Linear Systems (034032)

tutorial 5

1 Topics

Bilateral Laplace transform, bilateral z-transform, RoC, partial fraction expansion, solving differential equations, final and initial value theorems.

2 Definitions

2.1 Laplace Transform

The bilateral Laplace transform of a continuous time signal $x : \mathbb{R} \to \mathbb{F}$ is

$$X(s) = (\mathfrak{L}\{x\})(s) = \int_{-\infty}^{+\infty} x(t) \mathrm{e}^{-st} \mathrm{d}t$$

where the region of convergence (RoC) is all $s \in \mathbb{C}$ for which the integral converges. If $\operatorname{supp}(x) = \mathbb{R}_+$, then $\exists \alpha_x \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$\text{RoC} = \mathbb{C}_{\alpha_x} := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha_x\}$$

Specifically, for $\alpha_x \in \{\pm\infty\}$ we have that

$$\begin{aligned} \alpha_x = -\infty &\implies \text{RoC} = \mathbb{C} \\ \alpha_x = \infty &\implies \text{RoC} = \emptyset \end{aligned}$$

2.2 z Transform

The bilateral Z transform of a discrete time signal $x : \mathbb{Z} \to \mathbb{F}$ is

$$X(z) = (\mathfrak{Z}\{x\})(z) = \sum_{t=-\infty}^{+\infty} x[t] z^{-t}$$

where the region of convergence (RoC) is all $z \in \mathbb{C}$ for which the sum converges. If $\operatorname{supp}(x) = \mathbb{Z}_+$ then $\exists \alpha_x \in \mathbb{R} \cup \{\infty\}$ such that

$$\text{RoC} = \{z \in \mathbb{C} \mid |z| > \alpha_x\}$$

2.3 Partial Fraction Expansion

Given a rational proper ($n \geq m$) function F,

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} =: \frac{N(s)}{D(s)}$$

we can rewrite the function as,

$$F(s) = F(\infty) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{c_{ij}}{(s - p_i)^j}$$

where p_i is the ith distinct pole of F (the ith root of $D(s)$) of order n_i. For a simple pole (a pole p_i with order $n_i = 1$) we can calculate c_{i1} as

$$c_{i1} = \text{Res}(F(s), p_i) := \lim_{s \to p_i} (s - p_i) F(s)$$

For higher order poles we need to do a few tricks like using coefficient comparison.

2.4 Final and Initial Value Theorems

Given a continuous signal $x : \mathbb{R} \to \mathbb{F}$ with $\text{supp}(x) \subset \mathbb{R}_+$, the initial and final value theorems are as follows.

1. Initial value theorem:
$$\lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s),$$
assuming $x(0^+)$ exists.

2. Final value theorem:
$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) = \text{Res}(X, 0),$$
assuming x is converging.

Similarly, for a discrete signal $x : \mathbb{Z} \to \mathbb{F}$ with $\text{supp}(x) \subset \mathbb{Z}_+$,

1. Initial value theorem:
$$x[0] = \lim_{z \in \mathbb{R}, z \to \infty} X(z),$$
assuming $x[0]$ exists.

2. Final value theorem:
$$\lim_{t \to \infty} x[t] = \lim_{z \to 1} (z - 1) X(z) = \text{Res}(X, 1),$$
assuming x is converging.

3 Problems

Question 1. Consider the signal y shown in Fig. 1 defined as

$$y = \mathbb{S}_{1/2} \,\text{rect} - \mathbb{S}_{-1/2} \,\text{rect} \qquad \Longrightarrow \qquad y(t) = \text{rect}\big(t + \tfrac{1}{2}\big) - \text{rect}\big(t - \tfrac{1}{2}\big),$$

with

$$\text{rect}(t) = \begin{cases} 1 & |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}.$$

Find the Laplace transform of y and its RoC in three different ways,

1. by calculating the Laplace transform directly,
2. by using the Laplace transform properties,
3. by using the Fourier transform,

Solution.

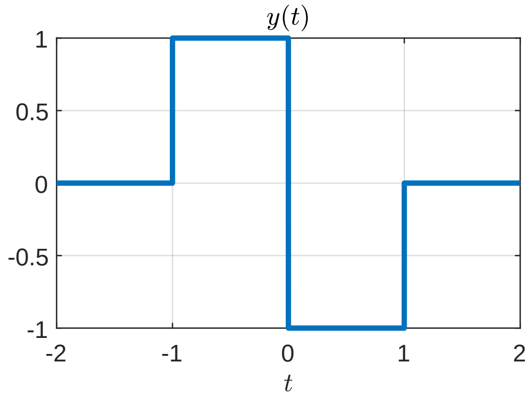

Fig. 1: Signal Used in Question 1.

1. Calculating the Laplace transform by definition.

   We express y directly as:

$$y(t) = \begin{cases} -1 & \text{if } 0 < t < 1 \\ 1 & \text{if } -1 < t < 0 \\ 0 & \text{if } |t| > 1 \end{cases}$$

   By the definition of the Laplace transform we get:

$$\begin{aligned} Y(s) &= \int_{-\infty}^{+\infty} y(t)\mathrm{e}^{-st}\mathrm{d}t = \int_{-1}^{0} \mathrm{e}^{-st}\mathrm{d}t - \int_{0}^{1} \mathrm{e}^{-st}\mathrm{d}t \\ &= \left[\frac{\mathrm{e}^{-st}}{-s}\right]_{-1}^{0} - \left[\frac{\mathrm{e}^{-st}}{-s}\right]_{0}^{1} = -\frac{1}{s} + \frac{\mathrm{e}^{s}}{s} - \left(-\frac{\mathrm{e}^{-s}}{s} + \frac{1}{s}\right) \\ &= \frac{\mathrm{e}^{s} + \mathrm{e}^{-s} - 2}{s} = \frac{\left(\mathrm{e}^{s/2}\right)^2 + \left(\mathrm{e}^{-s/2}\right)^2 - 2}{s} = \frac{\left(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}\right)^2}{s} \end{aligned}$$

   The RoC is defined as the the range of s where the integral converges. Here we see that RoC $= \mathbb{C}$

2. Using the Laplace properties - 2 options.

   Option 1 - Using the linearity and shift properties:

   Reminder from the lecture, the Laplace transform of the signal $x = \text{rect}$ is:

$$X(s) = (\mathfrak{L}\{\text{rect}\})(s) = \frac{\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}}{s}$$

   The Laplace transform of y is defined as:

$$Y(s) = (\mathfrak{L}\{y\})(s) = (\mathfrak{L}\{\mathbb{S}_{1/2}\,\text{rect} - \mathbb{S}_{-1/2}\,\text{rect}\})(s)$$

   thus, using linearity and shift:

$$Y(s) = \mathrm{e}^{s/2}X(s) - \mathrm{e}^{-s/2}X(s) = \left(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}\right)X(s) = \frac{\left(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}\right)^2}{s}$$

   To calculate the RoC we note that shift does not change the RoC, and because of linearity we get RoC $= \mathbb{C}_{\alpha_1} \cap \mathbb{C}_{\alpha_2}$. So for our case,

$$\text{RoC} = \mathbb{C}_{\alpha_y} = \mathbb{C}_{\alpha_x} \cap \mathbb{C}_{\alpha_x} = \mathbb{C}_{\alpha_x}$$

Again, the region of convergence of rect is $\text{RoC} = \mathbb{C}$, so that,

$$\text{RoC} = \mathbb{C}$$

for y as well.

Option 2 - Using the convolution and differentiation properties:

In Tutorial 3 we saw that

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\text{tent}(t)$$

The Laplace transform of the signal $x = \text{tent}$, using convolution:

$$X(s) = (\mathfrak{L}\{\text{tent}\})(s) = \mathfrak{L}\{\text{rect} * \text{rect}\})(s) = \frac{\left(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}\right)^2}{s^2}$$

which together with the derivative property yields:

$$Y(s) = sX(s) = \frac{\left(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}\right)^2}{s}$$

Here, again the differentiation property does not change the RoC and the convolution property gives $\text{RoC} = \mathbb{C}_{\alpha_x} \cap \mathbb{C}_{\alpha_y}$. So for our case,

$$\text{RoC} = \mathbb{C}_{\alpha_y} = \mathbb{C}_{\alpha_x} \cap \mathbb{C}_{\alpha_x} = \mathbb{C}_{\alpha_x}$$

The region of convergence of rect is $\text{RoC} = \mathbb{C}$, so that,

$$\text{RoC} = \mathbb{C}$$

for y as well.

3. Using the Fourier transform.

   In Tutorial 3 we saw that y is Fourier transformable and calculated its Fourier transform as

$$Y(\mathrm{j}\omega) = \mathrm{j}\omega\,\text{sinc}^2(\omega/2)$$

   We may thus expect that y is Laplace transformable at least for $s \in \mathrm{j}\mathbb{R}$. Mechanically substituting $\mathrm{j}\omega \to s$ yields that

$$Y(s) = s\,\text{sinc}^2(s/(2\mathrm{j})) = s\frac{\sin^2(s/(2\mathrm{j}))}{(s/(2j))^2} = s\frac{\left(\mathrm{e}^{\mathrm{j}s/(2\mathrm{j})} - \mathrm{e}^{-\mathrm{j}s/(2\mathrm{j})}\right)^2}{s^2} = \frac{\left(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}\right)^2}{s}$$

   In this approach we need to be careful through, as we cannot extrapolate the RoC of the Laplace transform, other than to acknowledge that $\mathrm{j}\mathbb{R} \subseteq \text{RoC}$.

Thus, all approaches produce the same Laplace transform but the RoC needs to be handled carefully. ∇

Question 2. Consider the mass-spring-damper system in Fig. 2 with

$$m = 1, \quad k = 6, \quad \text{and} \quad c = 5.$$

We suppose zero spring force at $x = 0$ and zero initial velocity and position. By Newton's second law

$$m\ddot{x}(t) = F(t) - f_{\text{damper}}(t) - f_{\text{spring}}(t) = F(t) - c\dot{x}(t) - kx(t)$$

which is equivalent to

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

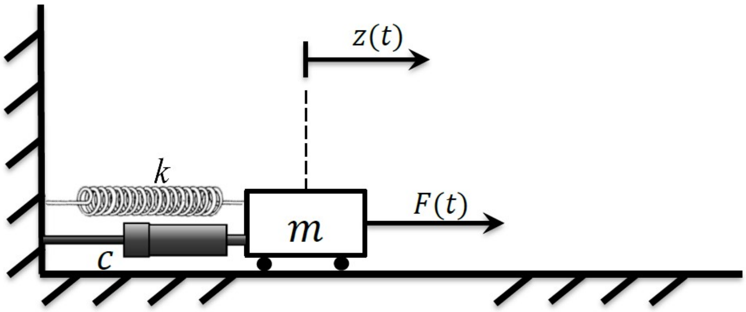

Fig. 2: The Mass-Spring-Damper System Used in Question 2.

1. Find the solution to the problem, i.e. the position of the mass in time, for the given input force $F = \mathbb{1}$.
2. The mass position is measured using a digital sensor with the sampling period h. Find the z-transform of the sampled signal $\bar{x}$.
3. What is the position of the mass after infinite time (use the Final Value Theorem)? What will the sensor show?

Solution.

1. We describe the differential equation of the system using the Laplace transform (using the differentiation property).

$$ms^2X(s) + csX(s) + kX(s) = F(s)$$
$$\Updownarrow$$
$$(ms^2 + cs + k)X(s) = F(s)$$
$$\Updownarrow$$
$$X(s) = \frac{1}{ms^2 + cs + k}F(s)$$

We substitute now the Laplace transform of the force. Reminder:

$$F(s) = (\mathfrak{L}\{\mathbb{1}\})(s) = \frac{1}{s}$$

Substituting into our dynamic equation, we have that

$$X(s) = \frac{1}{s(ms^2 + cs + k)} \tag{1}$$

Substituting the parameter values:

$$X(s) = \frac{1}{s(s^2 + 5s + 6)} \tag{2}$$

Next, calculate the partial fraction expansion of this X. To do so, we first find the roots of the denominator,

$$s(s^2 + 5s + 6) = 0 \implies s_1 = 0, s_2 = -2, s_3 = -3$$

so that

$$\frac{1}{s(s^2+5s+6)} = \frac{1}{s(s+2)(s+3)} = \frac{\mathrm{Res}(X,-2)}{s+2} + \frac{\mathrm{Res}(X,-3)}{s+3} + \frac{\mathrm{Res}(X,0)}{s}$$

The residues are computed as follows:

$$\begin{aligned}
\mathrm{Res}(X,-2) &= \lim_{s\to-2}(s+2)\frac{1}{s(s+2)(s+3)} = \frac{1}{-2(-2+3)} = -\frac{1}{2} \\
\mathrm{Res}(X,-3) &= \lim_{s\to-3}(s+3)\frac{1}{s(s+2)(s+3)} = \frac{1}{-3(-3+2)} = \frac{1}{3} \\
\mathrm{Res}(X,0) &= \lim_{s\to0} s\frac{1}{s(s+2)(s+3)} = \frac{1}{(0+2)(0+3)} = \frac{1}{6}
\end{aligned}$$

so that we finally get that

$$X(s) = -\frac{1}{2(s+2)} + \frac{1}{3(s+3)} + \frac{1}{6s}$$

Using our knowledge of Laplace transforms of the step function together with the modulation property, we can calculate the inverse Laplace transform for each element separately.

$$\begin{aligned}
\frac{1}{s} &= (\mathfrak{L}\{\mathbb{1}\})(s), & \mathrm{RoC} &= \mathbb{C}_0 \\
\frac{1}{s+2} &= (\mathfrak{L}\{\mathbb{1}\exp_{-2}\})(s), & \mathrm{RoC} &= \mathbb{C}_{-2} \\
\frac{1}{s+3} &= (\mathfrak{L}\{\mathbb{1}\exp_{-3}\})(s), & \mathrm{RoC} &= \mathbb{C}_{-3}
\end{aligned}$$

with the region of convergence

$$\mathrm{RoC} = \mathbb{C}_0 \cap \mathbb{C}_{-2} \cap \mathbb{C}_{-3} = \mathbb{C}_0.$$

Combining everything we end up with

$$x(t) = -1/2(\exp_{-2}\mathbb{1})(t) + 1/3(\exp_{-3}\mathbb{1})(t) + 1/6\mathbb{1}(t) = \left(-\frac{1}{2}\mathrm{e}^{-2t} + \frac{1}{3}\mathrm{e}^{-3t} + \frac{1}{6}\right)\mathbb{1}(t)$$

The system response is shown in Fig. 3.

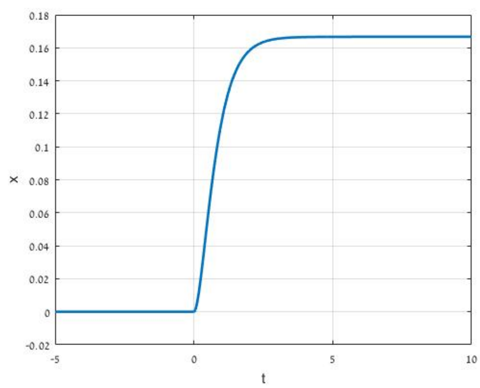

Fig. 3: The system response in question 2.

2. To calculate the z-transform, we first need to find the sampled signal:

$$\bar{x}[i] = x(ih) = \left(-\frac{1}{2}\mathrm{e}^{-2ih} + \frac{1}{3}\mathrm{e}^{-3ih} + \frac{1}{6}\right)\mathbb{1}(ih) = -\frac{1}{2}(\exp_{\mathrm{e}^{-2h}}\mathbb{1})[i] + \frac{1}{3}(\exp_{\mathrm{e}^{-3h}}\mathbb{1})[i] + \frac{1}{6}\mathbb{1}[i]$$

where,

$$\exp_\lambda = \lambda^t, t \in \mathbb{Z}$$

Now calculating the z-transform, we use linearity and modulation. Reminder:

$$(\mathfrak{Z}\{\mathbb{1}\})(z) = \frac{z}{z-1} \quad \text{and} \quad (\mathfrak{Z}\{\exp_\lambda \mathbb{1}\})(z) = \frac{z}{z-\lambda}$$

so for our case,

$$\bar{X}(z) = -\frac{1}{2}\frac{z}{z-\mathrm{e}^{-2h}} + \frac{1}{3}\frac{z}{z-\mathrm{e}^{-3h}} + \frac{1}{6}\frac{z}{z-1}$$

3. Using the final value theorem, we calculate the position of the mass after a long time:

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s)\\ &= \lim_{s\to 0} s\left(-\frac{1}{2(s+2)} + \frac{1}{3(s+3)} + \frac{1}{6s}\right)\\ &= \lim_{s\to 0} -\frac{s}{2(s+2)} + \frac{s}{3(s+3)} + \frac{1}{6} = 1/6\end{aligned}$$

Using the final value theorem on the sampled signal, we calculate the sensor's value after a long time,

$$\begin{aligned}\lim_{i\to\infty} \bar{x}[i] &= \lim_{z\to 1}(z-1)\bar{X}(z)\\ &= \lim_{z\to 1}(z-1)\left(-\frac{1}{2}\frac{z}{z-\mathrm{e}^{-2h}} + \frac{1}{3}\frac{z}{z-\mathrm{e}^{-3h}} + \frac{1}{6}\frac{z}{z-1}\right)\\ &= \lim_{z\to 1}\left(-\frac{1}{2}\frac{z(z-1)}{z-\mathrm{e}^{-2h}} + \frac{1}{3}\frac{z(z-1)}{z-\mathrm{e}^{-3h}} + \frac{1}{6}z\right) = \frac{1}{6}\end{aligned}$$

We see that we get the same result, which is also shown in the system response in Fig. 3 as the position the mass converges to after a long time.

▽

Question 3. Consider the signal from Fig. 4, i.e.

$$x(t) = \cos(\omega_x t)\mathbb{1}(t)$$

1. Find the Laplace transform of x with its RoC.
2. What about $-\mathbb{P}_{-1}x$?

Solution.

1. Reminder, for a harmonic signal, $x(t) = \sin(\omega_x t + \phi)\mathbb{1}(t)$,

$$X(s) = \frac{s\sin\phi + \omega_x\cos\phi}{s^2 + \omega_x^2}$$

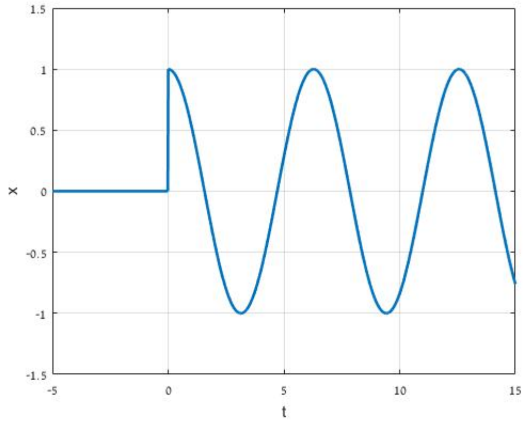

Fig. 4: Signal used in Question 3.

In our case $\phi = \pi/2$, thus,

$$X(s) = \frac{s}{s^2 + \omega_x^2}$$

The RoC is $\mathbb{C}_0$ (see lecture slides). We can also calculate the Laplace transform directly,

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \cos(\omega_x t)\mathbb{1}(t)\mathrm{e}^{-st}\mathrm{d}t = \int_0^{\infty} \cos(\omega_x t)\mathrm{e}^{-st}\mathrm{d}t \\ &= \int_0^{\infty} \frac{\mathrm{e}^{\mathrm{j}\omega_x t} + \mathrm{e}^{-\mathrm{j}\omega_x t}}{2}\mathrm{e}^{-st}\mathrm{d}t = \int_0^{\infty} \frac{\mathrm{e}^{(\mathrm{j}\omega_x - s)t} + \mathrm{e}^{(-\mathrm{j}\omega_x - s)t}}{2}\mathrm{d}t \\ &= \left[\frac{\mathrm{e}^{(\mathrm{j}\omega_x - s)t}}{2(\mathrm{j}\omega_x - s)} - \frac{\mathrm{e}^{-(\mathrm{j}\omega_x + s)t}}{2(\mathrm{j}\omega_x + s)}\right]_0^{\infty} = \frac{-1}{2(\mathrm{j}\omega_x - s)} - \frac{-1}{2(\mathrm{j}\omega_x + s)} \\ &= \frac{-(\cancel{\mathrm{j}\omega_x} + s) + (\cancel{\mathrm{j}\omega_x} - s)}{-2(s^2 + \omega_x^2)} = \frac{s}{s^2 + \omega_x^2} \end{aligned}$$

To find the RoC we need to calculate for what s does the integral converge. We demand that:

$$\lim_{t\to\infty} \mathrm{e}^{(\mathrm{j}\omega_x - s)t} = 0 \iff \mathrm{Re}(\mathrm{j}\omega_x - s) = -\mathrm{Re}\, s < 0$$
$$\lim_{t\to\infty} \mathrm{e}^{-(\mathrm{j}\omega_x + s)t} = 0 \iff \mathrm{Re}(\mathrm{j}\omega_x + s) = \mathrm{Re}\, s > 0$$

therefore the region of convergence is $\mathbb{C}_0$.

2. Now the signal is $-\mathbb{P}_{-1}x$, which is shown in Fig. 5.

$$\mathbb{P}_{-1}x = \cos(-\omega_x t)\mathbb{1}(-t)$$

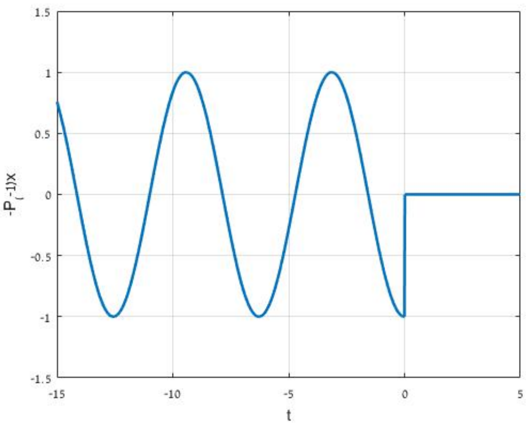

Fig. 5: The Flipped Signal from Question 3.

In order to calculate the Laplace transform, we need to use the definition,

$$\begin{aligned}
X(s) &= -\int_{-\infty}^{\infty} \cos(-\omega_x t)\mathbb{1}(-t)\mathrm{e}^{-st}\mathrm{d}t = -\int_{-\infty}^{0} \cos(\omega_x t)\mathrm{e}^{-st}\mathrm{d}t \\
&= -\int_{-\infty}^{0} \frac{\mathrm{e}^{\mathrm{j}\omega_x t} + \mathrm{e}^{-\mathrm{j}\omega_x t}}{2}\mathrm{e}^{-st}\mathrm{d}t = -\int_{-\infty}^{0} \frac{\mathrm{e}^{(\mathrm{j}\omega_x - s)t} + \mathrm{e}^{(-\mathrm{j}\omega_x - s)t}}{2}\mathrm{d}t \\
&= -\left[\frac{\mathrm{e}^{(\mathrm{j}\omega_x - s)t}}{2(\mathrm{j}\omega_x - s)} - \frac{\mathrm{e}^{-(\mathrm{j}\omega_x + s)t}}{2(\mathrm{j}\omega_x + s)}\right]_{-\infty}^{0} = -\frac{1}{2(\mathrm{j}\omega_x - s)} + \frac{1}{2(\mathrm{j}\omega_x + s)} \\
&= \frac{-(\cancel{\mathrm{j}\omega_x} + s) + (\cancel{\mathrm{j}\omega_x} - s)}{-2(s^2 + \omega_x^2)} = \frac{s}{s^2 + \omega_x^2}
\end{aligned}$$

We got the same Laplace transform, but now the integral converges for,

$$\begin{aligned}
\lim_{t\to-\infty} \mathrm{e}^{(\mathrm{j}\omega_x - s)t} = 0 &\iff \mathrm{Re}(\mathrm{j}\omega_x - s) = -\mathrm{Re}\, s > 0 \\
\lim_{t\to-\infty} \mathrm{e}^{-(\mathrm{j}\omega_x + s)t} = 0 &\iff \mathrm{Re}(\mathrm{j}\omega_x + s) = \mathrm{Re}\, s < 0
\end{aligned}$$

therefore the region of convergence is $\mathbb{C} \setminus \bar{\mathbb{C}}_0 := \{s \in \mathbb{C} \mid \mathrm{Re}\, s < 0\}$.

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4 Homework problems

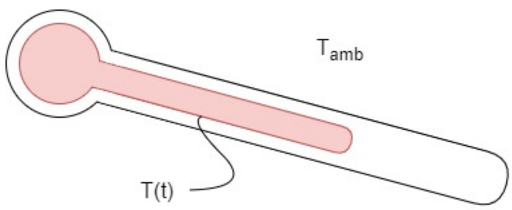

Fig. 6: The Thermometer system used in question 4.

Question 4. We are again given the system of the thermometer as in Lecture 5 (see Fig. 6). We assume the thermometer has some initial stable temperature θ_0 (for example, due to it being placed in a different room for a long period of time). After which, it is transferred to a different environment.

The dynamic equation of the system is as follows:

$$\tau\dot{\theta}(t) = \theta_{\text{amb}}(t) - \theta(t)$$

where θ_{amb} is the ambient temperature (which can change over time), and θ is the temperature of the thermometer.

1. Express the temperature shown by the thermometer as a function of time for the ambient temperature

$$\theta_{\text{amb}}(t) = \theta_1 t\,\mathbb{1}(t) + \theta_0$$

2. Assuming $\tau > 0$ find the stable temperature.

Solution.

1. We first describe the model parameters in terms of a deviation from θ_0,

$$\tilde{\theta}(t) = \theta(t) - \theta_0 \quad \text{and} \quad \tilde{\theta}_{\text{amb}}(t) = \theta_{\text{amb}}(t) - \theta_0$$

This allows us to analyze a model with $\operatorname{supp}(\tilde{\Theta}) \subset \mathbb{R}_+$,

$$\tau\dot{\tilde{\theta}}(t) = \tilde{\theta}_{\text{amb}}(t) - \tilde{\theta}(t)$$

Next, we calculate the Laplace transform of the model,

$$\tau s\tilde{\Theta}(s) = \tilde{\Theta}_{\text{amb}}(s) - \tilde{\Theta}(s)$$
$$\Updownarrow$$
$$\tau s\tilde{\Theta}(s) + \tilde{\Theta}(s) = \tilde{\Theta}_{\text{amb}}(s)$$
$$\Updownarrow$$
$$(\tau s + 1)\tilde{\Theta}(s) = \tilde{\Theta}_{\text{amb}}(s)$$
$$\Updownarrow$$
$$\tilde{\Theta}(s) = \frac{\tilde{\Theta}_{\text{amb}}(s)}{\tau s + 1}$$

For the ambient temperature of $\theta_{\text{amb}} = \theta_1 t\,\mathbb{1}(t) + \theta_0$ we have

$$\tilde{\theta}_{\text{amb}}(t) = \theta_1 t\,\mathbb{1}(t) \tag{3}$$

We now calculate its Laplace transform using the t-modulation property. Reminder,

$$(\mathfrak{L}\{x \text{ ramp}\})(s) = -\frac{\mathrm{d}}{\mathrm{d}s}X(s) \quad \text{and} \quad (\mathfrak{L}\{\mathbb{1}\})(s) = \frac{1}{s}$$

Thus, for our case

$$(\mathfrak{L}\{\tilde{\Theta}_{\text{amb}}\})(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{\theta_1}{s} = \frac{\theta_1}{s^2}$$

with RoC $= \mathbb{C}_0$. Plugging back into the model equation,

$$\tilde{\Theta}(s) = \frac{\theta_1}{s^2(\tau s + 1)}$$

The system has a simple pole at $s = -1/\tau$ and a pole of order 2 at $s = 0$. Using partial fraction expansion,

$$\tilde{\Theta}(s) = \frac{\operatorname{Res}(\tilde{\Theta}, -1/\tau)}{s + 1\tau} + \frac{f_1}{s} + \frac{f_0}{s^2} = \frac{\operatorname{Res}(\tilde{\Theta}, -1/\tau)}{s + 1\tau} + \frac{f(s)}{s^2},$$

where $f(s) = f_1 s + f_0$. To find $f(s)$, we first calculate the residue of $-1/\tau$:

$$\operatorname{Res}(\tilde{\Theta}, -1/\tau) = \lim_{s \to -1/\tau} (s + 1/\tau)\tilde{\Theta}(s) = \lim_{s \to -1/\tau} (s + 1/\tau)\frac{\theta_1}{s^2(\tau s + 1)} = \lim_{s \to -1/\tau} \frac{\theta_1}{\tau s^2} = \tau\theta_1$$

Now we can find $f(s)$,

$$\frac{f(s)}{s^2} = \frac{\theta_1}{s^2(\tau s + 1)} - \frac{\tau\theta_1}{s + 1/\tau} = \frac{\theta_1 - s^2\tau^2\theta_1}{s^2(\tau s + 1)} = \frac{\theta_1(1 - (\tau s)^2)}{s^2(1 + \tau s)} = \frac{\theta_1(1 - \tau s)}{s^2}$$

We therefore got that

$$f(s) = \theta_1(1 - \tau s)$$

Substituting back to the Laplace representation gives,

$$\tilde{\Theta}(s) = \frac{\tau\theta_1}{s + 1/\tau} + \frac{\theta_1(1 - \tau s)}{s^2} = \frac{\tau\theta_1}{s + 1/\tau} + \frac{\theta_1}{s^2} - \frac{\theta_1\tau}{s}$$

Now we can calculate the reverse Laplace transform using known functions. Reminder,

$$(\mathfrak{L}\{\mathbb{1}\})(s) = \frac{1}{s}, \quad (\mathfrak{L}\{t\,\mathbb{1}\})(s) = \frac{1}{s^2}, \quad \text{and} \quad (\mathfrak{L}\{\exp_\lambda \mathbb{1}\})(s) = \frac{1}{1 - \lambda}$$

So for our problem,

$$\tilde{\theta}(t) = \theta_1\tau \exp_{-1/\tau} \mathbb{1}(t) + \theta_1 t\,\mathbb{1}(t) - \theta_1\tau\,\mathbb{1}(t) = \theta_1\tau\left(\mathrm{e}^{-t/\tau} + \frac{t}{\tau} - 1\right)\mathbb{1}(t)$$

Finally, the absolute temperature θ is obtained by adding back θ_0,

$$\theta(t) = \theta_1\tau\left(\mathrm{e}^{-t/\tau} + \frac{t}{\tau} - 1\right)\mathbb{1}(t) + \theta_0$$

The graph in Fig. 7 shows the temperature change over time for these parameters:

$$\tau = 3, \quad \theta_1 = 10, \quad \text{and} \quad \theta_0 = 10.$$

2. We can see from Fig. 7, and also from the time response of the temperature, that

$$\theta(t) = \theta_1\tau\left(\mathrm{e}^{-t/\tau} + \frac{t}{\tau} - 1\right)\mathbb{1}(t) + \theta_0$$

that the signal diverges as $t \to \infty$. So we cannot use the final value theorem.

▽

Question 5. Given bellow a system $u \mapsto y$, described by the difference equation

$$y[k + 2] - 1.2y[k + 1] + 0.2y[k] = 0.8u[k]$$

Assume that $u = \mathbb{1}$. Find the discrete output y. Can you use the final value theorem to find the system output at $k \to \infty$? What about the initial value theorem for $k = 0$?

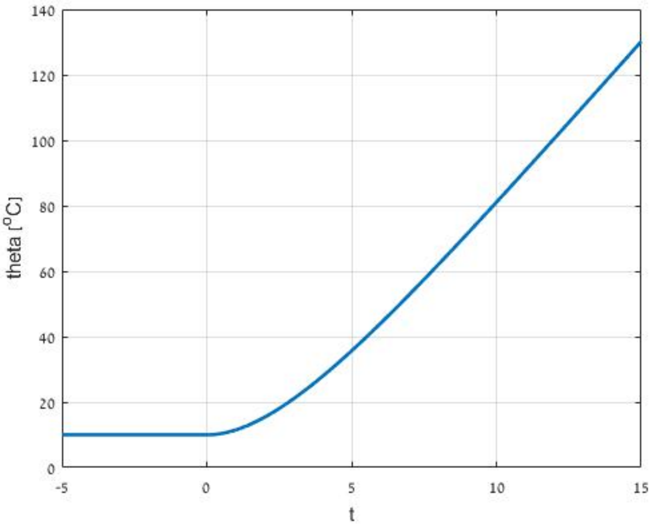

Fig. 7: The System Response to a Ramp Input.

Solution. To solve the difference equation we first calculate the Z-transform of the system:

$$z^2Y(z) - 1.2zY(z) + 0.2Y(z) = 0.8U(z)$$
$$(z^2 - 1.2z + 0.2)Y(z) = 0.8U(z)$$
$$Y(z) = \frac{0.8}{z^2 - 1.2z + 0.2}U(z)$$

We calculate the Z-transform of $u[k] = \mathbb{1}[k]$ which is:

$$U(z) = \frac{z}{z-1}$$

Inserting into the system's dynamic equation gives:

$$Y(z) = \frac{0.8z}{(z^2 - 1.2z + 0.2)(z-1)}$$

The poles of the denominator are:

$$z^2 - 1.2z + 0.2 = 0 \Rightarrow z_{1,2} = 1, z_3 = 0.2$$

so we get,

$$Y(z) = \frac{0.8z}{(z-1)^2(z-0.2)}$$

To calculate the partial fraction expansion we first divide both sides of the equation by z:

$$\frac{Y(z)}{z} = \frac{0.8}{(z-1)^2(z-0.2)} = \frac{Az + B}{(z-1)^2} + \frac{C}{z-0.2}$$

Performing coefficient comparison:

$$0.8 = (Az + B)(z - 0.2) + C(z-1)^2 = Az^2 + (B - 0.2A)z - 0.2B + Cz^2 - 2Cz + C$$

which yields,

$$\begin{aligned}
1 &: 0.8 = C - 0.2B \\
z &: 0 = B - 0.2A - 2C \\
z^2 &: 0 = A + C
\end{aligned}$$

Calculating we get,

$$A = -1.25, B = 2.25, C = 1.25$$

Inserting back into the equation,

$$\begin{aligned}
\frac{Y(z)}{z} &= \frac{-1.25z + 2.25}{(z-1)^2} + \frac{1.25}{z - 0.2} \\
&= -\frac{1.25(z-1)}{(z-1)^2} + \frac{1}{(z-1)^2} + \frac{1.25}{z-0.2} \\
&= -\frac{1.25}{z-1} + \frac{1}{(z-1)^2} + \frac{1.25}{z-0.2}
\end{aligned}$$

We multiply by z:

$$Y(z) = -\frac{1.25z}{z-1} + \frac{z}{(z-1)^2} + \frac{1.25z}{z-0.2}$$

We can find the inverse Z-transforms by finding the transforms of known functions. Reminder:

$$\begin{aligned}
(\mathfrak{Z}\{\mathbb{1}\})(z) &= \frac{z}{z-1} \\
(\mathfrak{Z}\{k\,\mathbb{1}\})(z) &= -z\frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{z-1} = \frac{z}{(z-1)^2} \\
(\mathfrak{Z}\{\exp_\lambda \mathbb{1}\})(z) &= \frac{z}{z-\lambda}
\end{aligned}$$

Also the discrete exponent function is defined as:

$$(\exp_\lambda)[k] = \lambda^k$$

We apply this to our problem and get:

$$y[k] = \Big(k + 1.25(0.2^k - 1)\Big)\,\mathbb{1}[k]$$

We can see from our result that y does not converge therefore we cannot use the final value theorem. The initial value theorem can be used on the other hand since y exists for $k = 0$.

$$y[0] = \lim_{z\to\infty} Y(z) = \lim_{z\to\infty} -\frac{1.25z}{z-1} + \frac{z}{(z-1)^2} + \frac{1.25z}{z-0.2} = 0$$

∇