Linear Systems (034032)

tutorial 5

1 Topics

Bilateral Laplace transform, bilateral z-transform, RoC, partial fraction expansion, solving differential equations, final and initial value theorems.

2 Definitions

2.1 Laplace Transform

The bilateral Laplace transform of a continuous time signal $x : \mathbb{R} \to \mathbb{F}$ is

$$X(s) = (\mathfrak{L}\{x\})(s) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-st}\mathrm{d}t$$

where the region of convergence (RoC) is all $s \in \mathbb{C}$ for which the integral converges. If $\mathrm{supp}(x) = \mathbb{R}_+$, then $\exists \alpha_x \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$\mathrm{RoC} = \mathbb{C}_{\alpha_x} := \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \alpha_x\}$$

Specifically, for $\alpha_x \in \{\pm\infty\}$ we have that

$$\begin{aligned} \alpha_x = -\infty &\implies \mathrm{RoC} = \mathbb{C} \\ \alpha_x = \infty &\implies \mathrm{RoC} = \emptyset \end{aligned}$$

2.2 z Transform

The bilateral Z transform of a discrete time signal $x : \mathbb{Z} \to \mathbb{F}$ is

$$X(z) = (\mathfrak{Z}\{x\})(z) = \sum_{t=-\infty}^{+\infty} x[t]z^{-t}$$

where the region of convergence (RoC) is all $z \in \mathbb{C}$ for which the sum converges. If $\mathrm{supp}(x) = \mathbb{Z}_+$ then $\exists \alpha_x \in \mathbb{R} \cup \{\infty\}$ such that

$$\mathrm{RoC} = \{z \in \mathbb{C} \mid |z| > \alpha_x\}$$

2.3 Partial Fraction Expansion

Given a rational proper ($n \geq m$) function F,

$$F(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} =: \frac{N(s)}{D(s)}$$

we can rewrite the function as,

$$F(s) = F(\infty) + \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{c_{ij}}{(s - p_i)^j}$$

where p_i is the ith distinct pole of F (the ith root of $D(s)$) of order n_i. For a simple pole (a pole p_i with order $n_i = 1$) we can calculate c_{i1} as

$$c_{i1} = \mathrm{Res}(F(s), p_i) := \lim_{s \to p_i} (s - p_i) F(s)$$

For higher order poles we need to do a few tricks like using coefficient comparison.

2.4 Final and Initial Value Theorems

Given a continuous signal $x : \mathbb{R} \to \mathbb{F}$ with $\mathrm{supp}(x) \subset \mathbb{R}_+$, the initial and final value theorems are as follows.

1. Initial value theorem:
$$\lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s),$$
assuming $x(0^+)$ exists.

2. Final value theorem:
$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) = \mathrm{Res}(X, 0),$$
assuming x is converging.

Similarly, for a discrete signal $x : \mathbb{Z} \to \mathbb{F}$ with $\mathrm{supp}(x) \subset \mathbb{Z}_+$,

1. Initial value theorem:
$$x[0] = \lim_{z \in \mathbb{R}, z \to \infty} X(z),$$
assuming $x[0]$ exists.

2. Final value theorem:
$$\lim_{t \to \infty} x[t] = \lim_{z \to 1} (z - 1) X(z) = \mathrm{Res}(X, 1),$$
assuming x is converging.

3 Problems

Question 1. Consider the signal y shown in Fig. 1 defined as

$$y = \mathbb{S}_{1/2} \,\mathrm{rect} - \mathbb{S}_{-1/2} \,\mathrm{rect} \quad \Longrightarrow \quad y(t) = \mathrm{rect}\big(t + \tfrac{1}{2}\big) - \mathrm{rect}\big(t - \tfrac{1}{2}\big),$$

with

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}.$$

Find the Laplace transform of y and its RoC in three different ways,

1. by calculating the Laplace transform directly,
2. by using the Laplace transform properties,
3. by using the Fourier transform,

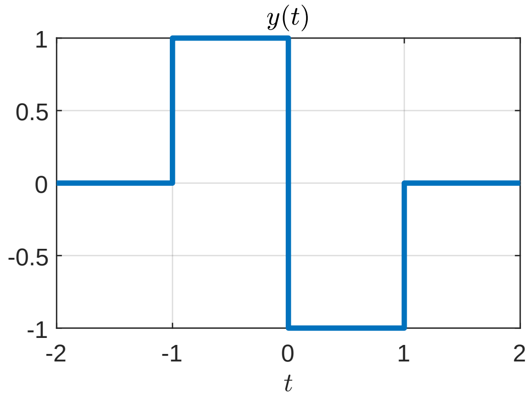

Fig. 1: Signal Used in Question 1.

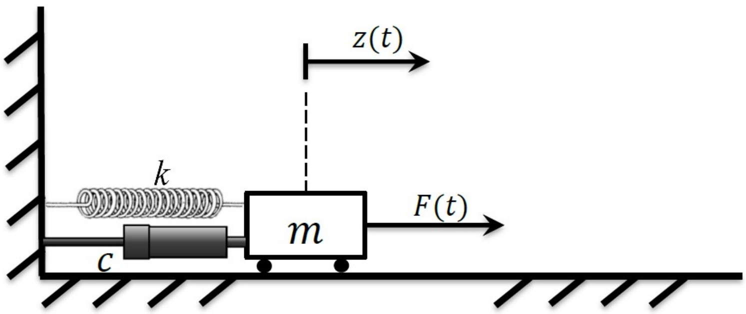

Fig. 2: The Mass-Spring-Damper System Used in Question 2.

Question 2. Consider the mass-spring-damper system in Fig. 2 with

$$m = 1, \quad k = 6, \quad \text{and} \quad c = 5.$$

We suppose zero spring force at $x = 0$ and zero initial velocity and position. By Newton's second law

$$m\ddot{x}(t) = F(t) - f_{\text{damper}}(t) - f_{\text{spring}}(t) = F(t) - c\dot{x}(t) - kx(t)$$

which is equivalent to

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

1. Find the solution to the problem, i.e. the position of the mass in time, for the given input force $F = \mathbb{1}$.
2. The mass position is measured using a digital sensor with the sampling period h. Find the z-transform of the sampled signal $\bar{x}$.
3. What is the position of the mass after infinite time (use the Final Value Theorem)? What will the sensor show?

Question 3. Consider the signal from Fig. 3, i.e.

$$x(t) = \cos(\omega_x t)\mathbb{1}(t)$$

1. Find the Laplace transform of x with its RoC.
2. What about $-\mathbb{P}_{-1}x$?

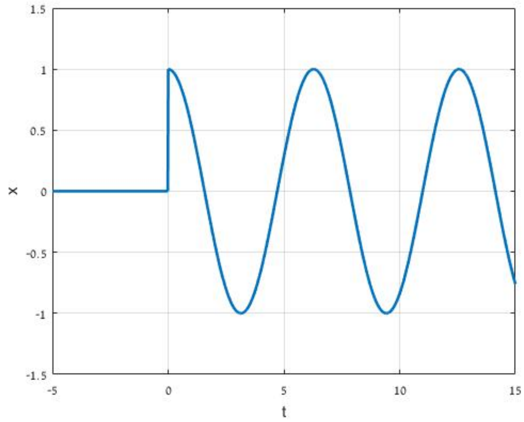

Fig. 3: Signal used in Question 3.

4 Homework problems

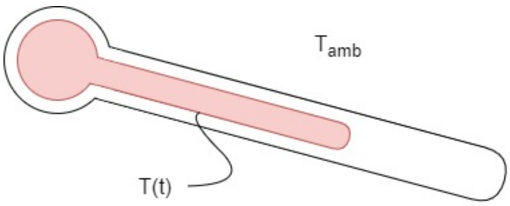

Fig. 4: The Thermometer system used in question 4.

Question 4. We are again given the system of the thermometer as in Lecture 5 (see Fig. 4). We assume the thermometer has some initial stable temperature θ_0 (for example, due to it being placed in a different room for a long period of time). After which, it is transferred to a different environment.

The dynamic equation of the system is as follows:

$$\tau\dot{\theta}(t) = \theta_{\text{amb}}(t) - \theta(t)$$

where θ_{amb} is the ambient temperature (which can change over time), and θ is the temperature of the thermometer.

1. Express the temperature shown by the thermometer as a function of time for the ambient temperature

$$\theta_{\text{amb}}(t) = \theta_1 t \mathbb{1}(t) + \theta_0$$

2. Assuming $\tau > 0$ find the stable temperature.

Question 5. Given bellow a system $u \mapsto y$, described by the difference equation

$$y[k+2] - 1.2y[k+1] + 0.2y[k] = 0.8u[k]$$

Assume that $u = \mathbb{1}$. Find the discrete output y. Can you use the final value theorem to find the system output at $k \to \infty$? What about the initial value theorem for $k = 0$?