Linear Systems (034032)

tutorial 8

1 Topics

Stability criteria for LTI systems with rational transfer functions in the Laplace domain and the Z-domain.

2 Background

2.1 Rational functions

A rational function can be written as

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

- $G(s)$ is proper if the order of the denominator is higher or equal to the order of the numerator, i.e. $n \geq m$.
- If $N(s)$ and $D(s)$ have no common roots, then the poles of $G(s)$ are the roots of $D(s)$.

2.2 Causality and stability

Theorem 1. *A continuous-time LTI system with rational transfer function $G(s)$ is causal and I/O stable iff*

- *$G(s)$ is proper and*
- *$G(s)$ has all its poles in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$.*

Example

$$G(s) = \frac{1}{s-a}$$

This system has one pole in $s = a$. Its impulse response is

$$g(t) = \mathrm{e}^{at}\,\mathbb{1}(t).$$

It can easily be seen that $g \in L_1$ only if $\operatorname{Re} a < 0$.[1]

Theorem 2. *A discrete-time LTI system with rational transfer function $G(z)$ is*

- *causal iff $G(z)$ is proper*
- *I/O stable iff $G(z)$ has all its poles in the open unit disk $\mathbb{D}_1 = \{z \in \mathbb{C} \mid |z| < 1\}$.*

Example

$$G(s) = \frac{z}{z-a}$$

This system has one pole in $z = a$. Its impulse response is

$$g[k] = a^k\,\mathbb{1}[k].$$

It can easily be seen that $g \in l_1$ only if $|a| < 1$.

[1] See Lecture 6: An LTI system G is BIBO stable iff $g \in L_1$ in the continous-time case and $g \in l_1$ in the discrete-time case.

2.3 Roots of a polynomial

In the case that our LTI system is represented by a rational function, the stability is determined by the locations of the roots of the denominator. It is not always easy to calculate these, but there are ways to test whether all the roots lie in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$ or in the open unit disk $\mathbb{D}_1$.

Terminology A polynomial is said to be

- Hurwitz if all its roots are in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$.
- Schur if all its roots are in $\mathbb{D}_1$.

Monic polynomials are polynomials whose leading coefficient $a_n = 1$, like

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

2.3.1 Continous-time case

Theorem 3 (necessary condition)**.** *The polynomial* $D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ *is Hurwitz only if* $a_i > 0$ *for all* $i \in \mathbb{Z}_{0..n-1}$. *In general, for polynomials that are not monic, we require that all coefficients are nonzero and have the same sign.*

Given the polynomial $D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, the associated Routh table is

$$\begin{array}{c|llll}
0 & r_{0,1} = a_n & r_{0,2} = a_{n-2} & r_{0,3} = a_{n-4} & \cdots \\
1 & r_{1,1} = a_{n-1} & r_{1,2} = a_{n-3} & r_{1,3} = a_{n-5} & \cdots \\
2 & r_{2,1} & r_{2,2} & r_{2,3} & \cdots \\
\vdots & \vdots & \vdots & & \\
n-2 & r_{n-2,1} & r_{n-2,2} & r_{n-2,2} & \\
n-1 & r_{n-1,1} & r_{n-1,2} & & \\
n & r_{n,1} & & &
\end{array}$$

where for each $i \in \mathbb{Z}_{2..n}$

$$\begin{bmatrix} r_{i,1} & r_{i,2} & \cdots \end{bmatrix} = \begin{bmatrix} r_{i-2,2} & r_{i-2,3} & \cdots \end{bmatrix} - \frac{r_{i-2,1}}{r_{i-1,1}} \begin{bmatrix} r_{i-1,2} & r_{i-1,3} & \cdots \end{bmatrix}$$

and if the last required column of an involved row is empty, 0 is taken. We have two cases according to what happens to the elements in the first column.

- Singular: $\exists i : r_{i,1} = 0$.
- Regular: $\forall i : r_{i,1} \neq 0$.

Theorem 4 (necessary and sufficient condition)**.** *Consider a polynomial in* s.

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- $D(s)$ *is Hurwitz iff the associated Routh table is regular and all the elements of the first column have the same sign.*
- *If the Routh table is regular, then* $D(s)$ *has no roots* $\mathrm{j}\mathbb{R}$ *and the number of poles in* $\mathbb{C}_0$ *equals the number of sign changes in the first column of the table.*
- *If the Routh table is singular, then* $D(s)$ *is not Hurwitz. Is this case, we cannot say anything about the location of the poles, except that there is at least one pole on* $\mathrm{j}\mathbb{R}$ *or in* $\mathbb{C}_0$.[2]

[2]There are tricks to deal with singularities, but these are not part of the course material.

Second order polynomial (memorize!) The Routh table for a second order polynomial $D(s) = a_2 s^2 + a_1 s + a_0$.

$$\begin{array}{c|cc} 0 & a_2 & a_0 \\ 1 & a_1 & \\ 2 & a_0 & \end{array}$$

Thus, for $D(s)$ to be Hurwitz, we must have that a_2, a_1, and a_0 are *nonzero and have the same sign*. Therefore, the necessary condition (see theorem 3) is in this case a sufficient condition as well.

Third order polynomial (memorize!) The Routh table for a third order polynomial $D(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$.

$$\begin{array}{c|cc} 0 & a_3 & a_1 \\ 1 & a_2 & a_0 \\ 2 & a_1 - \frac{a_3}{a_2} a_0 & \\ 3 & a_0 & \end{array}$$

Requiring that all elements in the first column have the same sign leads to the following condition for $D(s)$ to be Hurwitz:

- a_0, a_1, a_2, a_3 are nonzero and have the same sign and
- $a_1 a_2 > a_0 a_3$.

2.3.2 Discrete-time case

Given the polynomial $D(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, the associated Jury table is

$$\begin{array}{c|ccccc} 0 & j_{0,1} = a_n & j_{0,2} = a_{n-1} & \cdots & j_{0,n} = a_1 & j_{0,n+1} = a_0 \\ & j_{0,n+1} = a_0 & j_{0,n} = a_1 & \cdots & j_{0,2} = a_{n-1} & j_{0,1} = a_n \\ 1 & j_{1,1} & j_{1,2} & \cdots & j_{1,n} & \\ & j_{1,n} & j_{1,n-1} & \cdots & j_{1,1} & \\ \vdots & \vdots & \vdots & & & \\ n-1 & j_{n-1,1} & j_{n-1,2} & & & \\ & j_{n-1,2} & j_{n-1,1} & & & \\ n & j_{n,1} & & & & \end{array}$$

where for each $i \in \mathbb{Z}_{1..n}$

$$\begin{bmatrix} j_{i,1} & \cdots & j_{i,n+1-i} \end{bmatrix} = \begin{bmatrix} j_{i-1,1} & \cdots & j_{i-1,n+1-i} \end{bmatrix} - \frac{j_{i-1,n+2-i}}{j_{i-1,1}} \begin{bmatrix} j_{i-1,n+2-i} & \cdots & j_{i-1,2} \end{bmatrix}$$

(the ith row has $n + 1 - i$ elements). The Jury table is said to be *regular* if all $j_{i,1} \neq 0$. Otherwise, it is *singular*.

Theorem 5 (necessary and sufficient condition)**.** *Consider a polynomial in z with $a_n > 0$.*

$$D(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

- *$D(z)$ is Schur iff the associated Jury table is regular and all the elements of the first column are positive.*
- *If the Jury table is regular, then $D(z)$ has no roots on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ and the number of poles outside of the unit circle $\{z \in \mathbb{C} \mid |z| > 1\}$ equals the number of negative elements in the first column of the table.*

- *If the Jury table is singular, then $D(z)$ is not Schur. Is this case, we cannot say anything about the location of the poles, except that there is at least one pole in $\{z \in \mathbb{C} \mid |z| \geq 1\}$.*[3]

2.4 Matlab commands

Matlab commands to analyze the stability of a system:

- `isstable(G)` returns a logical value of 1 (true) if the dynamic system model `G` (e.g. obtained via the `tf` command) is stable and a logical value of 0 (false) otherwise.
- `isproper(G)` returns a logical value of 1 (true) if the transfer function of a dynamic system model `G` is proper and a logical value of 0 (false) otherwise.

3 Problems

Question 1. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{s^2 + 1}{3s + 2}$$

Is this system I/O stable?

Solution. This system is unstable because the transfer function is not proper, despite the fact that there are no poles in the right half plane $\mathbb{C}_0$. ∇

Question 2. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^3 + 7s^2 - 4s + 2}$$

Is this system I/O stable?

Solution. The polynomial of the denominator contains a coefficient (-4) with a different sign and will therefore have at least 1 root in the right half plane $\mathbb{C}_0$ (see theorem 3). Although the system is proper, it is not I/O stable because of this pole. ∇

Question 3. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^5 + 4s^4 + 2s^3 + 2s^2 + s + 10}$$

Is this system I/O stable? If not, then where are the poles placed?

Solution. The transfer function is proper. Also, all coefficients of the denominator have the same sign, which means that the system *might* be stable. Let's construct an associated Routh table.

0	1	2	1
1	4	2	10
2	$r_{2,1}$	$r_{2,2}$	
3	$r_{3,1}$	$r_{3,2}$	
4	$r_{4,1}$		
5	$r_{5,1}$		

[3] There are tricks to deal with singularities, but these are not part of the course material.

For row 2, we have

$$\begin{bmatrix} r_{2,1} & r_{2,2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 & 10 \end{bmatrix} = \begin{bmatrix} 1.5 & -1.5 \end{bmatrix}$$

$$\begin{array}{c|ccc} 0 & 1 & 2 & 1 \\ 1 & 4 & 2 & 10 \\ 2 & 1.5 & -1.5 & \\ 3 & r_{3,1} & r_{3,2} & \\ 4 & r_{4,1} & & \\ 5 & r_{5,1} & & \end{array}$$

For row 3, we have

$$\begin{bmatrix} r_{3,1} & r_{3,2} \end{bmatrix} = \begin{bmatrix} 2 & 10 \end{bmatrix} - \frac{4}{1.5}\begin{bmatrix} -1.5 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 10 \end{bmatrix}$$

$$\begin{array}{c|ccc} 0 & 1 & 2 & 1 \\ 1 & 4 & 2 & 10 \\ 2 & 1.5 & -1.5 & \\ 3 & 6 & 10 & \\ 4 & r_{4,1} & & \\ 5 & r_{5,1} & & \end{array}$$

Next, for row 4, we have

$$\begin{bmatrix} r_{4,1} \end{bmatrix} = \begin{bmatrix} -1.5 \end{bmatrix} - \frac{1.5}{6}\begin{bmatrix} 10 \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix}$$

$$\begin{array}{c|ccc} 0 & 1 & 2 & 1 \\ 1 & 4 & 2 & 10 \\ 2 & 1.5 & -1.5 & \\ 3 & 6 & 10 & \\ 4 & -4 & & \\ 5 & r_{5,1} & & \end{array}$$

Finally,

$$\begin{bmatrix} r_{5,1} \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix} - \frac{6}{-4}\begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}$$

$$\begin{array}{c|ccc} 0 & 1 & 2 & 1 \\ 1 & 4 & 2 & 10 \\ 2 & 1.5 & -1.5 & \\ 3 & 6 & 10 & \\ 4 & -4 & & \\ 5 & 10 & & \end{array}$$

We can see that there are two sign changes in the first column ($6 \to -4 \to 10$). Therefore, we must have two poles in the right half plane $\mathbb{C}_0$. As the Routh table is regular, there are no poles on the imaginary axis $j\mathbb{R}$. Using MATLAB, we can confirm this result as is shown in fig. 1. To conclude, this system is not I/O stable despite the fact that it is proper.

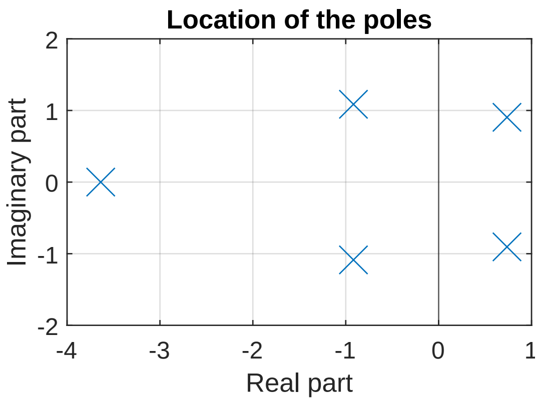

Fig. 1: Pole zero map of $G(s) = \dfrac{1}{s^5 + 4s^4 + 2s^3 + 2s^2 + s + 10}$.

▽

Question 4. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{-10}{s^2 + 5s + 2}$$

Is this system I/O stable?

Solution. Because there are no zeros in this transfer function, we need not concern ourselves with the -10 in the numerator. As for the denominator, for a second order polynomial, the necessary condition for stability (coefficients have the same sign) is also a sufficient condition. Because the system is proper and there are no poles in the right half plane $\mathbb{C}_0$, the system is I/O stable. ▽

Question 5. Consider the system shown in fig. 2.

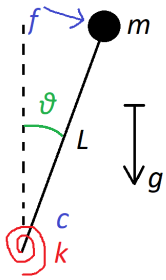

Fig. 2: Inverted pendulum with torsion spring.

The transfer function of the linearized system is given by

$$G(s) = \frac{L}{mL^2s^2 + cs + (k - mgL)}.$$

For those interested, a derivation is given in appendix A. When is this system I/O stable?

Solution. The transfer function is proper. Because we are working with a second degree polynomial, the criterion for stability is that all coefficients of the numerator have the same sign. In this case, all coefficients must be positive.

$$\begin{cases} mL^2 > 0 \\ c > 0 \\ k - mgL > 0 \end{cases}$$

The first two inequalities are always met. Therefore, the condition for stability becomes.

$$\boxed{k > mgL}$$

What this means intuitively is that the force of the spring ($k\theta$) should be larger than the moment produced by the weight of the mass ($mgL\sin\theta$).

$$k\theta > mgL\sin\theta \approx mgL\theta$$

For example, if the mass is too heavy, the rod will fall down. ∇

Question 6. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{6}{s^3 + 2s^2 + ks + 4}, \quad k \in \mathbb{R}$$

For which k is this system I/O stable? Describe the placement of the poles for different k.

Solution. Note that the transfer function is proper. Therefore, the criterion for I/O stability is that the poles be in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$. Because this is a third order system, we can use the condition which you memorized. All coefficients must have the same sign and $a_1 a_2 > a_0 a_3$. Therefore, for stability, we must have

$$\begin{cases} k > 0 \\ 2k > 1 \cdot 4 = 4 \end{cases} \implies \boxed{k > 2}$$

To understand better what will happen when $k \leq 2$, we have to construct a Routh table.

0	1	k
1	2	4
2	$k-2$	
3	4	

We have three cases:

- $k > 2$: the Routh table is regular and there are no sign changes. All the poles must be in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$, which means our system is I/O stable (the transfer function is proper).
- $k = 2$: the Routh table is singular. Therefore, there *might* be poles on the imaginary axis $j\mathbb{R}$. What we can say for certain that not all poles will be in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$. In conclusion, the system is not I/O stable despite the fact that it's proper.
- $k < 2$: the Routh table is regular and there are 2 sign changes ($2 \to k - 2 \to 4$). Therefore, there must be 2 poles in the right half plane $\mathbb{C}_0$. In conclusion, the system is not I/O stable despite the fact that it's proper.

We confirm this in fig. 3. For $k = 2$, we can indeed see that there are two poles on the imaginary axis $\mathrm{j}\mathbb{R}$, but this is not always necessarily the case when the Routh table is singular (see question 9).

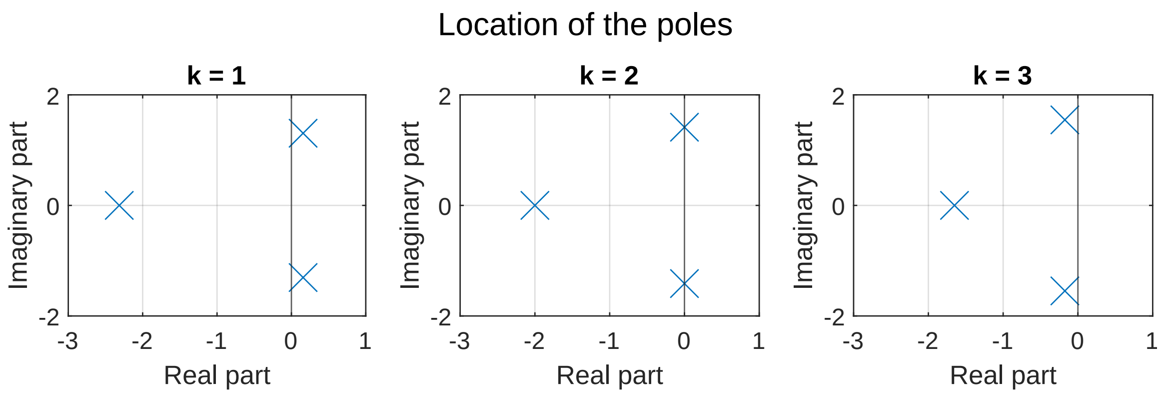

Fig. 3: Pole zero map of $G(s) = \dfrac{1}{s^3 + 2s^2 + ks + 4}$.

$\triangledown$

Question 7. Consider the discrete-time LTI system with transfer function $G(z)$.

$$G(s) = \frac{1}{z^3 + 4z^2 + 8z + 3}$$

Is this system I/O stable? If not, then where are the poles placed?

Solution. We will create an associated Jury table.

$$\begin{array}{c|cccc}
0 & 1 & 4 & 8 & 3 \\
 & 3 & 8 & 4 & 1 \\
1 & j_{1,1} & j_{1,2} & j_{1,3} & \\
 & j_{1,3} & j_{1,2} & j_{1,1} & \\
2 & j_{2,1} & j_{2,2} & & \\
 & j_{2,3} & j_{2,2} & & \\
3 & j_{3,1} & & &
\end{array}$$

For the row 1, we have

$$\begin{bmatrix} j_{1,1} & j_{1,2} & j_{1,3} & j_{1,4} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 8 & 3 \end{bmatrix} - \frac{3}{1}\begin{bmatrix} 3 & 8 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -8 & -20 & -4 & 0 \end{bmatrix}$$

$$\begin{array}{c|cccc}
0 & 1 & 4 & 8 & 3 \\
 & 3 & 8 & 4 & 1 \\
1 & -8 & -20 & -4 & \\
 & -4 & -20 & -8 & \\
2 & j_{2,1} & j_{2,2} & & \\
 & j_{2,3} & j_{2,2} & & \\
3 & j_{3,1} & & &
\end{array}$$

For row 2, we have

$$\begin{bmatrix} j_{2,1} & j_{2,2} & j_{2,3} \end{bmatrix} = \begin{bmatrix} -8 & -20 & -4 \end{bmatrix} - \frac{-4}{-8}\begin{bmatrix} -4 & -20 & -8 \end{bmatrix} = \begin{bmatrix} -6 & -10 & 0 \end{bmatrix}$$

0	1	4	8	3
	3	8	4	1
1	−8	−20	−4	
	−4	−20	−8	
2	−6	−10		
	−10	−6		
3	$j_{3,1}$			

Finally, we have

$$\begin{bmatrix} j_{3,1} & j_{3,2} \end{bmatrix} = \begin{bmatrix} -6 & -10 \end{bmatrix} - \frac{-10}{-6}\begin{bmatrix} -10 & -6 \end{bmatrix} = \begin{bmatrix} 32/3 & 0 \end{bmatrix}$$

0	1	4	8	3
	3	8	4	1
1	−8	−20	−4	
	−4	−20	−8	
2	−6	−10		
	−10	−6		
3	32/3			

The Jury table is regular. The first column has 2 negative numbers. Therefore, there will be 2 poles outside of the unit circle $\{z \in \mathbb{C} \mid |z| > 1\}$. As this is a third order system, there can only be 3 poles and there is one remaining pole inside the unit circle. This is confirmed in fig. 4. The system is not I/O stable because not all the poles are inside the unit circle $\mathbb{D}_1$.

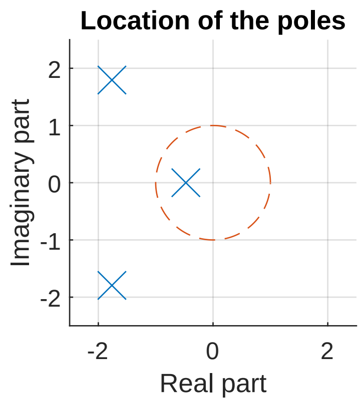

Fig. 4: Pole zero map of $G(z) = \dfrac{1}{z^3 + 4z^2 + 8z + 3}$.

▽

4 Homework problems

Question 8. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^4 + 2s^3 + ks^2 + 4s + k}, \quad k \in \mathbb{R}$$

For which k is this system I/O stable? Describe the placement of the poles for different k.

Solution. The associated Routh table is shown below.

$$\begin{array}{c|ccc} 0 & 1 & k & k \\ 1 & 2 & 4 & \\ 2 & k-2 & k & \\ 3 & 2\frac{k-4}{k-2} & & \\ 4 & k & & \end{array}$$

Note that the transfer function is proper. Therefore, the system is I/O stable if all the poles are in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$. We have a few cases:

Case	Regular or singular	Sign changes	Conclusion
$k > 4$	Regular	0	All poles in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$, so stable.
$k = 4$	Singular		Not all poles in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$, so unstable.
$2 < k < 4$	Regular	2	2 poles in $\mathbb{C}_0$, so unstable.
$k = 2$	Singular		Not all poles in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$, so unstable.
$0 < k < 2$	Regular	2	2 poles in $\mathbb{C}_0$, so unstable.
$k = 0$	Singular		Not all poles in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$, so unstable.
$k < 0$	Regular	3	3 poles in $\mathbb{C}_0$, so unstable.

$\triangledown$

Question 9. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 6s + 5}$$

Is this system I/O stable? If not, then where are the poles placed?

Solution. In addition to the system being proper, all coefficients of the denominator have the same sign, which means that the system *might* be I/O stable. Let's construct an associated Routh table.

$$\begin{array}{c|ccc} 0 & 1 & 3 & 5 \\ 1 & 2 & 6 & \\ 2 & r_{2,1} & r_{2,2} & \\ 3 & r_{3,1} & & \\ 4 & r_{4,1} & & \end{array}$$

For row 2, we have

$$\begin{bmatrix} r_{2,1} & r_{2,2} \end{bmatrix} = \begin{bmatrix} 3 & 5 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5 \end{bmatrix}$$

$$\begin{array}{c|ccc} 0 & 1 & 3 & 5 \\ 1 & 2 & 6 & \\ 2 & 0 & 5 & \\ 3 & r_{3,1} & & \\ 4 & r_{4,1} & & \end{array}$$

This Routh table is singular ($r_{2,1} = 0$), and we therefore cannot continue (we cannot divide by 0). This means that the system is unstable (despite properness) and that there *might* be poles on the imaginary axis $j\mathbb{R}$. However, in this particular case as can be seen in fig. 5, we have 2 poles in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$ and 2 poles in the right half plane $\mathbb{C}_0$ and no poles on the imaginary axis $j\mathbb{R}$.

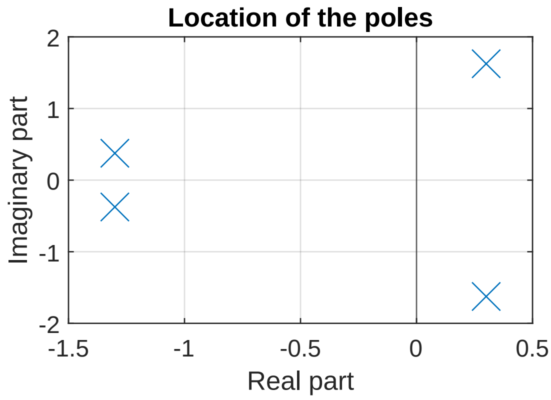

Fig. 5: Pole zero map of $G(s) = \dfrac{1}{s^4 + 2s^3 + 3s^2 + 6s + 5}$.

▽

Question 10. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^4 + 2s^3 + 8s^2 + 4s + 6}$$

Is this system I/O stable? If not, then where are the poles placed?

Solution. The transfer function is proper so we only need to show that the poles are in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$. In addition, all coefficients have the same sign, which means that the system *might* be I/O stable. Let's construct an associated Routh table.

$$\begin{array}{c|ccc} 0 & 1 & 8 & 6 \\ 1 & 2 & 4 & \\ 2 & r_{2,1} & r_{2,2} & \\ 3 & r_{3,1} & & \\ 4 & r_{4,1} & & \end{array}$$

For row 2, we have

$$\begin{bmatrix} r_{2,1} & r_{2,2} \end{bmatrix} = \begin{bmatrix} 8 & 6 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 6 \end{bmatrix}$$

$$\begin{array}{c|ccc} 0 & 1 & 8 & 6 \\ 1 & 2 & 4 & \\ 2 & 6 & 6 & \\ 3 & r_{3,1} & & \\ 4 & r_{4,1} & & \end{array}$$

For row 3, we have

$$\begin{bmatrix} r_{3,1} \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 6 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$\begin{array}{c|ccc} 0 & 1 & 8 & 6 \\ 2 & 2 & 4 & \\ 2 & 6 & 6 & \\ 3 & 2 & & \\ 4 & r_{4,1} & & \end{array}$$

Finally,

$$[r_{4,1}] = [6] - \frac{6}{2}[0] = [6]$$

$$\begin{array}{c|ccc} 0 & 1 & 8 & 6 \\ 1 & 2 & 4 & \\ 2 & 6 & 6 & \\ 3 & 2 & & \\ 4 & 6 & & \end{array}$$

This Routh table is regular. Moreover, there are no sign changes in the first column. All the poles are located in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$, as can be seen in fig. 6. Therefore, the system is I/O stable.

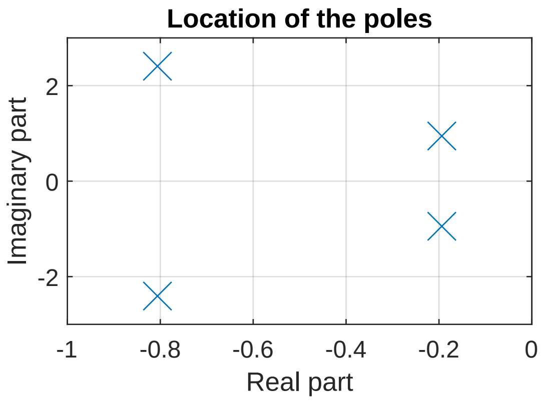

Fig. 6: Pole zero map of $G(s) = \dfrac{1}{s^4 + 2s^3 + 8s^2 + 4s + 6}$.

$\triangledown$

Question 11. Consider the discrete-time LTI system with transfer function $G(z)$.

$$G(z) = \frac{1}{z^2 + kz + 4}, \quad k \in \mathbb{R}$$

For which k is this system I/O stable? Describe the placement of the poles for different k.

Solution.

$$\begin{array}{c|ccc} 0 & 1 & k & 4 \\ & 4 & k & 1 \\ 1 & -15 & -3k & \\ & -3k & -15 & \\ 2 & -15 + \frac{3}{5}k^2 & & \end{array}$$

There is no k for which this system will be I/O stable (despite properness) because -15 is always negative, meaning that we will for sure have at least 1 pole outside of the unit circle $\{z \in \mathbb{C} \mid |z| > 1\}$. The last row is positive if

$$-15 + \frac{3}{5}k^2 > 0 \Rightarrow k^2 > 25 \Rightarrow |k| > 5$$

We can distinguish three cases.

- $|k| > 5$: the Jury table is regular with 1 negative element along the first column. Therefore, we have 1 pole outside the unit circle and 1 inside.
- $|k| = 5$: the Jury table is singular. We will have at least 1 pole in $\{z \in \mathbb{C} \mid |z| \geq 1\}$.
- $|k| < 5$: the Jury table is regular with 2 negative elements along the first column. Therefore, we have 2 poles outside of the unit circle. Because this is a second order system, this accounts for all the poles.

This is confirmed in fig. 7.

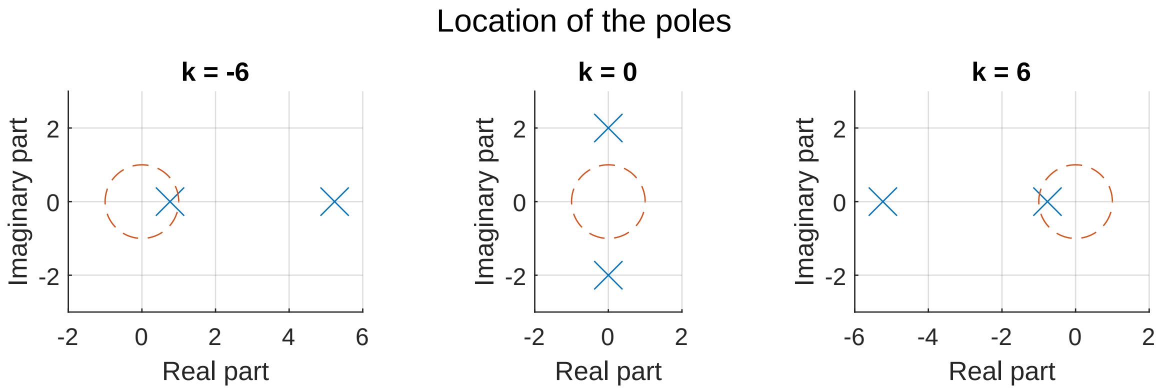

Fig. 7: Pole zero map of $G(z) = \dfrac{1}{z^2 + kz + 4}$.

∇

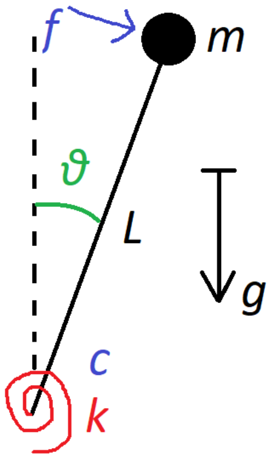

Fig. 8: Inverted pendulum with torsion spring.

A Inverted pendulum with a torsion spring [not part of the course material]

An inverted pendulum with length L is held in place by a torsion spring with constant k, as can be seen in fig. 8. A mass m is placed at the end of the rod. In addition, a torsion damper is set in place with constant c. Finally, an external force is acting on the mass perpendicularly to the rod.

We will use Lagrangian mechanics to derive the transfer function (which you probably have not seen yet). The position and velocity of the mass are given by

$$\vec{r} = \begin{bmatrix} L\sin\theta \\ L\cos\theta \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} L\cos\theta\dot{\theta} \\ -L\sin\theta\dot{\theta} \end{bmatrix}.$$

For the potential energy, we have

$$V = \frac{1}{2}k\theta^2 + mg(L\cos\theta).$$

For the kinetic energy, we have

$$K = \frac{1}{2}m(L\dot{\theta})^2.$$

Thus, the Lagrangian is

$$\mathcal{L} = K - V = \frac{1}{2}m(L\dot{\theta})^2 - \frac{1}{2}k\theta^2 - mg(L\cos\theta).$$

The force $\vec{f}$ acts perpendicularly to the rod.

$$\vec{f} = f\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

The dampening applies a torque $\tau = -c\dot{\theta}$. As it doesn't depend on θ, we can conclude that the force applied to the mass is perpendicular to the rod as well.

$$\vec{f_c} = \frac{-c\dot{\theta}}{L}\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

We can now combine these two non conservative forces into a generalized force.

$$Q = (\vec{f} + \vec{f_c}) \cdot \frac{\partial \vec{r}}{\partial \theta} = (f - \frac{c\dot{\theta}}{L}) \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \cdot \begin{bmatrix} L\cos\theta \\ -L\sin\theta \end{bmatrix} = fL - c\dot{\theta}$$

Now, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q \quad \Longrightarrow \quad mL^2\ddot{\theta} + k\theta - mgL\sin\theta = fL - c\dot{\theta}.$$

Upon linearization ($\sin\theta \approx \theta$), we get

$$\boxed{mL^2\ddot{\theta} + c\dot{\theta} + (k - mgL)\theta = fL}.$$

In the Laplace domain, the transfer function of the linearized system $G(s) = \Theta(s)/F(s)$ is

$$\boxed{G(s) = \frac{L}{mL^2s^2 + cs + (k - mgL)}}.$$