Linear Systems (034032)

tutorial 6

1 Topics

Systems: basic definitions (linearity, time invariance, causality), convolution representation, and transfer functions.

2 Background

Basic definitions (linearity, time invariance, causality, I/O stability). Convolution representation.

2.1 Linearity

A system $G : u \mapsto y$ is said to be linear if

$$G(a_1 u_1 + a_2 u_2) = a_1(G u_1) + a_2(G u_2)$$

for all admissible inputs u_1 and u_2 and all scalars a_1 and a_2.

2.2 Time (shift) invariance

A system $G : u \mapsto y$ is said to be time-invariant if

$$G(\mathbb{S}_\tau u) = \mathbb{S}_\tau (G u)$$

for all admissible inputs u and all $\tau \in \mathbb{R}$ (or $\tau \in \mathbb{Z}$ in the discrete-time case). In other words, every time shift of the input results in the same time shift of the output.

2.3 Causality

A system $G : u \mapsto y$ is said to be causal if the signal y at every time instance t_c can only depend on u at $t \le t_c$ and not at $t > t_c$.

2.4 Impulse response

If $G : u \mapsto y$ is LTI (linear time invariant), then

$$y(t) = \int_{\mathbb{R}} g(t - s) u(s) \mathrm{d}s,$$

where g is the response of G to the Dirac delta δ, i.e. the response of an LTI system to an input signal u amounts to convolving that input with the impulse response g of the system.

2.5 BIBO stability via impulse response

An LTI system G with impulse response g is BIBO stable iff $g \in L_1$ in the continuous-time case or $g \in \ell_1$ in the discrete-time case.

3 Problems

Question 1. Classify the models $G : u \mapsto y$ below in terms of linearity, time invariance, and (except the systems in items 1 and 2) causality.

1. $y(t) = t\dot{u}(t)$
2. $y(t) = u(t)\dot{u}(t)$
3. $y(t) = u(t+1)$
4. $y(t) = 2u(t) + 1$
5. $y(t) = \begin{cases} u(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Solution.

1. $y(t) = t\dot{u}(t)$

   LINEARITY: if $u = a_1u_1 + a_2u_2$, then

   $$y(t) = t\dot{u}(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\Big(a_1u_1(t) + a_2u_2(t)\Big) = a_1t\dot{u}_1(t) + a_2t\dot{u}_2(t) = a_1y_1(t) + a_2y_2(t)$$

   hence it's linear.

   TIME-INVARIANCE:

   $$(\mathbb{S}_\tau y)(t) = y(t+\tau) = (t+\tau)\dot{u}(t+\tau) \neq t\dot{u}(t+\tau) = \big(G(\mathbb{S}_\tau u)\big)(t).$$

   Hence, this system is *not* time-invariant.

2. $y(t) = u(t)\dot{u}(t)$

   LINEARITY: if $u = a_1u_1 + a_2u_2$, then

   $$y(t) = u(t)\dot{u}(t) = (a_1u_1(t) + a_2u_2(t))(a_1\dot{u}_1(t) + a_2\dot{u}_2(t)) \neq a_1u_1(t)\dot{u}_1(t) + a_2u_2(t)\dot{u}_2(t)$$

   Hence, this system is not linear.

   TIME-INVARIANCE:

   $$(\mathbb{S}_\tau y)(t) = y(t+\tau) = u(t+\tau)\dot{u}(t+\tau) = \big(G(\mathbb{S}_\tau u)\big)(t).$$

   Hence, the system is time-invariant.

3. $y(t) = u(t+1)$

   LINEARITY: if $u = a_1u_1 + a_2u_2$, then

   $$y(t) = a_1u_1(t+1) + a_2u_2(t+1) = a_1y_1(t) + a_2y_2(t)$$

   Hence, the system is linear.

   TIME-INVARIANCE:

   $$(\mathbb{S}_\tau y)(t) = y(t+\tau) = u(t+1+\tau) = \big(G(\mathbb{S}_\tau u)\big)(t).$$

   Hence, the system is time-invariant.

   CAUSALITY: The signal at time instance t_c depends on u at $t_\mathrm{c} + 1 > t_c$. Hence, the system is not causal.

4. $y(t) = 2u(t) + 1$

   Linearity: if $u = a_1u_1 + a_2u_2$, then

$$y(t) = 2(a_1u_1(t) + a_2u_2(t)) + 1 \neq a_1(2u_1(t) + 1) + a_2(2u_2(t) + 1)$$

   Hence the system is *not* linear (such systems are called *affine*).

   The system is time-invariant and causal (because it is static).

5. $y(t) = \begin{cases} u(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$

   Linearity: if $u = a_1u_1 + a_2u_2$, then

$$\begin{aligned} y(t) &= (Gu)(t) = \big(G(a_1u_1 + a_2u_2)\big)(t) \\ &= \begin{cases} (a_1u_1 + a_2u_2)(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases} = a_1 \begin{cases} u_1(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases} + a_2 \begin{cases} u_2(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases} \\ &= a_1(Gu_1)(t) + a_2(Gu_2)(t) \end{aligned}$$

   Hence, the system is linear.

   Time-invariance: the shifted output is

$$\big(\mathbb{S}_{-\tau}(Gu)\big)(t) = \mathbb{S}_{\tau}\left(\begin{cases} u(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}\right) = \begin{cases} u(t+\tau) & \text{if } 0 < t + \tau \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   The response to the time-shift input signal is

$$\big(G(\mathbb{S}_{\tau}u)\big)(t) = \begin{cases} u(t+\tau) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   We can see that $\mathbb{S}_{-\tau}(Gu) \neq G(\mathbb{S}_{\tau}u)$. Hence, the system is *not* time invariant.

   Causality: At every time instance $t = t_c$ the output is either 0 or $u(t_c)$. It therefore does not depend on the input signal in future times (actually, the system is static). Hence, the system is causal.

That's all … ∇

Question 2. Consider the system $G_{RLC} : v \mapsto i$ shown in Fig. 1 (in other words, the input is the applied

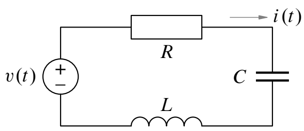

Fig. 1: RLC circuit

voltage v and the output is the circuit's current i). Here R, L, and C are constants, referred to as the resistance, inductance, and capacitance, respectively. Derive the impulse response of the system assuming that $L = 1$, $R = 3$, and $C = 0.5$. Is the system BIBO stable?

Solution. By Kirchhoff's voltage law,

$$v_R(t) + v_L(t) + v_C(t) = v(t),$$

where v_R, v_L, and v_C are the voltage drops at the resistor, inductor, and capacitor, respectively. It is known that

$$v_R(t) = Ri(t), \quad v_L(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}, \quad \text{and} \quad v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(s)\mathrm{d}s.$$

Hence,

$$Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int_{-\infty}^{t} i(s)\mathrm{d}s = v(t) \quad \Longrightarrow \quad R\frac{\mathrm{d}i(t)}{\mathrm{d}t} + L\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \frac{1}{C}i(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

In the Laplace domain, by the linearity and differentiation rules, we have that

$$RsI(s) + Ls^2 I(s) + \frac{1}{C}I(s) = sV(s) \quad \Longrightarrow \quad \Big(Ls^2 + Rs + \frac{1}{C}\Big)I(s) = sV(s)$$

Hence, the relation between the transformed current and voltage is

$$I(s) = \frac{s}{Ls^2 + Rs + 1/C}V(s).$$

The impulse response is the response of the system to the Dirac δ applies at $t = 0$. The Laplace transform of this impulse is $V(s) = 1$. Thus,

$$I(s) = \frac{s}{Ls^2 + Rs + 1/C}$$

Given that $L = 1$, $R = 3$, and $C = 0.5$, we get

$$I(s) = \frac{s}{s^2 + 3s + 2} = \frac{s}{(s+1)(s+2)} = \frac{\mathrm{Res}(I, -1)}{s+1} + \frac{\mathrm{Res}(I, -2)}{s+2} = -\frac{1}{s+1} + \frac{2}{s+2}.$$

Using the inverse Laplace transforms of each partial fraction, we get the impulse response

$$g_{RLC}(t) = -\mathrm{e}^{-t}\mathbb{1}(t) + 2\mathrm{e}^{-2t}\mathbb{1}(t),$$

An LTI system is BIBO stable iff its impulse response belongs to L_1. The L_1-norm of g_{RLC} is

$$\|g_{RLC}\|_1 = \int_{\mathbb{R}} |g_{RLC}(t)|\mathrm{d}t = \int_0^\infty |-\mathrm{e}^{-t} + 2\mathrm{e}^{-2t}|\mathrm{d}t \le \int_0^\infty |-\mathrm{e}^{-t}|\mathrm{d}t + \int_0^\infty |2\mathrm{e}^{-2t}|\mathrm{d}t = 1 + 1 = 2$$

(actually, $\|g_{RLC}\|_1 = 1/2$, but this is not important in this context). Hence, the system is stable. ▽

Question 3. Derive the impulse response of the discrete-time model

$$y[k+2] = 1.2y[k+1] - 0.2y[k] + 0.8u[k].$$

of a system $G : u \mapsto y$. Is this system BIBO stable?

Solution. Using z-transform and its shift property (i.e. $(\mathfrak{Z}\{\mathbb{S}_\tau x\})(z) = z^\tau X(z)$), we have

$$(z^2 - 1.2z + 0.2)Y(z) = 0.8U(z).$$

The z-transform of the pulse is 1, so that the impulse response in the z-domain is

$$Y(z) = \frac{0.8}{z^2 - 1.2z + 0.2} = \frac{0.8}{(z-1)(z-0.2)} = \frac{\mathrm{Res}(Y, 1)}{z-1} + \frac{\mathrm{Res}(Y, 0.2)}{z-0.2} = \frac{1}{z-1} - \frac{1}{z-0.2}$$

We know the inverse z-transforms of functions $z/(z-\lambda)$, but not $1/(z-\lambda)$. To circumvent this problem, consider the function

$$Y_1(z) := zY(z) = \frac{z}{z-1} - \frac{z}{z-0.2}.$$

Its time-domain counterpart is

$$y_1[k] = \mathbb{1}[k] - 0.2^k\,\mathbb{1}[k] = \mathbb{1}[k-1] - 0.2^k\,\mathbb{1}[k-1]$$

(because $y_1[0] = 0$ anyway). Now, returning to the original output y and noting that $y_1[k] = y[k+1] \iff y[k] = y_1[k-1]$, we end up with the impulse response of G in the form

$$\boxed{g[k] = (1 - 0.2^{k-1})\mathbb{1}[k-2]}.$$

An LTI system with the impulse response g is BIBO stable iff $g \in \ell_1$. The second term of g above decays exponentially ($|0.2| < 1$), but the first term does not, it converges to 1. Hence, $g \notin \ell_1$ and this system is unstable. ∇

Question 4. The Fibonacci sequence is obtained by repeatedly summing the 2 previous values in the sequence, i.e. it satisfies $f_t = f_{t-1} + f_{t-2}$, for $f_0 = 0$ and $f_1 = 1$.

- Represent it as the impulse response of a 2nd order discrete system $G_{\text{F}} : u \mapsto y$, so that $y[t] = f_t$.
- Is this system BIBO stable?

Solution.

1. A general 2-order discrete system satisfies

$$y[t+2] + a_1 y[t+1] + a_0 y[t] = b_2 u[t+2] + b_1 u[t+1] + b_0 u[t].$$

To be its impulse response, the Fibonacci numbers must satisfy

$$f_{t+2} + a_1 f_{t+1} + a_0 f_t = b_2\delta[t+2] + b_1\delta[t+1] + b_0\delta[t].$$

The right-hand side of this relation is zero for all $t > 0$ and $f_{t+2} = f_{t+1} + f_t$. Hence, we must have

$$f_{t+2} + a_1 f_{t+1} + a_0 f t = f_{t+1} + f_t + a_1 f_{t+1} + a_0 f_t = 0.$$

This relation implies that $\boxed{a_1 = a_0 = -1}$. Now, to match the Fibonacci sequence under $u = \delta$ and $f_t = 0$ for all $t \le 0$ we need the following conditions to be true:

$t = -2$: $f_0 - f_{-1} - f_{-2} = 0 = b_2\delta[0] + b_1\delta[-1] + b_0\delta[-2] = b_2 \implies \boxed{b_2 = 0}$

$t = -1$: $f_1 - f_0 - f_{-1} = 1 = b_1\delta[0] + b_0\delta[-1] = b_1 \implies \boxed{b_1 = 1}$

$t = 0$: $f_2 - f_1 - f_0 = 0 = \delta[1] + b_0\delta[0] = b_0 \implies \boxed{b_0 = 0}$

The impulse response itself can be calculated via z-transform as follows. Rewrite the equation of the resulted system,

$$y[t+2] - y[t+1] - y[t] = u[t+1]$$

in the z-domain as

$$z^2Y(z) - zY(z) - Y(z) = zU(z) \iff Y(z) = \frac{z}{z^2 - z - 1}U(z)$$

Thus, the transfer function of the system G_{F} is

$$G_{\mathrm{F}}(z) = \frac{z}{z^2 - z - 1} = \frac{\mathrm{Res}(G_{\mathrm{F}}(z), p_1)}{z - p_1} + \frac{\mathrm{Res}(G_{\mathrm{F}}(z), p_2)}{z - p_2},$$

where the roots of $z^2 - z - 1 = 0$, $p_1 = \phi$ and $p_2 = -1/\phi$ under $\phi = (1 + \sqrt{5})/2 \approx 1.618 > 1$ (the golden ratio). We have that

$$\mathrm{Res}(G_{\mathrm{F}}(z), p_1) = \lim_{z \to p_1} (z - p_1) G_{\mathrm{F}}(z) = \lim_{z \to p_1} \frac{p_1}{z - p_2} = \frac{p_1}{p_1 - p_2} = \frac{\phi}{\sqrt{5}}$$
$$\mathrm{Res}(G_{\mathrm{F}}(z), p_2) = \lim_{z \to p_2} (z - p_2) G_{\mathrm{F}}(z) = \lim_{z \to p_2} \frac{p_2}{z - p_1} = \frac{p_2}{p_2 - p_1} = \frac{1}{\phi\sqrt{5}}$$

Hence,

$$G_{\mathrm{F}}(z) = \frac{\phi/\sqrt{5}}{z - \phi} - \frac{1/(\phi\sqrt{5})}{z - 1/\phi} = z^{-1}\frac{1}{\sqrt{5}}\Big(\frac{\phi z}{z - \phi} + \frac{z/\phi}{z + 1/\phi}\Big),$$

The inverse z-transform of this function, taking into account that z^{-1} corresponds to the shift $\mathbb{S}_{-1}$ in the time domain, is

$$\begin{aligned} g_{\mathrm{F}}[t] &= \frac{1}{\sqrt{5}}\Big(\phi\phi^{t-1} + \frac{1}{\phi}(-\frac{1}{\phi})^{t-1}\Big)\mathbb{1}[t-1] \\ &= \frac{1}{\sqrt{5}}\Big(\phi^t - \frac{1}{(-\phi)^t}\Big)\mathbb{1}[t-1] = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\} \end{aligned}$$

as expected.

2. An LTI system with the impulse response g is BIBO stable iff $g \in \ell_1$. Because $|p_1| = \phi > 1$, we have that

$$\|g_{\mathrm{F}}\|_1 = \sum_{k=-\infty}^{\infty} |g_{\mathrm{F}}[k]| = \infty$$

hence $g_{\mathrm{F}} \notin \ell_1$ and this system is not BIBO stable. This conclusion agrees with the fact that Fibonacci numbers diverge.

That's all . . . ▽

4 Homework problems

Question 5. Classify the following models in terms of linearity, time invariance, and causality.

1. $y(t) = 5$
2. $y(t) = x(0) + \int_{t-1}^{t} x(\tau)\mathrm{d}\tau$
3. $y[t] = u[t+1]$
4. $\ddot{y}(t) = u(t-1) + \dot{u}(t)$
5. $y(t) = u(at)$

Solution.

1. Linear: No. Time-invariant: Yes. Causal: Yes.
2. Linear: Yes. Time-invariant: No. Causal: No.
3. Linear: Yes. Time-invariant: Yes. Causal: No.
4. Linear: Yes. Time-invariant: Yes. Causal: Yes.
5. Linear: Yes. Time-invariant: No. Causal: for $a = 1$.

$\triangledown$

Question 6. Suppose you have just started a savings program to save for a down payment on a house. You plan to save 500 ILS per month, starting with the opening teh program. The bank will also add 1% interest to your account at the end of each month.

- Write the difference equation that models the amount of money you will have in your savings account at the end of each month, assuming you do not withdraw any money.
- What is its impulse response?
- Is this system BIBO stable?

Solution. Let $y[k]$ be the amount of money in your savings account at the end of the kth month, where $k = 0$ corresponds to the beginning of the program. The amount of money in your savings account at the end of the $(k + 1)$th month is given by:

$$y[k] = y[k-1] + 0.01y[k-1] + u[k] \quad \text{or, equivalently,} \quad y[k+1] = y[k] + 0.01y[k] + u[k+1]$$

where $u = 500\mathbb{1}$ in our case. Using z-transform we get

$$zY(z) = 1.01Y(z) + zU(z)$$
$$(z - 1.01)Y(z) = zU(z)$$
$$Y(z) = \frac{z}{z - 1.01}U(z)$$

The impulse response g_{saving} in the z-domain corresponds to $(\mathfrak{Z}\{\})(z) = 1$, so

$$G_{\text{saving}}[z] = \frac{z}{z - 1.01}$$

The inverse z-transform of this signal is

$$g_{\text{saving}}[k] = 1.01^k\,\mathbb{1}[k]$$

An LTI system with the impulse response g_{saving} is BIBO stable iff $g_{\text{saving}} \in \ell_1$. Because $|1.01^k| = 1.01^k > 1$ for all $k \in \mathbb{N}$,

$$\|g_{\text{saving}}[\|_1 = \sum_{k=-\infty}^{\infty} |g_{\text{saving}}[k]| = \sum_{k=0}^{\infty} |1.01^k| \geq \sum_{k=0}^{\infty} 1 = \infty$$

hence $g_{\text{saving}} \notin \ell_1$ and this system is not BIBO stable. $\triangledown$

Question 7. Derive the impulse response of the following continuous time model and check if it is BIBO stable.

$$\ddot{y}(t) + \dot{y}(t) = 2\dot{u}(t) + u(t)$$

Solution.

$$Y(s) = \frac{2s+1}{s(s+1)}U(s)$$

Hence, the impulse response in the Laplace domain, which corresponds to $U(s) = 1$, is

$$G(s) = \frac{2s+1}{s(s+1)} = \frac{\mathrm{Res}(Y,0)}{s} + \frac{\mathrm{Res}(Y,-1)}{s+1} = \frac{1}{s} + \frac{1}{s+1}$$

Its inverse Laplace transform is

$$g(t) = \mathbb{1}(t) + e^{-t}\,\mathbb{1}(t).$$

This $g \notin \ell_1$ (it does not decay), so that this system is not BIBO stable. ∇