



## LINEAR SYSTEMS (034032)

### TUTORIAL 7

## 1 Topics

Transfer functions, zeros and poles, 1st order and 2nd order systems, step response.

## 2 Background

### 2.1 Rational transfer functions

An LTI system  $G : u \mapsto y$  can be represented by its transfer function  $G(s)$ . In some important cases (viz. systems described by ODEs) transfer functions are of the form of a quotient of two polynomials, like

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

for some  $n, m \in \mathbb{Z}_+$  and real coefficients  $a_i$  and  $b_i$ . Such transfer functions are said to be *rational*. Their poles are the roots of the denominator polynomial (assuming that the numerator and denominator polynomials are coprime, i.e. have no common roots). The roots of the numerator are called zeros of  $G(s)$ . The system is said to be proper if  $n \geq m$ , strictly proper if  $n > m$ , bi-proper if  $n = m$ , non-proper if  $n < m$ .

### 2.2 First order system

#### Canonical form of first order system

The transfer function of a general first-order system takes the form

$$G(s) = \frac{k_{st}}{\tau s + 1} \quad (1)$$

where  $k_{st}$  is the static gain and  $\tau$  is the time constant. The single pole of the system is  $p = -1/\tau \in \mathbb{R}_-$ .

#### Step response of first order system

The step response of a first order system is

$$y_{step}(t) = k_{st}(1 - e^{-t/\tau})\mathbb{1}(t) \quad (2)$$

The static gain  $k_{st}$  scales the response amplitude. When  $t = \tau$  and  $t = 3\tau$

$$y_{step}(\tau) = k_{st}(1 - e^{-1}) \approx 0.63k_{st} \quad \text{and} \quad y_{step}(3\tau) = k_{st}(1 - e^{-3}) \approx 0.95k_{st},$$

respectively. The time constant  $\tau$  dictates the responsiveness of the system. For the effect of changing  $k_{st}$  and  $\tau$  see Fig. 1.

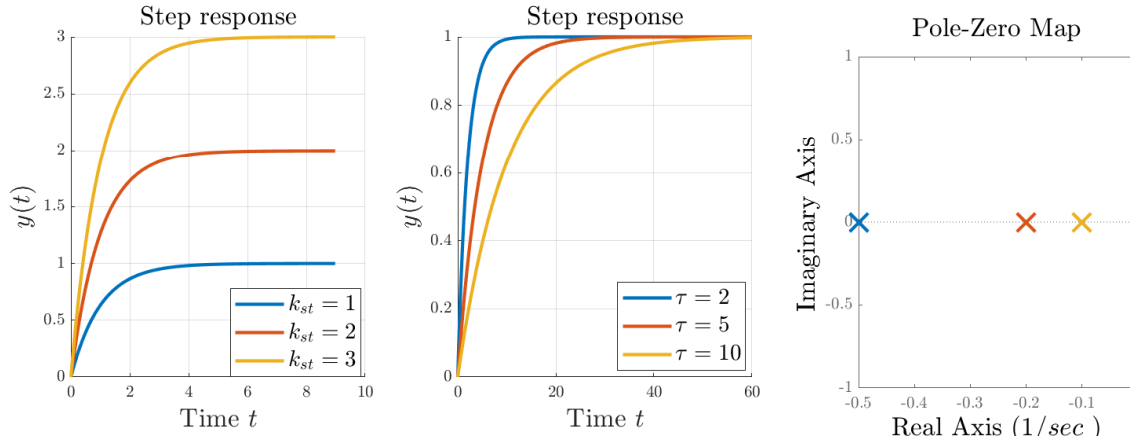


Fig. 1: First order system

## 2.3 Second order system

### Canonical form of second order system

The transfer function of a general second-order system takes the form

$$G(s) = \frac{k_{st}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (3)$$

where  $k_{st}$  is the static gain,  $\zeta$  is the damping ratio, and  $\omega_n$  is the natural frequency. The poles of the system are

$$s_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$$

and three cases shall be distinguished:

1.  $\zeta > 1$ : the system is called *overdamped*, the poles  $s_{1,2} \in \mathbb{R}$  and such that  $s_1 \neq s_2$
2.  $\zeta = 1$ : the system is called *critically damped* and the poles  $s_{1,2} \in \mathbb{R}$ , with  $s_1 = s_2$
3.  $0 \leq \zeta < 1$ : the system is called *underdamped* and the poles  $s_{1,2} \in \mathbb{C}$  and  $s_1 = \bar{s}_2$

### Step response of an overdamped second order system

The step response in this case is

$$y(t) = k_{st}(1 - \beta e^{s_1 t} + (\beta - 1)e^{s_2 t})\mathbb{1}(t), \quad \text{where } \beta = \frac{1}{2} \left( \frac{\zeta}{\sqrt{\zeta^2 - 1}} + 1 \right) > 1 \quad (4)$$

Increasing  $\omega_n$  will cause a faster response. Increasing  $\zeta$  will cause a slower response, see Fig. 2.

### Step response of a critically damped second order system

In this case the poles  $s_{1,2} = -\omega_n$  and the step response is

$$y(t) = k_{st}(1 - (1 + \omega_n t)e^{-\omega_n t})\mathbb{1}(t) \quad (5)$$

Increasing  $\omega_n$  will cause a faster response (as in overdamped system).

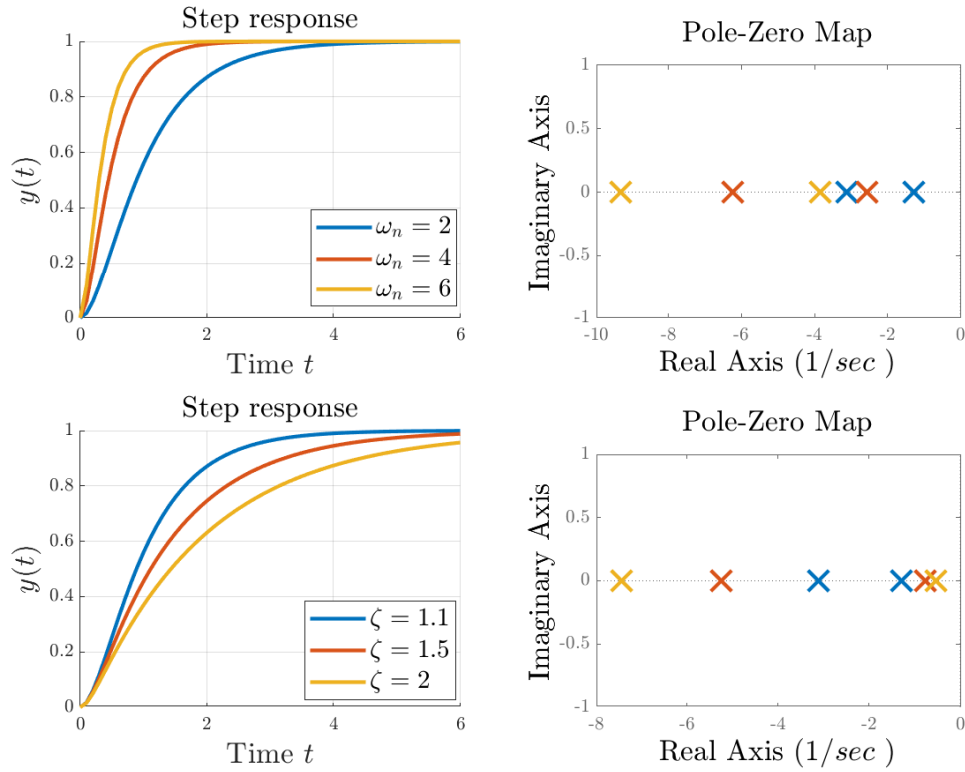


Fig. 2: Overdamped second order system

### Step response of an underdamped second order system

In this case the poles  $s_1, s_2 \in \mathbb{C}$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d,$$

where  $\omega_d := \omega_n\sqrt{1-\zeta^2}$  is the damped natural frequency. The step response in this case is

$$y(t) = k_{st} \left( 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \arccos \zeta) \right) \mathbb{1}(t) \quad (6)$$

We can notice that the response is composed of an exponential decay with  $-\zeta\omega_n$  and an oscillation with the frequency  $\omega_d$  (and thus the period  $2\pi/\omega_d$ ).

### Step response properties

1. Static gain:  $G(0)$  is called the static gain of  $G$ . The static gain of the first and second order system is  $k_{st}$ .
2. Overshoot:

$$\text{OS} = \frac{y_{os}}{y_{ss}} = \frac{y_{\max} - y_{ss}}{y_{ss}} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Sometimes it is useful to obtain the damping ratio from OS via the relation

$$\zeta = \frac{|\ln \text{OS}|}{\sqrt{\pi^2 + \ln^2 \text{OS}}} \quad (7)$$

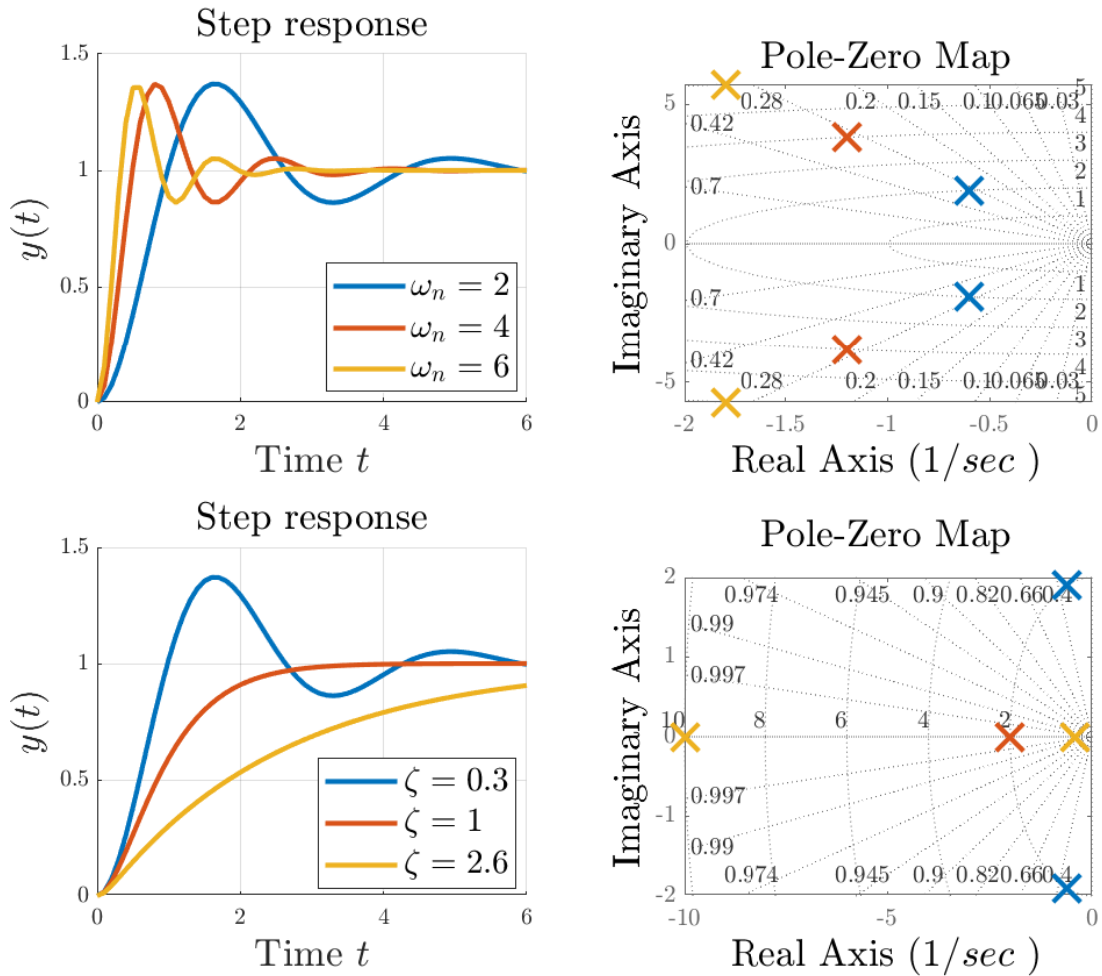


Fig. 3: **Upper plots:** Underdamped second order system  $\zeta = 0.3$ . **Lower plots:** Underdamped ( $\zeta = 0.3$ ), critically-damped ( $\zeta = 1$ ), and overdamped ( $\zeta = 2.6$ ) systems, where  $\omega_n = 4$

3. Initial slope: We may use the initial value theorem to check the initial slope of first and second order systems

$$\lim_{t \rightarrow 0} \dot{y}(t) = \lim_{s \rightarrow \infty} s^2 Y(s) = \lim_{s \rightarrow \infty} s^2 G(s) \frac{1}{s} = \begin{cases} k_{st}/\tau & \text{1st order} \\ 0 & \text{2nd order} \end{cases}$$

4. Rise time: The time it takes the response to reach from 10% to 90% of its steady-state value. For an underdamped system an approximated formula is given by

$$t_r \approx \frac{1.6\zeta^3 - 0.17\zeta^2 + 0.92\zeta + 1.02}{\omega_n} \quad (8)$$

5. Peak time: The time it takes the response to reach the first peak, which is  $t_p = \pi/\omega_d$ .
6. Settling time: The time it takes the response to converge into an error sleeve around the steady state value. An upper bound on  $t_s$ , which is based on the exponential envelope

$$|y(t) - y_{ss}| \leq \frac{k_{st}}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t},$$

is

$$t_s \leq -\frac{\ln \delta + \ln \sqrt{1 - \zeta^2}}{\zeta \omega_n} \quad (9)$$

where  $\delta$  is the desired error bound (relatively to  $y_{ss}$ ).

## 2.4 Matlab commands

Some Matlab commands to create transfer functions and their step response:

- $G = \text{tf}(vN, vD)$ ; generates a continuous-time real-rational transfer function  $G(s)$  from two vectors,  $vN = [b_m \cdots b_1 b_0]$  and  $vD = [1 a_{n-1} \cdots a_1 a_0]$ .  
For example,  $G = \text{tf}([-2 \ 1], [1 \ 2 \ 3 \ 1])$ ; generates  $G(s) = \frac{-2s + 1}{s^3 + 2s^2 + 3s + 1}$ .
- $G = \text{tf}(vN, vD, h)$ ; generates a discrete-time real-rational transfer function  $G(z)$  from two vectors,  $vN = [b_m \cdots b_1 b_0]$  and  $vD = [1 a_{n-1} \cdots a_1 a_0]$  and the sampling period  $h$ .  
For example,  $G = \text{tf}([1 \ -1], [1 \ -2 \ 1], 0.1)$ ; generates  $G(z) = \frac{z - 1}{z^2 - 2z + 1}$ , where the discrete time instance  $i$  corresponds to  $0.1i$  in the continuous time.
- $\text{pzmap}(G)$  draws the complex plane with poles (denoted as “x”) and zeros (denoted as “o”) of  $G(s)$  or  $G(z)$  on it.
- $\text{step}(G)$  draws the step response of a system with the transfer function  $G(s)$  (or  $G(z)$ ). The right-click on the plot offers several options, including showing various transients characteristics on it, like  $t_p$ , OS,  $t_r$ ,  $t_s$ , as well as  $y_{ss}$ .
- $\text{step}(G, t)$  draws the step response of a system with the transfer function  $G(s)$  (or  $G(z)$ ) at time instances defined by the vector  $t$ .
- $[y, t] = \text{step}(G)$ ; returns a vector  $y$  containing the step response of a system with the transfer function  $G(s)$  (or  $G(z)$ ) and the vector  $t$  of time instances for which  $y$  was generated.
- $y = \text{step}(G, t)$ ; returns a vector  $y$  containing the step response of a system with the transfer function  $G(s)$  (or  $G(z)$ ) at time instances defined by the vector  $t$ .
- $\text{dcgain}(G)$  returns the static gain of  $G$  (i.e.  $G(0)$  in the continuous time or  $G(1)$  in the discrete time).
- $\text{impz}(G)$  draws the impulse response of a system with the transfer function  $G(s)$  (or  $G(z)$ ). Variants similar to those of the `step` command are available here as well.
- $t = \text{linspace}(t1, t2, N)$ ; returns a row vector with  $N$  evenly spaced points in the interval  $[t1, t2]$ .  
For example,  $t = \text{linspace}(1, 3, 5)$ ; returns the row vector  $[1 \ 1.5 \ 2 \ 2.5 \ 3]$ .

### 3 Problems

**Question 1.** Consider the system  $G_{RLC} : v_{in} \mapsto i_R$  shown in Fig. 4. In other words, the input is the applied

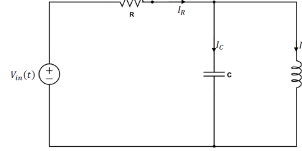


Fig. 4:  $RLC$  circuit

voltage  $v_{in}$  and the output is the resistor's current  $i_R$ . Here  $R = 1$ ,  $L = 1$ , and  $C = 1/8$  are constants, referred to as the resistance, inductance, and capacitance, respectively.

1. Write the transfer function  $G_{RLC}(s)$  of the system.
2. Determine if the system is proper / strictly proper / bi-proper / non-proper.
3. Find the zeros and poles of  $G_{RLC}(s)$  and associate them with parts of the complex plane  $\mathbb{C}$ .
4. What is the system's steady-state value for step input  $v_{in} = 5 \cdot \mathbb{1}(t)$
5. Calculate the step response  $v_{in} = \mathbb{1}$

**Question 2.** Consider the system  $G_T : q_{in} \mapsto h$  shown in Fig. 5. Its input is the volumetric flow  $q_{in}$  to a

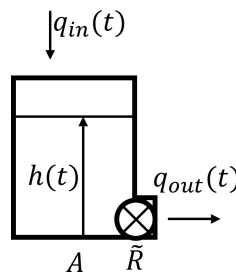


Fig. 5: Tank system

tank with cross-section  $A$  and the output  $h(t)$  is the liquid level in the tank. Assume that  $q_{out}(t) = h(t)/R$ , where  $q_{out}$  is the outlet volumetric flow and  $R$  is the flow resistance.

1. Derive the transfer function  $G_T(s)$  of the system and determine if it is proper / strictly proper/ bi-proper / non-proper.
2. Find the zeros and poles of  $G_T(s)$  and associate them with specific parts of the complex plane  $\mathbb{C}$ .
3. Calculate and plot the response for step input  $q_{in} = \mathbb{1}(t)$ . Find the steady-state value, initial slope, time it takes to reach  $\approx 63\%$  and  $\approx 99\%$  from its steady state value. Assume that  $R = 2$  and  $A = 3$ .

**Question 3.** Consider the system  $G_R : \tau \mapsto \theta$  shown in Fig. 6. The mass, whose moment of inertia,  $J$ , is attached to a torsion spring, whose torsion coefficient is  $k_T$ . An external torque  $\tau$  acts on the mass and friction between the mass and the cylinder is assumed to generate a viscous friction torque  $\tau_c = -c_T \dot{\theta}$ .

1. Derive the transfer function  $G_R(s)$  of the system.

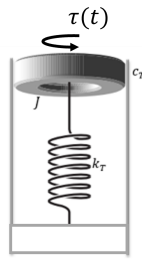


Fig. 6: Rotational mass-spring-damper system

2. Determine if the system is strictly proper / bi-proper/ non-proper
3. Find the zeros and poles of  $G_R(s)$  and place them in the  $\mathbb{C}$  plane.
4. Find  $k_{st}$ ,  $\omega_n$ ,  $\zeta$ ,  $\omega_d$ ,  $t_r$ ,  $t_p$ , OS, and  $t_s$  for  $\delta = 0.05$  assuming that  $J = 2$ ,  $c_T = 3$ , and  $k_T = 10$ .

## 4 Homework problems

**Question 4.** Consider the system  $G_{sw} : F \mapsto \theta$  shown in Fig. 7. A force  $f$  is applied on a massless rod,

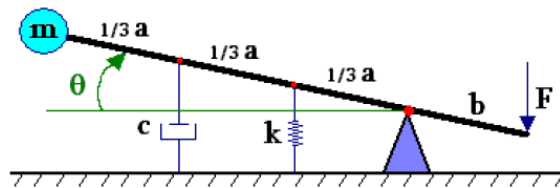


Fig. 7: Swing system

and mass  $m$  is attached at the tip. The rod is attached to the ground via a spring with the stiffness  $k$  and a viscous damper with the coefficient  $c$ . Assume throughout that  $m = 1$ ,  $a = 1$ , and  $b = 1/3$ .

1. Derive the transfer function  $G_{sw}(s)$  of the following system and determine if the system is proper / strictly proper / bi-proper / non-proper. Assume that deviations of the swing rotates from  $\theta = 0$  are small.
2. Find the static gain  $k_{st}$ , the natural frequency  $\omega_n$ , the steady state  $y_{ss}$ , and expression to damping ratio  $\zeta$  under the spring stiffness  $k = 9$ .
3. Given  $c = 3$ , determine whether the system is overdamped, underdamped, or critically damped.
4. Given that  $c = 2.25$  find the damped frequency  $\omega_d$  and peak time  $t_p$ .
5. Find the range of  $c$  where overshoot  $OS \leq 20\%$
6. Find the range of  $c$  in which the response has no overshoot.

**Question 5.** Consider the following discrete-time model of a system  $G : u \mapsto y$ .

$$y[k + 2] = 1.2y[k + 1] - 0.2y[k] + 0.8u[k].$$

1. Write the transfer function  $G(z)$  of the following system.
2. Determine if the system is strictly proper/ bi-proper/ non-proper
3. Find the zeros and poles of  $G(z)$  and place them in the  $\mathbb{C}$  plane.
4. Find its step response using  $z$ -transform