lag of $-90°$.

- Every pole in $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$ adds a phase lead of $+90°$.

- Every zero in $\mathbb{C}\backslash\mathbb{C}_0 = \{s \in \mathbb{C} \mid \mathrm{Re}\, s \leq 0\}$ adds a phase lead of $+90°$.

- Every zero in $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$ adds a phase lag of $-90°$.

2.3 Polar Diagram

The polar diagram is another way to represemt the frequency response of the system. Similarly to the Bode diagram, the polar diagram shows $G(j\omega)$. But, unlike the Bode diagram which is comprised of two parts: the magnitude as a function of frequency $|G(j\omega)|$ and the phase as a function of frequncy $\arg(G(j\omega))$, the polar diagram is comprised of only one graph where we can see the real and imaginary parts $\Re\{G(j\omega)\}$, $\Im\{G(j\omega)\}$ as a function of the frequency which **isn't** shown directly on the graph.

For example, Fig. 2 shows the polar diagram of the transfer function:
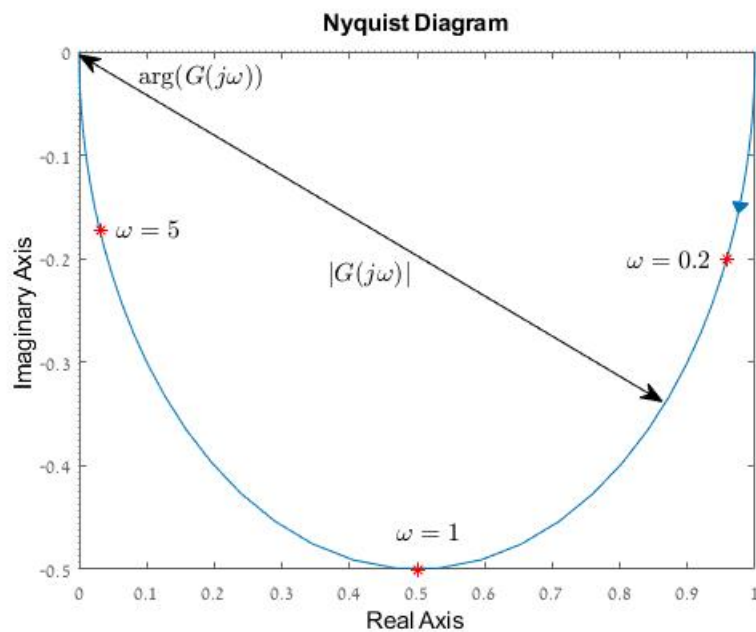
$$G(s) = \frac{1}{s+1}$$

Fig. 2: Polar Diagram

The blue arrow denotes the direction in which the frequency increases. The red points denote the frequency response at chosen frequencies.

2.3.1 The Magnitude and Phase in the Polar Diagram

Similarly to the Bode diagram, we can extract the magnitude and phase of the system for a given frequency from the polar diagram. But, here we do not know the actual frequency. The magnitude at a given point $|G(j\omega)|$ is the distance of that point from the origin. The phase $\arg(G(j\omega))$ is the angle between the line connecting that point to the origin, and the positive direction of the real axis.

When looking back at the Bode diagram of the same system in Fig. 1, we can see that the magnitude decreases monotonically. This can also be seen in the polar diagram as the distance from the origin decreases until it reaches 0. The system's phase also decreases which can again be seen in the polar diagram.

2.4 Filters

Using the frequency response we can design filters to shape the spectra of signals. 4 categories of filters are generally used:

1. Low-pass Filters:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \leq \omega_b$$

2. High-pass Filter:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \geq \omega_c$$

3. Band-pass Filter:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \in [\omega_1, \omega_2]$$

4. Band-stop Filter:
$$|G(j\omega)| \geq 1/\sqrt{2} \iff \omega \notin (\omega_1, \omega_2)$$

2.4.1 Butterworth Filter

A filter based on the Butterwoth polynomials:

$$B_n(s) = \begin{cases} \prod_{i=1}^{n/2}(s^2 + 2\zeta_i s + 1) & \text{if } n \text{ is even} \\ (s + 1)\prod_{i=1}^{(n-1)/2}(s^2 + 2\zeta_i s + 1) & \text{if } n \text{ is odd} \end{cases}$$

where,

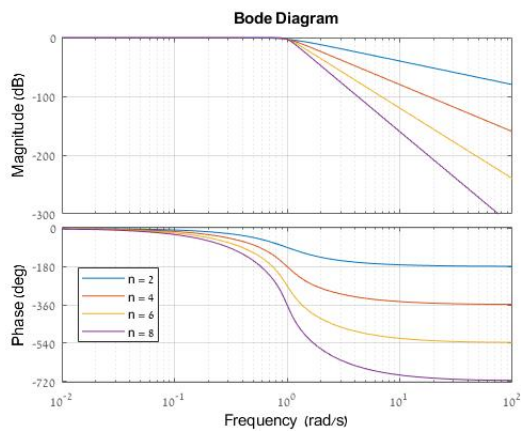$$\zeta_i := \sin\left(\frac{2i-1}{2n}\pi\right) \in (0, 1)$$

The low-pass Butterworth filter is:

$$F(s) = \frac{1}{B_n(s/\omega_b)} \implies |F(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_b)^{2n}}}$$
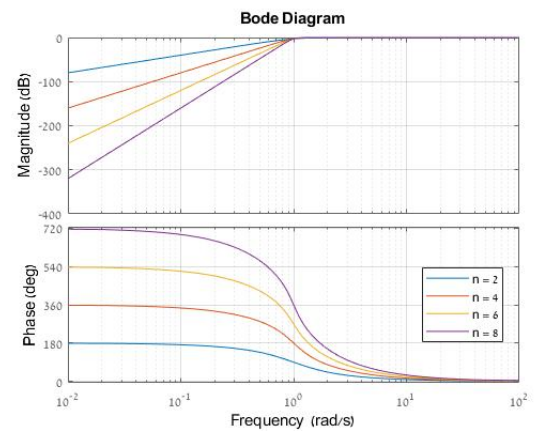
The high-pass Butterworth filter is:

$$F(s) = \frac{s/\omega_c}{B_n(s/\omega_c)} \implies |F(j\omega)| = \frac{(\omega/\omega_c)^n}{\sqrt{1 + (\omega/\omega_c)^{2n}}}$$

Their Bode diagrams are shown in figures 3(a) and 3(b).

(a) Low-pass Butterworth Filter, $\omega_b = 1$

(b) High-pass Butterworth Filter, $\omega_c = 1$

Fig. 3: Butterworth Filters Bode Diagrams

2.4.2 Notch Filter

A narrow stopband filter of the form:

$$F(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \implies |F(s)| = \sqrt{\frac{(\omega^2 - \omega_0^2)^2}{(\omega^2 - \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$

It has a stopband of (ω_1, ω_2) with:

$$\omega_1 := \left(\sqrt{1 + \zeta^2} - \zeta\right)\omega_0$$
$$\omega_2 := \left(\sqrt{1 + \zeta^2} + \zeta\right)\omega_0 = 1/\omega_1$$
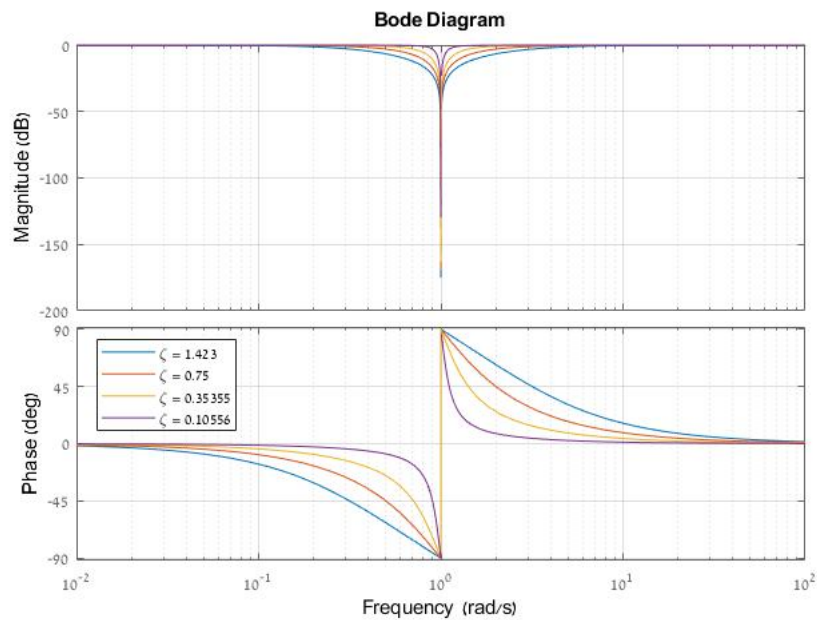
The Bode diagram is shown in Fig. 4

Fig. 4: Notch Filter Bode Diagram

2.5 Matlab Commands

Some useful Matlab functions:

1. `mag2db` and `db2mag` convert the magnitude to and from dB.

2. `bode(G)` and `bodeplot(G)` plot the bode diagram of the system G. `bodeplot(G)` has more plot options.

3. `nyquist(G)` and `nyquistplot(G)` plot a diagram of the system G in polar coordinates (called a Nyquist diagram). To create a plain polar plot ($\omega > 0$), use:
   `setoptions(nyquistplot(G),'ShowFullContour','off')`.

4. `[num,den] = butter(n,wb,'s');` generates the numerator and denominator of the low-pass Butterworth filter of order `n` with the bandwidth `wn`.

3 Problems

Question 1. Draw the asymptotic Bode magnitude plots of the transfer function

$$G(s) = \frac{k}{(\tau_1 s + 1)(\tau_2 s + 1)},$$

where $\tau_1 > 0$ and $\tau_2 > 0$.

Question 2. Draw the Bode and polar plots for the following transfer functions:

1. $G_1(s) = \dfrac{1}{(\tau s + 1)^2}$ for $\tau > 0$;

2. $G_2(s) = \dfrac{k}{s(\tau s + 1)}$ for $\tau > 0$ and $k > 0$;

3. $G_3(s) = \dfrac{\tau_2 s + 1}{\tau_1 s + 1}$ for $\tau_1 = \frac{1}{3}$ and $\tau_2 = \frac{5}{3}$ and then for $\tau_1 = \frac{5}{3}$ and $\tau_2 = \frac{1}{3}$.
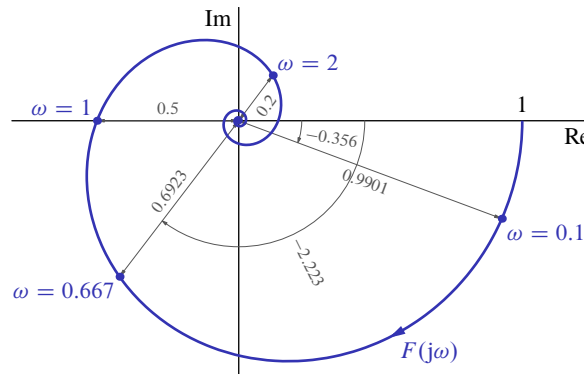

Fig. 5: Polar plot of $F(j\omega)$ in Question 3

Question 3. A signal u passes a stable system $F(s)$, whose frequency response is presented by its polar plot in Fig. 5. The magnitude $|F(j\omega)|$ is a monotonically decreasing function of ω. Denote by y the resulting output signal, i.e. $y = F(s)u$.

1. Find $y(t)$ for $u(t) = 2\sin(t)$.

2. Find $y(t)$ for $u(t) = \sin(\frac{2}{3}t + 2) + 3\sin(\frac{1}{10}t + 0.356)$.

3. In what frequency range harmonic u's are attenuated by at least a factor of 5?

4. What information about the bandwidth of $F(s)$ can be extracted from Fig. 5?

Question 4. Three sensors, $H_1(s)$, $H_2(s)$, and $H_3(s)$, were tested on the same signal:

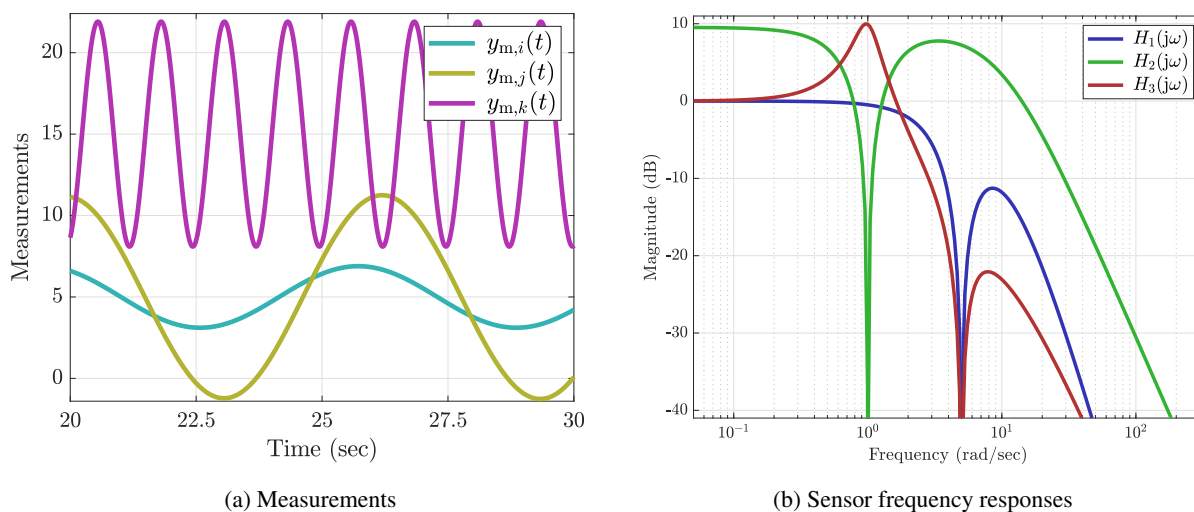(a) Measurements

(b) Sensor frequency responses
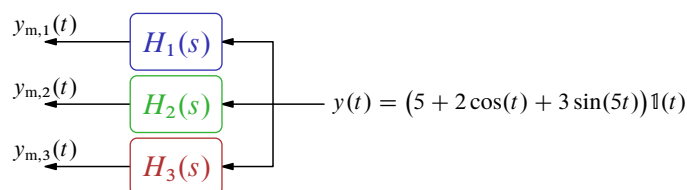
Fig. 6: Plots for Question 4

The results (measurements) were saved, see parts of them, in the time interval $t \in [20, 30]$, in Fig. 6(a). Unfortunately, the information about what sensor each measurement belongs to got lost. Fortunately, we still have frequency response plots of each sensor, see Fig. 6(b). Use it to reconstruct the lost information.

Question 5. During an experiment we measured the below signal.

$$y(t) = \big(5 + 3\cos(2\pi \cdot t) + 0.1\cos(2\pi \cdot 60t)\big)\mathbb{1}(t)$$

But, this signal has some noise due to the electrical line (a sinusoidal noise at 50 Hz) and some noise due to the environment as a sinusoidal signal with frequency 600 Hz. The signal with the noise is as described below and is shown in the time domain in Fig. 7.

$$y(t) = \big(5 + 3\cos(2\pi \cdot t) + 0.1\cos(2\pi \cdot 60t) + \sin(2\pi \cdot 50t) + \sin(2\pi \cdot 600t)\big)\mathbb{1}(t)$$

We wish to filter out the noise by using notch filters and Butterworth filters. The noise must have an attenuation to at most 1%. The signal itself is allowed to have an attenuation to at least 89%. Design the notch filters with the largest stop-band possible and the Butterworth filters with the smallest order that meet the requirements.

4 Homework Problems

Question 6. Given is a system represented by an ODE:

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = \dot{u}(t) + 2u(t)$$

and the input:

$$u(t) = \big(\delta(t) + (t - 4) + 2\sin 2t\big)\mathbb{1}$$

Find the system response in steady state to the input u.

Question 7. Given is the below transfer function:

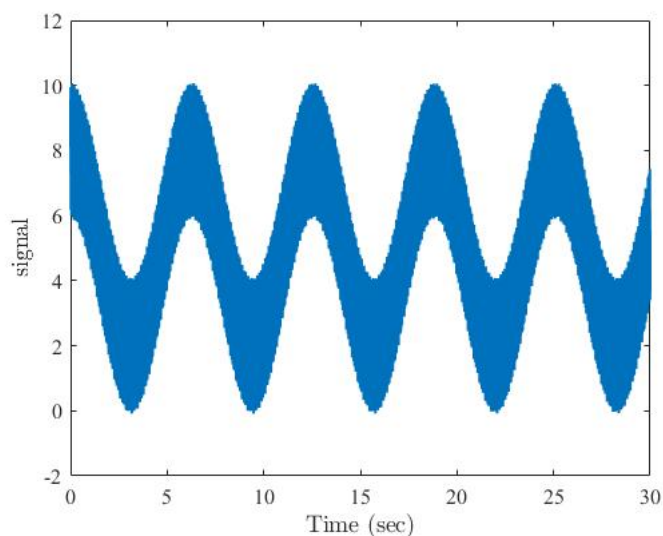$$G(s) = \frac{256s}{(s + 2)(s + 8)^2}$$

Fig. 7: The Noisy Signal in Question 5

1. Plot the asymptotic magnitude Bode diagram.

2. What is the deviation of the asymptotic diagram from the real frequency response at $\omega = 8 \left[\frac{rad}{sec} \right]$, and what is the phase delay at that frequency?
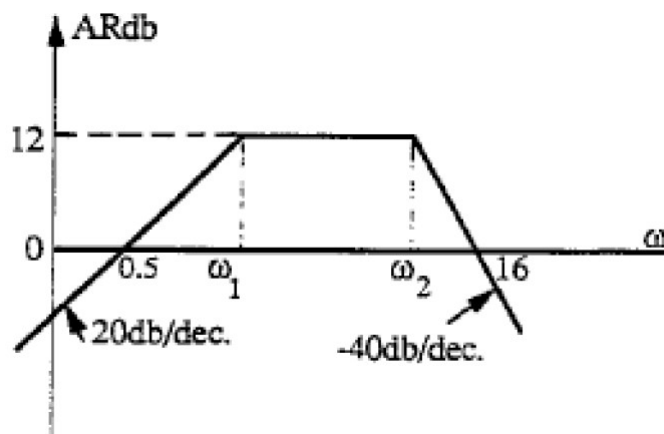

Fig. 8: Asymptotic Bode plot of the system in question 8

Question 8. Given in Fig. 8 is the asymptotic Bode diagram of a system.

1. Find the order of the system.

2. Find ω_1, ω_2 and the transfer function of the system.