Linear Systems (034032)

tutorial 6

1 Topics

Systems: basic definitions (linearity, time invariance, causality), convolution representation, and transfer functions.

2 Background

Basic definitions (linearity, time invariance, causality, I/O stability). Convolution representation.

2.1 Linearity

A system $G : u \mapsto y$ is said to be linear if

$$G(a_1u_1 + a_2u_2) = a_1(Gu_1) + a_2(Gu_2)$$

for all admissible inputs u_1 and u_2 and all scalars a_1 and a_2.

2.2 Time (shift) invariance

A system $G : u \mapsto y$ is said to be time-invariant if

$$G(\mathbb{S}_\tau u) = \mathbb{S}_\tau(Gu)$$

for all admissible inputs u and all $\tau \in \mathbb{R}$ (or $\tau \in \mathbb{Z}$ in the discrete-time case). In other words, every time shift of the input results in the same time shift of the output.

2.3 Causality

A system $G : u \mapsto y$ is said to be causal if the signal y at every time instance t_c can only depend on u at $t \le t_c$ and not at $t > t_c$.

2.4 Impulse response

If $G : u \mapsto y$ is LTI (linear time invariant), then

$$y(t) = \int_{\mathbb{R}} g(t-s)u(s)\mathrm{d}s,$$

where g is the response of G to the Dirac delta δ, i.e. the response of an LTI system to an input signal u amounts to convolving that input with the impulse response g of the system.

2.5 BIBO stability via impulse response

An LTI system G with impulse response g is BIBO stable iff $g \in L_1$ in the continuous-time case or $g \in \ell_1$ in the discrete-time case.

3 Problems

Question 1. Classify the models $G : u \mapsto y$ below in terms of linearity, time invariance, and (except the systems in items 1 and 2) causality.

1. $y(t) = t\dot{u}(t)$
2. $y(t) = u(t)\dot{u}(t)$
3. $y(t) = u(t + 1)$
4. $y(t) = 2u(t) + 1$
5. $y(t) = \begin{cases} u(t) & \text{if } 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Question 2. Consider the system $G_{RLC} : v \mapsto i$ shown in Fig. 1 (in other words, the input is the applied

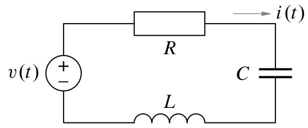

Fig. 1: RLC circuit

voltage v and the output is the circuit's current i). Here R, L, and C are constants, referred to as the resistance, inductance, and capacitance, respectively. Derive the impulse response of the system assuming that $L = 1$, $R = 3$, and $C = 0.5$. Is the system BIBO stable?

Question 3. Derive the impulse response of the discrete-time model

$$y[k + 2] = 1.2y[k + 1] - 0.2y[k] + 0.8u[k].$$

of a system $G : u \mapsto y$. Is this system BIBO stable?

Question 4. The Fibonacci sequence is obtained by repeatedly summing the 2 previous values in the sequence, i.e. it satisfies $f_t = f_{t-1} + f_{t-2}$, for $f_0 = 0$ and $f_1 = 1$.

- Represent it as the impulse response of a 2nd order discrete system $G_{\mathrm{F}} : u \mapsto y$, so that $y[t] = f_t$.
- Is this system BIBO stable?

4 Homework problems

Question 5. Classify the following models in terms of linearity, time invariance, and causality.

1. $y(t) = 5$
2. $y(t) = x(0) + \int_{t-1}^{t} x(\tau)\mathrm{d}\tau$
3. $y[t] = u[t + 1]$
4. $\ddot{y}(t) = u(t - 1) + \dot{u}(t)$

5. $y(t) = u(at)$

Question 6. Suppose you have just started a savings program to save for a down payment on a house. You plan to save 500 ILS per month, starting with the opening teh program. The bank will also add 1% interest to your account at the end of each month.

- Write the difference equation that models the amount of money you will have in your savings account at the end of each month, assuming you do not withdraw any money.
- What is its impulse response?
- Is this system BIBO stable?

Question 7. Derive the impulse response of the following continuous time model and check if it is BIBO stable.

$$\ddot{y}(t) + \dot{y}(t) = 2\dot{u}(t) + u(t)$$