Linear Systems (034032)

tutorial 8

1 Topics

Stability criteria for LTI systems with rational transfer functions in the Laplace domain and the Z-domain.

2 Background

2.1 Rational functions

A rational function can be written as

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

- $G(s)$ is proper if the order of the denominator is higher or equal to the order of the numerator, i.e. $n \geq m$.
- If $N(s)$ and $D(s)$ have no common roots, then the poles of $G(s)$ are the roots of $D(s)$.

2.2 Causality and stability

Theorem 1. *A continuous-time LTI system with rational transfer function $G(s)$ is causal and I/O stable iff*

- *$G(s)$ is proper and*
- *$G(s)$ has all its poles in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$.*

Example

$$G(s) = \frac{1}{s-a}$$

This system has one pole in $s = a$. Its impulse response is

$$g(t) = \mathrm{e}^{at}\,\mathbb{1}(t).$$

It can easily be seen that $g \in L_1$ only if $\operatorname{Re} a < 0$.[1]

Theorem 2. *A discrete-time LTI system with rational transfer function $G(z)$ is*

- *causal iff $G(z)$ is proper*
- *I/O stable iff $G(z)$ has all its poles in the open unit disk $\mathbb{D}_1 = \{z \in \mathbb{C} \mid |z| < 1\}$.*

Example

$$G(s) = \frac{z}{z-a}$$

This system has one pole in $z = a$. Its impulse response is

$$g[k] = a^k\,\mathbb{1}[k].$$

It can easily be seen that $g \in l_1$ only if $|a| < 1$.

[1] See Lecture 6: An LTI system G is BIBO stable iff $g \in L_1$ in the continous-time case and $g \in l_1$ in the discrete-time case.

2.3 Roots of a polynomial

In the case that our LTI system is represented by a rational function, the stability is determined by the locations of the roots of the denominator. It is not always easy to calculate these, but there are ways to test whether all the roots lie in the left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$ or in the open unit disk $\mathbb{D}_1$.

Terminology A polynomial is said to be

- Hurwitz if all its roots are in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$.
- Schur if all its roots are in $\mathbb{D}_1$.

Monic polynomials are polynomials whose leading coefficient $a_n = 1$, like

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

2.3.1 Continous-time case

Theorem 3 (necessary condition)**.** *The polynomial* $D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ *is Hurwitz only if* $a_i > 0$ *for all* $i \in \mathbb{Z}_{0..n-1}$*. In general, for polynomials that are not monic, we require that all coefficients are nonzero and have the same sign.*

Given the polynomial $D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, the associated Routh table is

$$\begin{array}{c|llll}
0 & r_{0,1} = a_n & r_{0,2} = a_{n-2} & r_{0,3} = a_{n-4} & \cdots \\
1 & r_{1,1} = a_{n-1} & r_{1,2} = a_{n-3} & r_{1,3} = a_{n-5} & \cdots \\
2 & r_{2,1} & r_{2,2} & r_{2,3} & \cdots \\
\vdots & \vdots & \vdots & & \\
n-2 & r_{n-2,1} & r_{n-2,2} & r_{n-2,2} & \\
n-1 & r_{n-1,1} & r_{n-1,2} & & \\
n & r_{n,1} & & &
\end{array}$$

where for each $i \in \mathbb{Z}_{2..n}$

$$\begin{bmatrix} r_{i,1} & r_{i,2} & \cdots \end{bmatrix} = \begin{bmatrix} r_{i-2,2} & r_{i-2,3} & \cdots \end{bmatrix} - \frac{r_{i-2,1}}{r_{i-1,1}} \begin{bmatrix} r_{i-1,2} & r_{i-1,3} & \cdots \end{bmatrix}$$

and if the last required column of an involved row is empty, 0 is taken. We have two cases according to what happens to the elements in the first column.

- Singular: $\exists i : r_{i,1} = 0$.
- Regular: $\forall i : r_{i,1} \neq 0$.

Theorem 4 (necessary and sufficient condition)**.** *Consider a polynomial in* s*.*

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- $D(s)$ *is Hurwitz iff the associated Routh table is regular and all the elements of the first column have the same sign.*
- *If the Routh table is regular, then* $D(s)$ *has no roots* $\mathrm{j}\mathbb{R}$ *and the number of poles in* $\mathbb{C}_0$ *equals the number of sign changes in the first column of the table.*
- *If the Routh table is singular, then* $D(s)$ *is not Hurwitz. Is this case, we cannot say anything about the location of the poles, except that there is at least one pole on* $\mathrm{j}\mathbb{R}$ *or in* $\mathbb{C}_0$*.*[2]

[2] There are tricks to deal with singularities, but these are not part of the course material.

Second order polynomial (memorize!) The Routh table for a second order polynomial $D(s) = a_2 s^2 + a_1 s + a_0$.

$$\begin{array}{c|cc} 0 & a_2 & a_0 \\ 1 & a_1 & \\ 2 & a_0 & \end{array}$$

Thus, for $D(s)$ to be Hurwitz, we must have that a_2, a_1, and a_0 are *nonzero and have the same sign*. Therefore, the necessary condition (see theorem 3) is in this case a sufficient condition as well.

Third order polynomial (memorize!) The Routh table for a third order polynomial $D(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$.

$$\begin{array}{c|cc} 0 & a_3 & a_1 \\ 1 & a_2 & a_0 \\ 2 & a_1 - \frac{a_3}{a_2} a_0 & \\ 3 & a_0 & \end{array}$$

Requiring that all elements in the first column have the same sign leads to the following condition for $D(s)$ to be Hurwitz:

- a_0, a_1, a_2, a_3 are nonzero and have the same sign and
- $a_1 a_2 > a_0 a_3$.

2.3.2 Discrete-time case

Given the polynomial $D(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, the associated Jury table is

$$\begin{array}{c|ccccc} 0 & j_{0,1} = a_n & j_{0,2} = a_{n-1} & \cdots & j_{0,n} = a_1 & j_{0,n+1} = a_0 \\ & j_{0,n+1} = a_0 & j_{0,n} = a_1 & \cdots & j_{0,2} = a_{n-1} & j_{0,1} = a_n \\ 1 & j_{1,1} & j_{1,2} & \cdots & j_{1,n} & \\ & j_{1,n} & j_{1,n-1} & \cdots & j_{1,1} & \\ \vdots & \vdots & \vdots & & & \\ n-1 & j_{n-1,1} & j_{n-1,2} & & & \\ & j_{n-1,2} & j_{n-1,1} & & & \\ n & j_{n,1} & & & & \end{array}$$

where for each $i \in \mathbb{Z}_{1..n}$

$$\begin{bmatrix} j_{i,1} & \cdots & j_{i,n+1-i} \end{bmatrix} = \begin{bmatrix} j_{i-1,1} & \cdots & j_{i-1,n+1-i} \end{bmatrix} - \frac{j_{i-1,n+2-i}}{j_{i-1,1}} \begin{bmatrix} j_{i-1,n+2-i} & \cdots & j_{i-1,2} \end{bmatrix}$$

(the ith row has $n + 1 - i$ elements). The Jury table is said to be *regular* if all $j_{i,1} \neq 0$. Otherwise, it is *singular*.

Theorem 5 (necessary and sufficient condition)**.** *Consider a polynomial in z with $a_n > 0$.*

$$D(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

- *$D(z)$ is Schur iff the associated Jury table is regular and all the elements of the first column are positive.*
- *If the Jury table is regular, then $D(z)$ has no roots on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ and the number of poles outside of the unit circle $\{z \in \mathbb{C} \mid |z| > 1\}$ equals the number of negative elements in the first column of the table.*

- *If the Jury table is singular, then $D(z)$ is not Schur. Is this case, we cannot say anything about the location of the poles, except that there is at least one pole in $\{z \in \mathbb{C} \mid |z| \geq 1\}$.*[3]

2.4 Matlab commands

Matlab commands to analyze the stability of a system:

- `isstable(G)` returns a logical value of 1 (true) if the dynamic system model `G` (e.g. obtained via the `tf` command) is stable and a logical value of 0 (false) otherwise.
- `isproper(G)` returns a logical value of 1 (true) if the transfer function of a dynamic system model `G` is proper and a logical value of 0 (false) otherwise.

3 Problems

Question 1. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{s^2 + 1}{3s + 2}$$

Is this system I/O stable?

Question 2. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^3 + 7s^2 - 4s + 2}$$

Is this system I/O stable?

Question 3. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^5 + 4s^4 + 2s^3 + 2s^2 + s + 10}$$

Is this system I/O stable? If not, then where are the poles placed?

Question 4. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{-10}{s^2 + 5s + 2}$$

Is this system I/O stable?

Question 5. Consider the system shown in fig. 1.

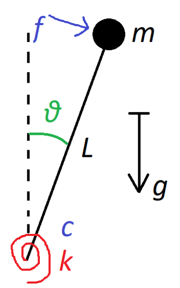

Fig. 1: Inverted pendulum with torsion spring.

[3]There are tricks to deal with singularities, but these are not part of the course material.

The transfer function of the linearized system is given by

$$G(s) = \frac{L}{mL^2s^2 + cs + (k - mgL)}.$$

For those interested, a derivation is given in appendix A. When is this system I/O stable?

Question 6. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{6}{s^3 + 2s^2 + ks + 4}, \; k \in \mathbb{R}$$

For which k is this system I/O stable? Describe the placement of the poles for different k.

Question 7. Consider the discrete-time LTI system with transfer function $G(z)$.

$$G(s) = \frac{1}{z^3 + 4z^2 + 8z + 3}$$

Is this system I/O stable? If not, then where are the poles placed?

4 Homework problems

Question 8. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^4 + 2s^3 + ks^2 + 4s + k}, \ k \in \mathbb{R}$$

For which k is this system I/O stable? Describe the placement of the poles for different k.

Question 9. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 6s + 5}$$

Is this system I/O stable? If not, then where are the poles placed?

Question 10. Consider the continuous-time LTI system with transfer function $G(s)$.

$$G(s) = \frac{1}{s^4 + 2s^3 + 8s^2 + 4s + 6}$$

Is this system I/O stable? If not, then where are the poles placed?

Question 11. Consider the discrete-time LTI system with transfer function $G(z)$.

$$G(z) = \frac{1}{z^2 + kz + 4}, \ k \in \mathbb{R}$$

For which k is this system I/O stable? Describe the placement of the poles for different k.

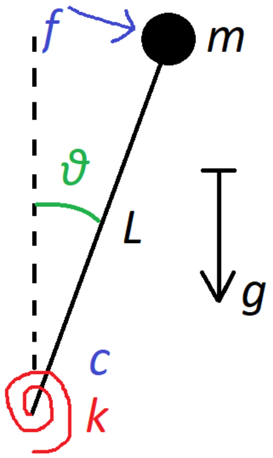

Fig. 2: Inverted pendulum with torsion spring.

A Inverted pendulum with a torsion spring [not part of the course material]

An inverted pendulum with length L is held in place by a torsion spring with constant k, as can be seen in fig. 2. A mass m is placed at the end of the rod. In addition, a torsion damper is set in place with constant c. Finally, an external force is acting on the mass perpendicularly to the rod.

We will use Lagrangian mechanics to derive the transfer function (which you probably have not seen yet). The position and velocity of the mass are given by

$$\vec{r} = \begin{bmatrix} L\sin\theta \\ L\cos\theta \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} L\cos\theta\dot{\theta} \\ -L\sin\theta\dot{\theta} \end{bmatrix}.$$

For the potential energy, we have

$$V = \frac{1}{2}k\theta^2 + mg(L\cos\theta).$$

For the kinetic energy, we have

$$K = \frac{1}{2}m(L\dot{\theta})^2.$$

Thus, the Lagrangian is

$$\mathcal{L} = K - V = \frac{1}{2}m(L\dot{\theta})^2 - \frac{1}{2}k\theta^2 - mg(L\cos\theta).$$

The force $\vec{f}$ acts perpendicularly to the rod.

$$\vec{f} = f\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

The dampening applies a torque $\tau = -c\dot{\theta}$. As it doesn't depend on θ, we can conclude that the force applied to the mass is perpendicular to the rod as well.

$$\vec{f_c} = \frac{-c\dot{\theta}}{L}\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

We can now combine these two non conservative forces into a generalized force.

$$Q = (\vec{f} + \vec{f_c}) \cdot \frac{\partial \vec{r}}{\partial \theta} = (f - \frac{c\dot{\theta}}{L}) \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \cdot \begin{bmatrix} L\cos\theta \\ -L\sin\theta \end{bmatrix} = fL - c\dot{\theta}$$

Now, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q \qquad \Longrightarrow \qquad mL^2\ddot{\theta} + k\theta - mgL\sin\theta = fL - c\dot{\theta}.$$

Upon linearization ($\sin\theta \approx \theta$), we get

$$\boxed{mL^2\ddot{\theta} + c\dot{\theta} + (k - mgL)\theta = fL}.$$

In the Laplace domain, the transfer function of the linearized system $G(s) = \Theta(s)/F(s)$ is

$$\boxed{G(s) = \frac{L}{mL^2s^2 + cs + (k - mgL)}}.$$