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LINEAR SYSTEMS (034032)

TUTORIAL 7

</div>

1 Topics

Transfer functions, zeros and poles, 1st order and 2nd order systems, step response.

2 Background

2.1 Rational transfer functions

An LTI system $G : u \mapsto y$ can be represented by its transfer function $G(s)$. In some important cases (viz. systems described by ODEs) transfer functions are of the form of a quotient of two polynomials, like

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

for some $n, m \in \mathbb{Z}_+$ and real coefficients a_i and b_i. Such transfer functions are said to be *rational*. Their poles are the roots of the denominator polynomial (assuming that the numerator and denominator polynomials are coprime, i.e. have no common roots). The roots of the numerator are called zeros of $G(s)$. The system is said to be proper if $n \geq m$, strictly proper is $n > m$, bi-proper if $n = m$, non-proper if $n < m$.

2.2 First order system

Canonical form of first order system

The transfer function of a general first-order system takes the form

$$G(s) = \frac{k_{\text{st}}}{\tau s + 1} \tag{1}$$

where k_{st} is the static gain and τ is the time constant. The single pole of the system is $p = -1/\tau \in \mathbb{R}_-$.

Step response of first order system

The step response of a first order system is

$$y_{\text{step}}(t) = k_{\text{st}}(1 - e^{-t/\tau}) \mathbb{1}(t) \tag{2}$$

The static gain k_{st} scales the response amplitude. When $t = \tau$ and $t = 3\tau$

$$y_{\text{step}}(\tau) = k_{\text{st}}(1 - e^{-1}) \approx 0.63 k_{\text{st}} \quad \text{and} \quad y_{\text{step}}(3\tau) = k_{\text{st}}(1 - e^{-3}) \approx 0.95 k_{\text{st}},$$

respectively. The time constant τ dictates the responsiveness of the system. For the effect of changing k_{st} and τ see Fig. 1.
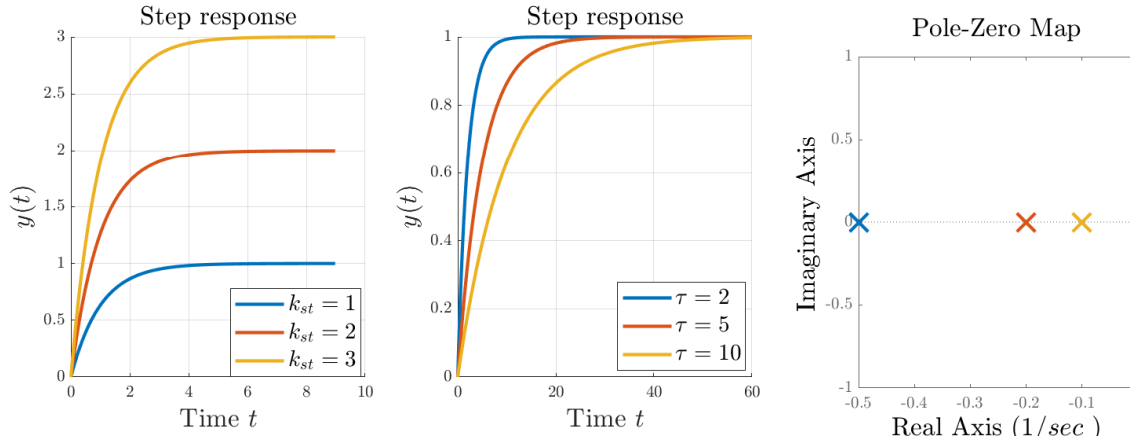
Fig. 1: First order system

2.3 Second order system

Canonical form of second order system

The transfer function of a general second-order system takes the form

$$G(s) = \frac{k_{st}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

where k_{st} is the static gain, ζ is the damping ratio, and ω_n is the natural frequency. The poles of the system are

$$s_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$$

and three cases shall be distinguished:

1. $\zeta > 1$: the system is called *overdamped*, the poles $s_{1,2} \in \mathbb{R}$ and such that $s_1 \neq s_2$

2. $\zeta = 1$: the system is called *critically damped* and the poles $s_{1,2} \in \mathbb{R}$, with $s_1 = s_2$

3. $0 \leq \zeta < 1$: the system is called *underdamped* and the poles $s_{1,2} \in \mathbb{C}$ and $s_1 = \overline{s_2}$

Step response of an overdamped second order system

The step response in this case is

$$y(t) = k_{st}(1 - \beta e^{s_1 t} + (\beta - 1)e^{s_2 t})\mathbb{1}(t), \quad \text{where } \beta = \frac{1}{2}\left(\frac{\zeta}{\sqrt{\zeta^2 - 1}} + 1\right) > 1 \tag{4}$$

Increasing ω_n will cause a faster response. Increasing ζ will cause a slower response, see Fig. 2.

Step response of a critically damped second order system

In this case the poles $s_{1,2} = -\omega_n$ and the step response is

$$y(t) = k_{st}\big(1 - (1 + \omega_n t)e^{-\omega_n t}\big)\mathbb{1}(t) \tag{5}$$

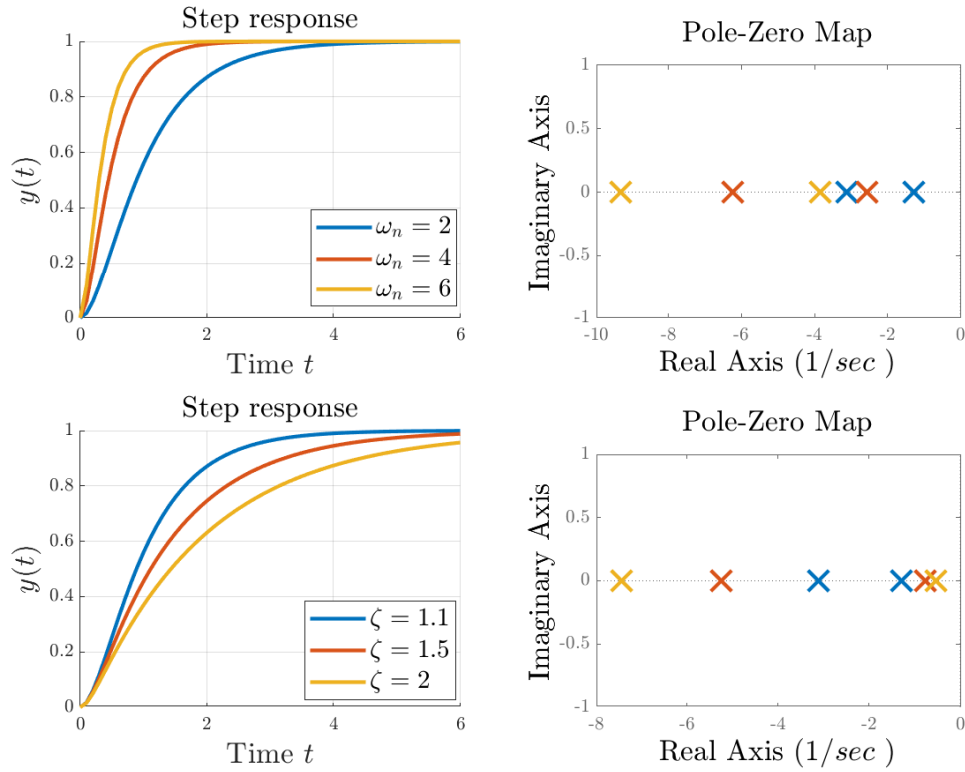Increasing ω_n will cause a faster response (as in overdamped system).

Fig. 2: Overdamped second order system

Step response of an underdamped second order system

In this case the poles $s_1, s_2 \in \mathbb{C}$

$$s_{1,2} = -\zeta\omega_\mathrm{n} \pm \mathrm{j}\omega_\mathrm{n}\sqrt{1-\zeta^2} = -\zeta\omega_\mathrm{n} \pm \mathrm{j}\omega_\mathrm{d},$$

where $\omega_\mathrm{d} := \omega_\mathrm{n}\sqrt{1-\zeta^2}$ is the damped natural frequency. The step response in this case is

$$y(t) = k_\mathrm{st}\left(1 - \frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\sin\left(\omega_\mathrm{d}t + \arccos\zeta\right)\right)\mathbb{1}(t) \tag{6}$$

We can notice that the response is composed of an exponential decay with $-\zeta\omega_\mathrm{n}$ and an oscillation with the frequency ω_d (and thus the period $2\pi/\omega_\mathrm{d}$).

Step response properties

1. Static gain: $G(0)$ is called the static gain of G. The static gain of the first and second order system is k_st.

2. Overshoot:

$$\mathrm{OS} = \frac{y_\mathrm{os}}{y_\mathrm{ss}} = \frac{y_\mathrm{max} - y_\mathrm{ss}}{y_\mathrm{ss}} = \mathrm{e}^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Sometimes it is useful to obtain the damping ratio from OS via the relation

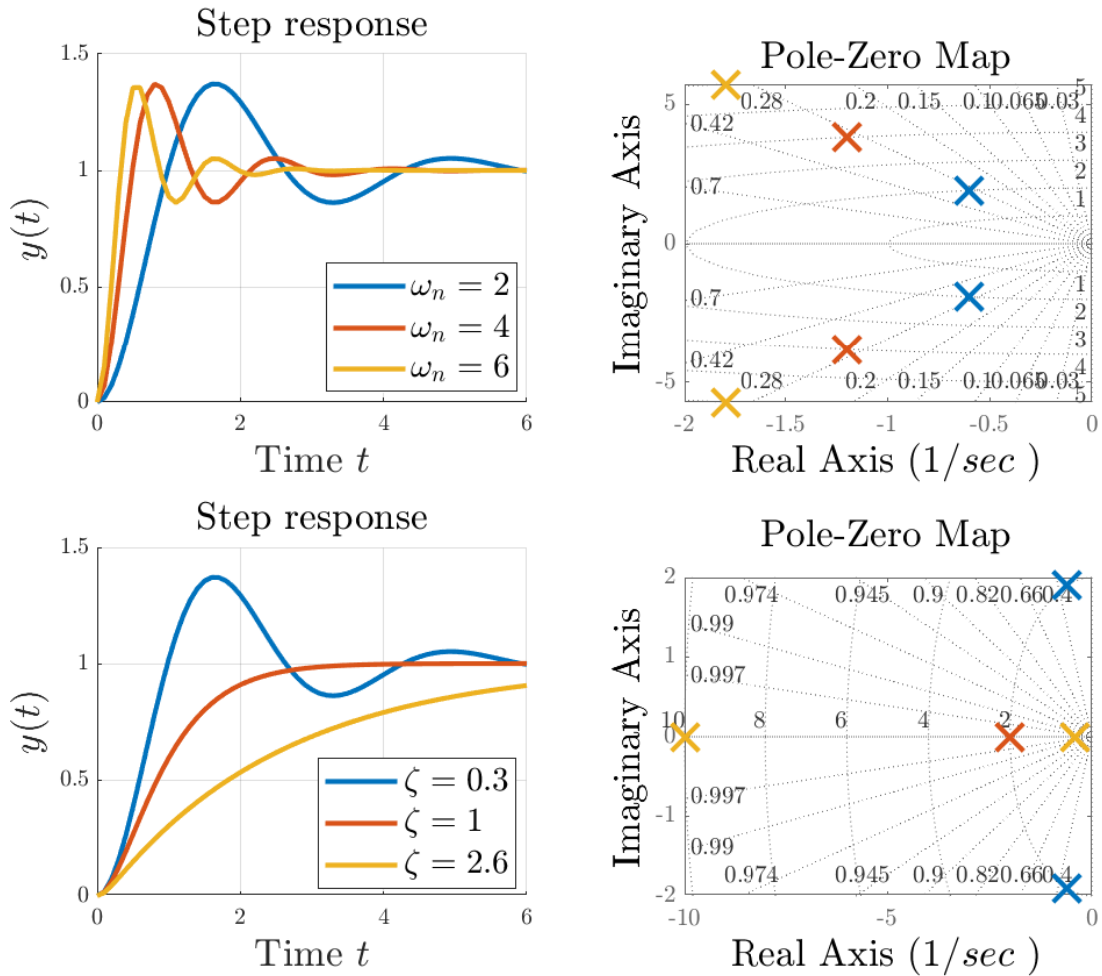$$\zeta = \frac{|\ln \mathrm{OS}|}{\sqrt{\pi^2 + \ln^2 \mathrm{OS}}} \tag{7}$$

Fig. 3: **Upper plots:** Underdamped second order system $\zeta = 0.3$. **Lower plots:** Underdamped ($\zeta = 0.3$), critically-damped ($\zeta = 1$), and overdamped ($\zeta = 2.6$) systems, where $\omega_n = 4$

3. Initial slope: We may use the initial value theorem to check the initial slope of first and second order systems

$$\lim_{t \to 0} \dot{y}(t) = \lim_{s \to \infty} s^2 Y(s) = \lim_{s \to \infty} s^2 G(s) \frac{1}{s} = \begin{cases} k_{\text{st}}/\tau & \text{1st order} \\ 0 & \text{2nd order} \end{cases}$$

4. Rise time: The time it takes the response to reach from 10% to 90% of its steady-state value. For an underdamped system an approximated formula is given by

$$t_{\text{r}} \approx \frac{1.6\zeta^3 - 0.17\zeta^2 + 0.92\zeta + 1.02}{\omega_{\text{n}}} \tag{8}$$

5. Peak time: The time it takes the response to reach the first peak, which is $t_{\text{p}} = \pi/\omega_{\text{d}}$.

6. Settling time: The time it takes the response to converge into an error sleeve around the steady state value. An upper bound on t_{s}, which is based on the exponential envelope

$$|y(t) - y_{\text{ss}}| \leq \frac{k_{\text{st}}}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_{\text{n}} t},$$

is

$$t_{\mathrm{s}} \leq -\frac{\ln \delta + \ln \sqrt{1 - \zeta^2}}{\zeta \omega_{\mathrm{n}}} \tag{9}$$

where δ is the desired error bound (relatively to y_{ss}).

2.4 Matlab commands

Some Matlab commands to create transfer functions and their step response:

- `G = tf(vN,vD);` generates a continuous-time real-rational transfer function $G(s)$ from two vectors, $\mathtt{vN} = \begin{bmatrix} b_m & \cdots & b_1 & b_0 \end{bmatrix}$ and $\mathtt{vD} = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}$.

  For example, `G = tf([-2 1],[1 2 3 1]);` generates $G(s) = \dfrac{-2s + 1}{s^3 + 2s^2 + 3s + 1}$.

- `G = tf(vN,vD,h);` generates a discrete-time real-rational transfer function $G(z)$ from two vectors, $\mathtt{vN} = \begin{bmatrix} b_m & \cdots & b_1 & b_0 \end{bmatrix}$ and $\mathtt{vD} = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}$ and the sampling period `h`.

  For example, `G = tf([1 -1],[1 -2 1],0.1);` generates $G(z) = \dfrac{z - 1}{z^2 - 2z + 1}$, where the discrete time instance i corresponds to $0.1i$ in the continuous time.

- `pzmap(G)` draws the complex plane with poles (denoted as "×") and zeros (denoted as "o") of $G(s)$ or $G(z)$ on it.

- `step(G)` draws the step response of a system with the transfer function $G(s)$ (or $G(z)$). The right-click on the plot offers several options, including showing various transients characteristics on it, like t_{p}, OS, t_{r}, t_{s}, as well as y_{ss}.

- `step(G,t)` draws the step response of a system with the transfer function $G(s)$ (or $G(z)$) at time instances defined by the vector `t`.

- `[y,t] = step(G);` returns a vector `y` containing the step response of a system with the transfer function $G(s)$ (or $G(z)$) and the vector `t` of time instances for which `y` was generated.

- `y = step(G,t);` returns a vector `y` containing the step response of a system with the transfer function $G(s)$ (or $G(z)$) at time instances defined by the vector `t`.

- `dcgain(G)` returns the static gain of G (i.e. $G(0)$ in the continuous time or $G(1)$ in the discrete time).

- `impulse(G)` draws the impulse response of a system with the transfer function $G(s)$ (or $G(z)$). Variants similar to those of the `step` command are available here as well.

- `t = linspace(t1,t2,N);` returns a row vector with `N` evenly spaced points in the interval $[\mathtt{t1}, \mathtt{t2}]$.

  For example, `t = linspace(1,3,5);` returns the row vector `[1 1.5 2 2.5 3]`.

3 Problems

Question 1. Consider the system $G_{RLC} : v_{\text{in}} \mapsto i_R$ shown in Fig. 4. In other words, the input is the applied
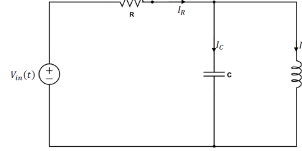

Fig. 4: RLC circuit

voltage v_{in} and the output is the resistor's current i_R. Here $R = 1$, $L = 1$, and $C = 1/8$ are constants, referred to as the resistance, inductance, and capacitance, respectively.

1. Write the transfer function $G_{RLC}(s)$ of the system.

2. Determine if the system is proper / strictly proper / bi-proper / non-proper.

3. Find the zeros and poles of $G_{RLC}(s)$ and associate them with parts of the complex plane $\mathbb{C}$.

4. What is the system's steady-state value for step input $v_{\text{in}} = 5 \cdot \mathbb{1}(t)$

5. Calculate the step response $v_{\text{in}} = \mathbb{1}$

Solution.

1. By Kirchhoff's voltage law,

$$v_{\text{in}}(t) = v_R(t) + v_L(t) = v_R(t) + v_C(t)$$

where v_R, v_L, and v_C are the voltage drops at the resistor, inductor, and capacitor, respectively. It is known that

$$v_R(t) = Ri_R(t), \quad v_L(t) = L\frac{di_L(t)}{dt}, \quad \text{and} \quad v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(s)ds.$$

Hence,

$$v_{\text{in}}(t) = Ri_R(t) + L\frac{di_L(t)}{dt}$$

and

$$v_{\text{in}}(t) = Ri_R(t) + \frac{1}{C}\int_{-\infty}^{t} i_C(s)ds \implies \dot{v}_{\text{in}}(t) = R\frac{di_R(t)}{dt} + \frac{1}{C}i_C(t)$$

In the Laplace domain, by the linearity and differentiation rules, we have that

$$V_{\text{in}}(s) = RI_R(s) + sLI_L(s) \implies I_L(s) = \frac{V_{\text{in}}(s) - RI_R(s)}{Ls}$$

$$sV_{\text{in}}(s) = RsI_R(s) + \frac{1}{C}I_C(s) \implies I_C(s) = Cs(V_{\text{in}}(s) - RI_R(s))$$

By Kirchhoff's current law,

$$I_R(s) = I_L(s) + I_C(s) = \frac{V_{\text{in}}(s) - RI_R(s)}{Ls} + Cs(V_{\text{in}}(s) - RI_R(s)),$$

from which

$$RLCs^2 I_R(s) + LsI_R(s) + RI_R(s) = LCs^2 V(s) + V(s).$$

Hence, the transfer function is

$$G_{RLC}(s) = \frac{I_R(s)}{V(s)} = \frac{LCs^2 + 1}{RLCs^2 + Ls + R}$$

Substituting $R = 1$, $L = 1$, and $C = 1/8$ we end up with

$$G_{RLC}(s) = \frac{(1/8)s^2 + 1}{(1/8)s^2 + s + 1} = \frac{s^2 + 8}{s^2 + 8s + 8}.$$

2. Because $n = m = 2$, the transfer function $G_{RLC}(s)$ is proper, not strictly proper, bi-proper.

3. The zeros, which are the roots of the numerator, are $z_{1,2} = \pm 2j\sqrt{2} \in j\mathbb{R}$ (the imaginary axis). The poles, which are the roots of the denominator, are $p_{1,2} = -4 \pm 2\sqrt{2} \in \mathbb{R}$ (the real axis).
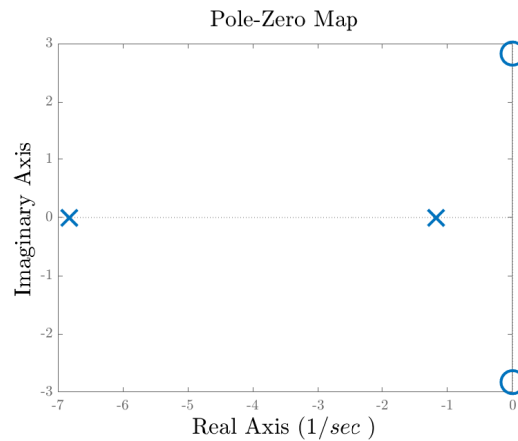

Fig. 5: Pole-zero map for Question 1

4. By the final value theorem,

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s)U(s) = \lim_{s \to 0} sG(s)\frac{5}{s} = 5G(0) = 5$$

5. There are two alternatives to this item.

(a) The step response is

$$I_{\text{step}}(s) = G_{RLC}(s)\frac{1}{s} = \frac{s^2 + 8}{s(s^2 + 8s + 8)} = \frac{\text{Res}(I_{\text{step}}, 0)}{s} + \frac{\text{Res}(I_{\text{step}}, p_1)}{s - p_1} + \frac{\text{Res}(I_{\text{step}}, p_2)}{s - p_2}$$

$$= \frac{1}{s} - \frac{\sqrt{2}}{s - (-4 + 2\sqrt{2})} + \frac{\sqrt{2}}{s - (-4 - 2\sqrt{2})}$$

where $p_1 = -4 + 2\sqrt{2}$ and $p_2 = -4 - 2\sqrt{2}$.

$$i_{\text{step}}(t) = \mathbb{1}(t) - \sqrt{2}e^{-(4 - 2\sqrt{2})t}\mathbb{1}(t) + \sqrt{2}e^{-(4 + 2\sqrt{2})t}\mathbb{1}(t).$$

(b) Note that $G_{RLC}(s) = G_0(s) + (1/8)s^2 G_0(s)$, where

$$G_0(s) = \frac{8}{s^2 + 8s + 8}.$$

Hence, the step response i_{step} of G_{RLC} can be obtained as a superposition of the step response y_0 of G_0 and its second derivative (this is the time-domain counterpart of multiplying by s^2 in the Laplace domain). $G_0(s)$ is the canonical second-order transfer function with $k_{\text{st}} = 1$, $\omega_n = \sqrt{8}$, and $\zeta = \sqrt{2}$ (for $2 \cdot \sqrt{2} \cdot \sqrt{8} = 8$). This is an overdamped system, whose step response is as in (4). Hence,

$$i_{\text{step}}(t) = y_0(t) + \frac{1}{8}\ddot{y}_0(t) = k_{\text{st}}\left(1 - \left(1 + \frac{s_1^2}{8}\right)\beta e^{s_1 t} + \left(1 + \frac{s_2^2}{8}\right)(\beta - 1)e^{s_2 t}\right)\mathbb{1}(t)$$

$$= \mathbb{1}(t) - \left(1 + \frac{(-4 + 2\sqrt{2})^2}{8}\right)\frac{1 + \sqrt{2}}{2}e^{-(4 - 2\sqrt{2})t}\mathbb{1}(t)$$

$$+ \left(1 + \frac{(-4 - 2\sqrt{2})^2}{8}\right)\left(\frac{1 + \sqrt{2}}{2} - 1\right)e^{-(4 + 2\sqrt{2})t}\mathbb{1}(t)$$

$$= \mathbb{1}(t) - \sqrt{2}e^{-(4 - 2\sqrt{2})t}\mathbb{1}(t) + \sqrt{2}e^{-(4 + 2\sqrt{2})t}\mathbb{1}(t),$$

which coincides with what we had with the first approach.

That's all . . . $\triangledown$

Question 2. Consider the system $G_T : q_{\text{in}} \mapsto h$ shown in Fig. 6. Its input is the volumetric flow q_{in} to a
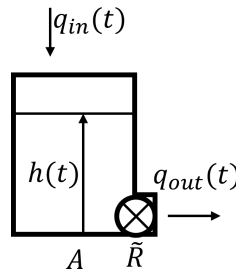

Fig. 6: Tank system

tank with cross-section A and the output $h(t)$ is the liquid level in the tank. Assume that $q_{\text{out}}(t) = h(t)/R$, where q_{out} is the outlet volumetric flow and R is the flow resistance.

1. Derive the transfer function $G_T(s)$ of the system and determine if it is proper / strictly proper/ bi-proper / non-proper.

2. Find the zeros and poles of $G_T(s)$ and associate them with specific parts of the complex plane $\mathbb{C}$.

3. Calculate and plot the response for step input $q_{\text{in}} = \mathbb{1}(t)$. Find the steady-state value, initial slope, time it takes to reach $\approx 63\%$ and $\approx 99\%$ from its steady state value. Assume that $R = 2$ and $A = 3$.

Solution.

1. The tank volume is $v(t) = Ah(t)$, so the rate of its change

$$\dot{v}(t) = A\dot{h}(t) = q_{\text{in}}(t) - q_{\text{out}}(t) = q_{\text{in}}(t) - \frac{1}{R}h(t).$$

In the Laplace domain,
$$sAH(s) = Q_{in}(s) - H(s)/R$$

from which
$$G_T(s) = \frac{H(s)}{Q_{in}(s)} = \frac{R}{RAs + 1} = \frac{k_{st}}{\tau s + 1}$$

for $k_{st} = R$ and $\tau = RA$. This transfer function is strictly proper ($1 = n > m = 0$).

2. The single pole is $p = -1/\tau \in \mathbb{R}$ (the real axis), see Fig. 7(a).
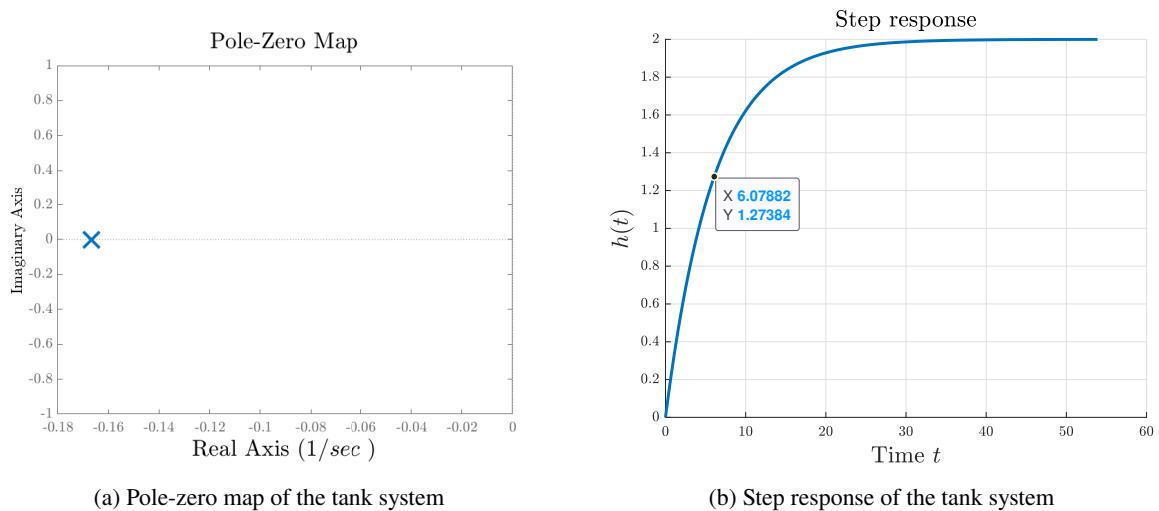

(a) Pole-zero map of the tank system

(b) Step response of the tank system

Fig. 7: Plots for Question 2

3. By (2), the solution is

$$h_{step}(t) = k_{st}\left(1 - e^{-t/\tau}\right) = R\left(1 - e^{-t/(RA)}\right)\mathbb{1}(t)$$

shown in Fig. 7(b). We know that the steady-state value is $G(0) = R = 2$ (can also be derived by the final value theorem). The system reaches $\approx 63\%$ its steady state value in τ seconds, and reaches $\approx 99\%$ its steady state value in 5τ seconds. Also, we notice that the initial slope is not zero, as expected in first order system.

That's all ...   ▽

Question 3. Consider the system $G_R : \tau \mapsto \theta$ shown in Fig. 8. The mass, whose moment of inertia, J,
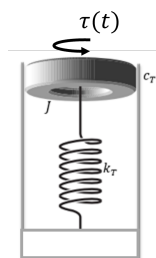

Fig. 8: Rotational mass-spring-damper system

is attached to a torsion spring, whose torsion coefficient is k_T. An external torque τ acts on the mass and friction between the mass and the cylinder is assumed to generate a viscous friction torque $\tau_c = -c_T\dot{\theta}$.

1. Derive the transfer function $G_R(s)$ of the system.

2. Determine if the system is strictly proper / bi-proper/ non-proper

3. Find the zeros and poles of $G_R(s)$ and place them in the $\mathbb{C}$ plane.

4. Find k_{st}, ω_n, ζ, ω_d, t_r, t_p, OS, and t_s for $\delta = 0.05$ assuming that $J = 2$, $c_T = 3$, and $k_T = 10$.

Solution. The Newtonian equation of motion of the mass is $J\ddot{\theta} = \tau_{\text{net}}$, where τ_{net} is the net torque applied to it. In our case,

$$\tau_{\text{net}} = \tau - k_T\theta - c_T\dot{\theta}.$$

Hence, we the relation describing the system $\tau \mapsto \theta$.

$$J\ddot{\theta}(t) + c_T\dot{\theta}(t) + k_T\theta(t) = \tau(t) \implies (Js^2 + c_Ts + k_T)\Theta(s) = T(s)$$

Thus, the transfer function

$$G_R(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + c_Ts + k_T} = \frac{1/J}{s^2 + (c_T/J)s + k_T/J} = \frac{k_{\text{st}}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(it is obviously a strictly proper second-order system). Assuming that $J = 2$, $c_T = 3$, and $k_T = 10$, the static gain

$$k_{\text{st}} = G_R(0) = \frac{1}{k_T} = 0.1,$$

the natural frequency

$$\omega_n^2 = \frac{k_T}{J} \implies \omega_n = \sqrt{\frac{k_T}{J}} = \sqrt{5} = 2.23,$$

and the damping ratio satisfies

$$2\zeta\omega_n = \frac{c_T}{J} \implies \zeta = \frac{c_T}{2\omega_n J} \approx 0.33 < 1.$$

Hence the system is underdamped and the damped natural frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 2.1$$

The map of poles and zeros and the step response, calculated by (6), are shown in Fig. 9.

From the plot in Fig. 9(b) we can calculate characteristics of the transient response,

$$t_r = 0.825 - 0.213 = 0.612, \quad t_p = 1.491, \quad OS = \frac{0.133 - 0.1}{0.1} \cdot 100\% = 33\%, \quad \text{and} \quad t_s = 3.542.$$

Note that the approximate rise time in (8) is $t_r \approx 0.612$, which is quite accurate, within a per mille. The bound in (9) yields $t_s \leq 4.074$, which is about 15% larger. ∇
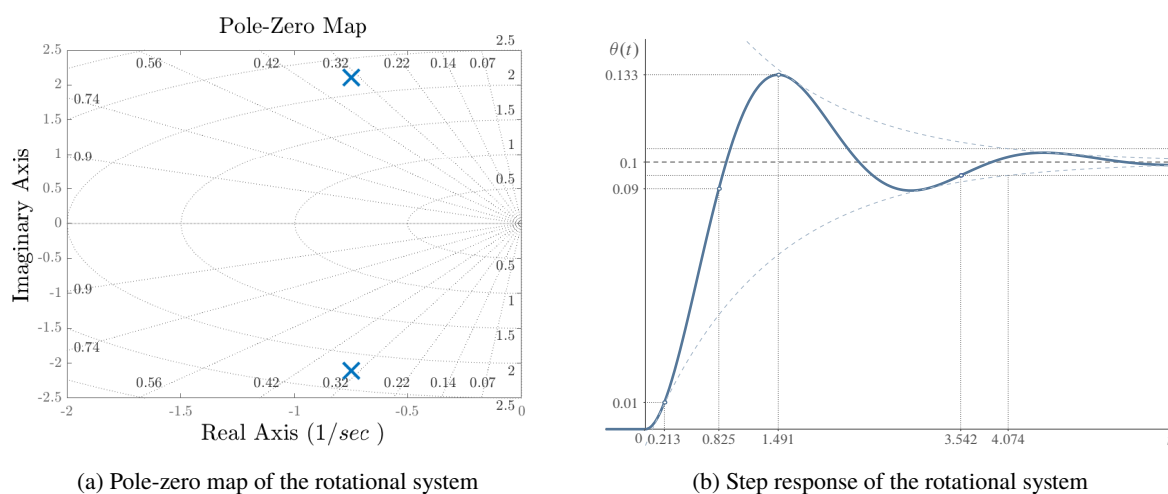
(a) Pole-zero map of the rotational system

(b) Step response of the rotational system
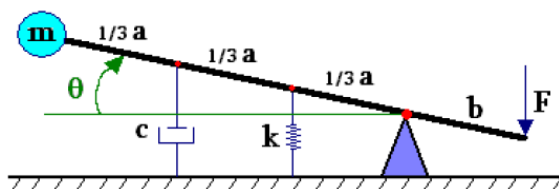
Fig. 9: Plots for Question 3

Fig. 10: Swing system

4 Homework problems

Question 4. Consider the system $G_{sw} : F \mapsto \theta$ shown in Fig. 10. A force f is applied on a massless rod, and mass m is attached at the tip. The rod is attached to the ground via a spring with the stiffness k and a viscous damper with the coefficient c. Assume throughout that $m = 1$, $a = 1$, and $b = 1/3$.

1. Derive the transfer function $G_{sw}(s)$ of the following system and determine if the system is proper / strictly proper / bi-proper / non-proper. Assume that deviations of the swing rotates from $\theta = 0$ are small.

2. Find the static gain k_{st}, the natural frequency ω_n, the steady state y_{ss}, and expression to damping ratio ζ under the spring stiffness $k = 9$.

3. Given $c = 3$, determine whether the system is overdamped, underdamped, or critically damped.

4. Given that $c = 2.25$ find the damped frequency ω_d and peak time t_p.

5. Find the range of c where overshoot $OS \leq 20\%$

6. Find the range of c in which the response has no overshoot.

Solution.

1. The equation of motion is
$$J\ddot{\theta}(t) = \sum M(t).$$

The moment of inertial of this system $J = ma^2$ and the net torque

$$\sum M(t) = -\frac{2}{3}a \cdot \frac{2}{3}ac\dot{\theta}(t) - \frac{1}{3}a \cdot \frac{1}{3}ak\theta(t) + bf(t).$$

Hence,

$$ma^2\ddot{\theta}(t) = -\frac{2}{3}a \cdot \frac{2}{3}ac\dot{\theta}(t) - \frac{1}{3}a \cdot \frac{1}{3}ak\theta(t) + bf(t)$$

or, with the numerical values of m, a, and b,

$$\ddot{\theta}(t) + \frac{4}{9}c\dot{\theta}(t) + \frac{1}{9}k\theta(t) = \frac{1}{3}f(t)$$

In the Laplace domain this equation reads

$$\Theta(s) = \frac{1/3}{s^2 + (4/9)cs + (1/9)k}F(s) \implies G_{sw}(s) = \frac{\Theta(s)}{F(s)} = \frac{1/3}{s^2 + (4/9)cs + (1/9)k}$$

It is strictly proper.

2. If $k = 9$, then

$$G_{sw}(s) = \frac{1/3}{s^2 + (4/9)cs + 1}$$

Comparing this $G_{sw}(s)$ with the canonical form (3), the following relations can be established:

$$k_{st} = G_{sw}(0) = \frac{1/3}{1} = \frac{1}{3}, \quad \omega_n = 1, \quad \text{and} \quad 2\zeta\omega_n = \frac{4}{9}c \implies \zeta = \frac{2c}{9}$$

3. If $c = 3$, then

$$\zeta = \frac{2c}{9} = \frac{2}{3} < 1$$

so the system is underdamped.

4. If $c = 2.25$, then $\zeta = 0.5$, so that

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \frac{\sqrt{3}}{2} \quad \text{and} \quad t_p = \pi/\omega_d.$$

5. We know that the overshoot grows if the damping ratio reduces. Hence, (7) yields that to have OS ≤ 0.2 we need

$$\zeta \geq \frac{|\ln 0.2|}{\sqrt{\pi^2 + \ln^2 0.2}} = \zeta_{min} = 0.456$$

Because $\zeta = 2c/9$, the damping coefficient should satisfy

$$c \geq \frac{9}{2}\zeta_{min} = 2.052$$

6. Underdamped second-order systems always have a nonzero overshoot, whereas under- and critically damped systems have zero overshoot. This, all we need is to have $\zeta \geq 1$, which is equivalent to the condition $c \geq 4.5$.

That's all . . .   ▽

Question 5. Consider the following discrete-time model of a system $G : u \mapsto y$.

$$y[k + 2] = 1.2y[k + 1] - 0.2y[k] + 0.8u[k].$$

1. Write the transfer function $G(z)$ of the following system.

2. Determine if the system is strictly proper/ bi-proper/ non-proper

3. Find the zeros and poles of $G(z)$ and place them in the $\mathbb{C}$ plane.

4. Find its step response using z-transform

Solution. Using z-transform and its shift property (i.e. $(\mathfrak{Z}\{\mathbb{S}_\tau x\})(z) = z^\tau X(z)$), we have that

$$(z^2 - 1.2z + 0.2)Y(z) = 0.8U(z) \quad \Longrightarrow \quad G(z) = \frac{0.8}{z^2 - 1.2z + 0.2} = \frac{0.8}{(z - 0.2)(z - 1)}.$$

It is strictly proper, it has no zeros and two poles, at $z = 0.2$ and $z = 1$ (both are real).

The time-domain $u = \mathbb{1}$ corresponds to $U(z) = z/(z - 1)$. Hence, the step response in the z domain

$$Y(z) = \frac{0.8}{z^2 - 1.2z + 0.2} \frac{z}{z - 1} = \frac{0.8}{(z - 1)^2(z - 0.2)} = \frac{0.25}{z - 0.2} - \frac{0.25}{z - 1} + \frac{1}{(z - 1)^2}$$

We know the inverse z-transforms of functions $z/(z - \lambda)$. Hence, introduce

$$Y_1(z) = zY(z) = -\frac{0.25z}{z - 1} + \frac{0.25z}{z - 0.2} + \frac{z}{(z - 1)^2}$$

Using inverse z-transform
$$y_1[t] = -0.25\mathbb{1}[t] + 0.25 \cdot 0.2^t \mathbb{1}[t] + t \mathbb{1}[t],$$

so that
$$y[t] = y_1[t - 1] = -0.25\mathbb{1}[t - 1] + 0.25 \cdot 0.2^{t-1}\mathbb{1}[t - 1] + (t - 1)\mathbb{1}[t - 1],$$

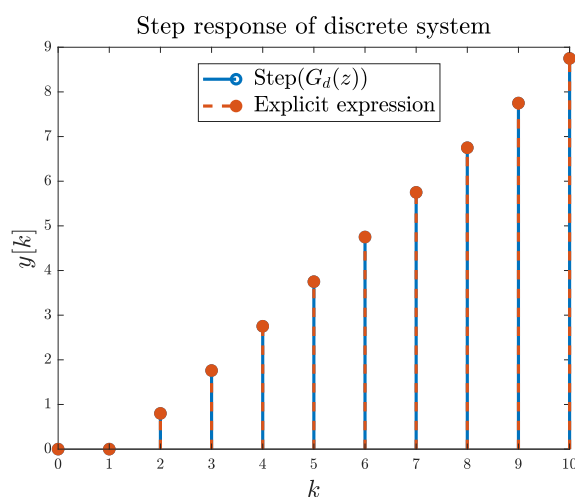see the blue plot in Fig. 11. The same result (the red plot in Fig. 11) is obtained with the direct use of

Fig. 11: Discrete system

Matlab's commands `G = tf([0.8],[1,-1.2,0.2],1);` and `step(G)`. ▽