Linear Systems (034032)

tutorial 4

1 Topics

Discrete-time Fourier transform (DTFT). Sampling in the frequency domain. Zero-order hold (ZOH).

2 Background results

The discrete-time Fourier transform (DTFT) is defined as

$$X(e^{j\theta}) = (\mathfrak{F}\{x\})(e^{j\theta}) = \sum_{t=-\infty}^{+\infty} x[t]e^{-j\theta t}.$$

Under some technical conditions (see slides), the inverse discrete-time Fourier transform results in

$$x[t] = (\mathfrak{F}^{-1}\{X\})[t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{j\theta t}d\theta.$$

The periodic summation of a signal x with period T is

$$x_T := \sum_{i=-\infty}^{\infty} \mathbb{S}_{iT}x \qquad \Longrightarrow \qquad x_T(t) = \sum_{i=-\infty}^{\infty} x(t+iT).$$

The spectrum of the sampled version of x, i.e. the discrete signal $\bar{x}$ such that $\bar{x}[i] = x(ih)$, is

$$\bar{X} = \frac{1}{h}\mathbb{P}_{1/h}X_{2\pi/h} \qquad \Longrightarrow \qquad \bar{X}(e^{j\theta}) = \frac{1}{h}X_{2\pi/h}(j\theta/h).$$

Let $\bar{x}$ be a discrete signal and x_{ZOH} be a continuous-signal obtained from $\bar{x}$ via the zero-order hold, i.e.

$$x_{\text{ZOH}}(t) = \bar{x}[i], \quad \forall t \in (ih, (i+1)h)$$

for a given sampling period $h > 0$. The spectrum of x is then

$$X(j\omega) = h\,\text{sinc}\Big(\frac{\omega h}{2}\Big)e^{-j\omega h/2}\bar{X}(e^{j\omega h}).$$

Theorem 1 (the Sampling Theorem)**.** *If* $\text{supp}(X) \subset [-\omega_N, \omega_N]$, *then x can be perfectly recovered from its sampled measurements as*

$$x = \sum_{i=-\infty}^{\infty} x(ih)\mathbb{S}_{-ih}\mathbb{P}_{\omega_N}\,\text{sinc} \qquad \Longrightarrow \qquad x(t) = \sum_{i=-\infty}^{\infty} x(ih)\,\text{sinc}((t-ih)\omega_N)$$

known as the sinc-interpolator (sinc hold).

3 Problems

Question 1. Let $x = \mathrm{rect}_{2N}$, i.e.

$$x[t] = \begin{cases} 1 & \text{if } |t| \leq N \\ 0 & \text{otherwise} \end{cases}$$

see Fig. 1. Find the DTFT of x.

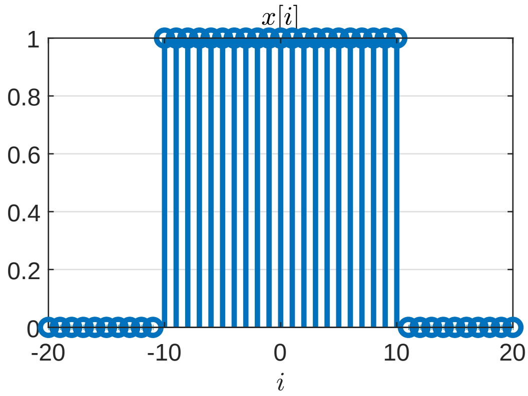


Fig. 1: Discrete-time signal $x[i]$

Solution. Trying to represent x using known signals and properties

$$x = \mathbb{S}_N \mathbb{1} - \mathbb{S}_{-N-1} \mathbb{1}$$

Reminder that

$$\mathfrak{F}\{\mathbb{1}\}\left(e^{j\theta}\right) = \frac{1}{1 - e^{-j\theta}} + \pi\delta(\theta)$$

and the time shift property of the DTFT

$$y = \mathbb{S}_\tau x \Rightarrow Y(e^{j\theta}) = e^{j\theta\tau} X(e^{j\theta}).$$

Using these we have

$$X(e^{j\theta}) = e^{j\theta N} \frac{1}{1 - e^{-j\theta}} + e^{j\theta N} \pi\delta(\theta) - e^{-j\theta(N+1)} \frac{1}{1 - e^{-j\theta}} - e^{-j\theta(N+1)} \pi\delta(\theta)$$

Using the property of the delta dirac $x(t)\delta(t) = x(0)\delta(t)$

$$\begin{aligned} X(e^{j\theta}) &= e^{j\theta N} \frac{1}{1 - e^{-j\theta}} + \pi\delta(\theta) - e^{-j\theta(N+1)} \frac{1}{1 - e^{-j\theta}} - \pi\delta(\theta) = \\ &= e^{j\theta N} \frac{1}{1 - e^{-j\theta}} - e^{-j\theta(N+1)} \frac{1}{1 - e^{-j\theta}} = \frac{e^{j\theta N} - e^{-j\theta(N+1)}}{1 - e^{-j\theta}} \end{aligned}$$

Multiplying the numerator and denominator by $e^{j\theta/2}$

$$X(e^{j\theta}) = \frac{e^{j\theta(N+1/2)} - e^{-j\theta(N+1/2)}}{e^{j\theta/2} - e^{-j\theta/2}}$$

using the exponential representation for sin

$$X(e^{j\theta}) = \frac{e^{j\theta(N+1/2)} - e^{-j\theta(N+1/2)}}{j2} \frac{j2}{e^{j\theta/2} - e^{-j\theta/2}} = \frac{\sin(\theta(N + 1/2))}{\sin(\theta/2)}$$

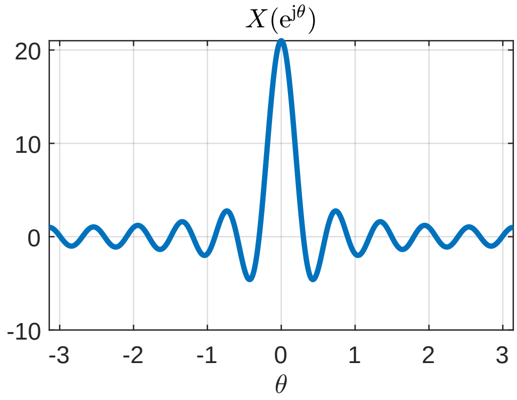


Fig. 2: Discrete-time Fourier transform, $X(e^{j\theta})$

For $N = 10$ the plot is in Fig. 2. ∇

Question 2. Let y be such that

$$y(t) = \frac{1}{2\pi}\,\mathrm{sinc}^2(t/2)$$

1. What is the minimal sampling frequency, ω_s, to fulfil the sampling theorem? What is the corresponding sampling period, h?

2. What is the DTFT of sampled signal $\tilde{y}$ sampled at $\tilde{\omega}_s = 3/4\omega_s$ of the frequency in section 1?

3. What is the DTFT of sampled signal $\bar{y}$ sampled at the frequency in section 1, ω_s?

4. Let $\hat{y}$ be the signal resulting from performing zero-order hold on $\bar{y}$. What is the Fourier transform of $\hat{y}$?

Solution.

1. First the spectrum needs to be found. Performing the Fourier transform

$$Y(j\omega) = \mathfrak{F}\left\{\frac{1}{2\pi}\,\mathrm{sinc}^2(t/2)\right\}$$

Using the duality property, $y = X|_{\omega=t} \implies Y(j\omega) = 2\pi x(-\omega)$

$$Y(j\omega) = \frac{1}{2\pi}2\pi\,\mathrm{tent}(-\omega) = \mathrm{tent}(\omega)$$

Reminder that tent is

$$\mathrm{tent}(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & |t| > 1 \end{cases}.$$

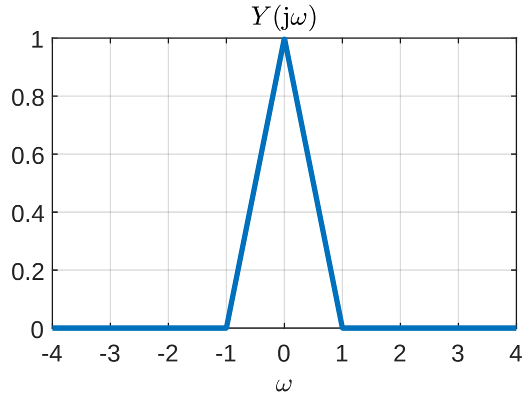
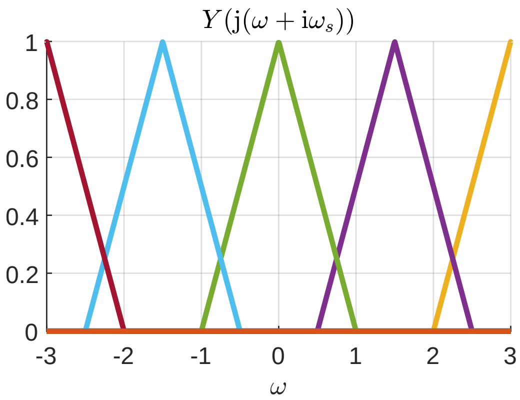


Fig. 3: Fourier transform, $Y(\mathrm{j}\omega)$

The spectrum is shown in Fig. 3. We can see that $\mathrm{supp}(Y) = [-1, 1]$, so that the minimum Nyquist frequency to satisfy the Sampling Theorem is $\omega_{\mathrm{N}} = 1$. In $[-\omega_{\mathrm{N}}, \omega_{\mathrm{N}}]$ the spectrum is nonzero and outside of the interval the spectrum is zero. The units are radians per time unit. The period and sampling frequency can then be recovered via the relations

$$\omega_{\mathrm{N}} = \frac{\pi}{h} \implies h = \pi \implies \omega_{\mathrm{S}} = \frac{2\pi}{h} = 2 = 2\omega_{\mathrm{N}}.$$



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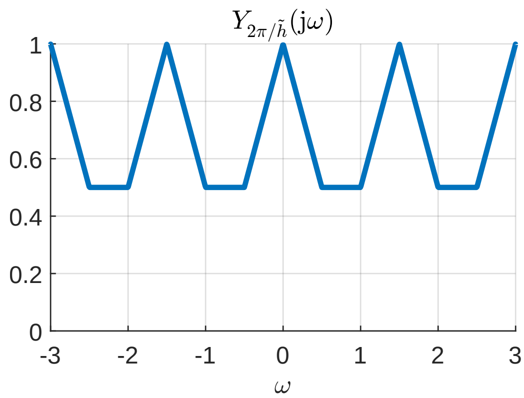

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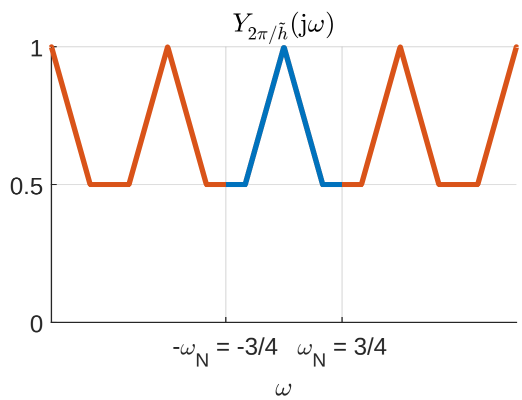

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Fig. 4: Shifted signals and periodic summation of $Y(\mathrm{j}\omega)$

2. The shifted signals and periodic summation is shown in Fig. 4.

$$\tilde{\omega}_{\mathrm{S}} = \omega_{\mathrm{S}}\frac{3}{4} = \frac{3}{2} \qquad \Longrightarrow \qquad \tilde{h} = \frac{2\pi}{\tilde{\omega}_{\mathrm{S}}} = 4\pi/3 \quad \omega_{\mathrm{N}} = \frac{3}{4}$$

From the lecture we saw

$$\tilde{Y}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{\tilde{h}} Y_{2\pi/\tilde{h}}(\mathrm{j}\theta/\tilde{h})$$

In general we look at the spectrum of discrete time signals in the interval $\theta \in [-\pi, \pi]$. This corresponds to $\omega \in [-\tilde{\omega}_{\mathrm{N}}, \tilde{\omega}_{\mathrm{N}}]$. The periodic summation in this interval is made up of,

$$\begin{aligned} Y(\mathrm{j}\omega) + Y(\mathrm{j}(\omega - \tilde{\omega}_{\mathrm{S}})) + Y(\mathrm{j}(\omega + \tilde{\omega}_{\mathrm{S}})) &= \mathrm{tent}(\omega) + \mathrm{tent}(\omega - \tilde{\omega}_{\mathrm{S}}) + \mathrm{tent}(\omega + \tilde{\omega}_{\mathrm{S}}) \\ &= \begin{cases} 1 - |\omega + \tilde{\omega}_{\mathrm{S}}| & -5/2 \le \omega < -1 \\ 1 - |\omega| + 1 - |\omega + \tilde{\omega}_{\mathrm{S}}| & -1 \le \omega < -1/2 \\ 1 - |\omega| & -1/2 \le \omega < 1/2 \\ 1 - |\omega| + 1 - |\omega - \tilde{\omega}_{\mathrm{S}}| & 1/2 \le \omega \le 1 \\ 1 - |\omega - \tilde{\omega}_{\mathrm{S}}| & 1 \le \omega \le 5/2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

For the interval $\omega \in [-\tilde{\omega}_{\mathrm{N}}, \tilde{\omega}_{\mathrm{N}}]$, where $\tilde{\omega}_{\mathrm{N}} = 3/4$:

$$\begin{aligned} \begin{cases} 1 - |\omega| + 1 - |\omega + \tilde{\omega}_{\mathrm{S}}| & -3/4 \le \omega < -1/2 \\ 1 - |\omega| & -1/2 \le \omega < 1/2 \\ 1 - |\omega| + 1 - |\omega - \tilde{\omega}_{\mathrm{S}}| & 1/2 \le \omega \le 3/4 \end{cases} &= \begin{cases} 1 + \omega + 1 - \omega - \tilde{\omega}_{\mathrm{S}} & -3/4 \le \omega < -1/2 \\ 1 - |\omega| & -1/2 \le \omega < 1/2 \\ 1 - \omega + 1 + \omega - \tilde{\omega}_{\mathrm{S}} & 1/2 \le \omega \le 3/4 \end{cases} \\ = \begin{cases} 2 - \tilde{\omega}_{\mathrm{S}} & -3/4 \le \omega < -1/2 \\ 1 - |\omega| & -1/2 \le \omega < 1/2 \\ 2 - \tilde{\omega}_{\mathrm{S}} & 1/2 \le \omega \le 3/4 \end{cases} &= \begin{cases} 1/2 & -3/4 \le \omega < -1/2 \\ 1 - |\omega| & -1/2 \le \omega < 1/2 \\ 1/2 & 1/2 \le \omega \le 3/4 \end{cases}. \end{aligned}$$

Then in the interval $\theta \in [-\pi, \pi]$, $Y_{2\pi/\tilde{h}}(\mathrm{j}\theta/\tilde{h})$ is

$$\begin{aligned} Y(\mathrm{j}\theta/\tilde{h}) = Y_{2\pi/\tilde{h}}(3\mathrm{j}\theta/(4\pi)) &= \begin{cases} 1/2 & -3/4 \le 3\theta/(4\pi) < -1/2 \\ 1 - |3\theta/(4\pi)| & -1/2 \le 3\theta/(4\pi) < 1/2 \\ 1/2 & 1/2 \le 3\theta/(4\pi) \le 3/4 \end{cases} \\ &= \begin{cases} 1/2 & -\pi \le \theta < -2\pi/3 \\ 1 - |3\theta/(4\pi)| & -2\pi/3 \le \theta < 2\pi/3 \\ 1/2 & 2\pi/3 \le \theta \le \pi \end{cases} \end{aligned}$$

Lastly, with the scaling

$$\tilde{Y}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{\tilde{h}} Y_{2\pi/\tilde{h}}(\mathrm{j}\theta/\tilde{h}) = \begin{cases} 3/(8\pi) & -\pi \le \theta < -2\pi/3 \\ 3/(4\pi)(1 - |3\theta/(4\pi)|) & -2\pi/3 \le \theta < 2\pi/3 \\ 3/(8\pi) & 2\pi/3 \le \theta \le \pi \end{cases}$$

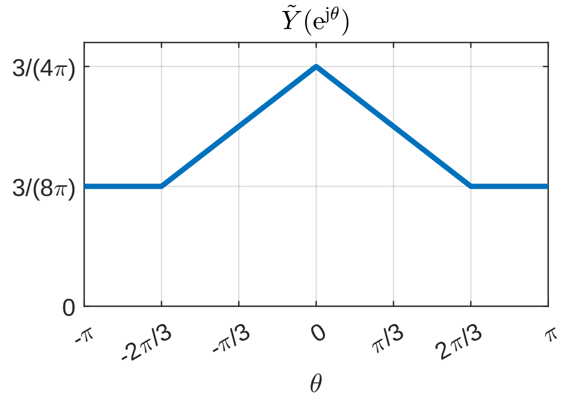


Fig. 5: Discrete-time Fourier transform, $\tilde{Y}(\mathrm{e}^{\mathrm{j}\theta})$

The spectrum is shown in Fig. 5.

3. The sampling frequency in the first section was $\omega_s = 2$. In that case, at every discrete frequency θ_0 the frequency responses at every aliased frequency $\omega_i = \theta_0/h + i\omega_s$ is zero for all $i \neq 0$. Therefore, the sampled spectrum, $\bar{Y}$, is the same as that of the analog signal in $\omega \in [-\omega_N, \omega_N]$, modulo scaling by $1/h = 1/\pi$, i.e.

$$\bar{Y}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{h} Y_{2\pi/h}(\mathrm{j}\theta/h) = \frac{1}{h} Y(\mathrm{j}\theta/h) = \frac{1}{\pi}(1 - |\theta/\pi|) = \frac{1}{\pi}\,\mathrm{tent}_\pi(\theta)$$

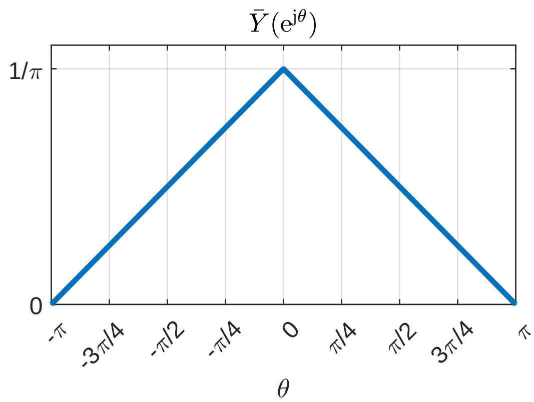


Fig. 6: Discrete-time Fourier transform, $\bar{Y}(\mathrm{e}^{\mathrm{j}\theta})$

The spectrum is shown in Fig. 6.

4. From the lecture we saw

$$\hat{Y}(\mathrm{j}\omega) = h\,\mathrm{sinc}(\omega h/2)\mathrm{e}^{-\mathrm{j}\omega h/2}\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h})$$

Here we are interested in $\bar{Y}$ over $\mathbb{R}$, which is

$$Y(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{i=-\infty}^{\infty} \frac{1}{\pi} \$_{2\pi i}\,\mathrm{tent}_\pi(\theta) = \frac{1}{\pi} \sum_{i=-\infty}^{\infty} \mathrm{tent}(\theta/\pi + 2i))$$

Now in the form $\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h})$, with $h = \pi$

$$\bar{Y}(\mathrm{e}^{\mathrm{j}\omega\pi}) = \frac{1}{\pi} \sum_{i=-\infty}^{\infty} \mathrm{tent}(\omega + 2i)$$

Substituting this back in

$$\hat{Y}(\mathrm{j}\omega) = h\,\mathrm{sinc}(\omega h/2)\mathrm{e}^{-\mathrm{j}\omega h/2}\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h}) = \pi\,\mathrm{sinc}(\omega\pi/2)\mathrm{e}^{-\mathrm{j}\omega\pi/2}\frac{1}{\pi}\sum_{i=-\infty}^{\infty}\mathrm{tent}(\omega + 2i)$$

$$\hat{Y}(\mathrm{j}\omega) = \mathrm{sinc}(\omega\pi/2)\mathrm{e}^{-\mathrm{j}\omega\pi/2}\sum_{i=-\infty}^{\infty}\mathrm{tent}(\omega + 2i)$$

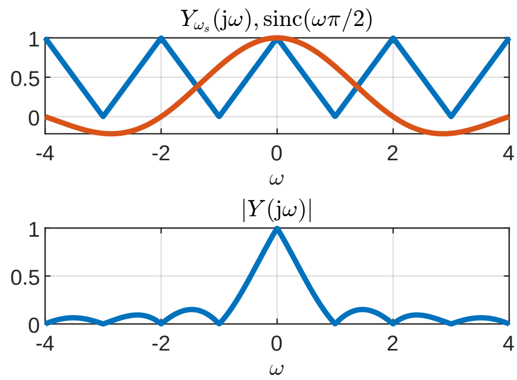


Fig. 7: Discrete-time Fourier transform, $\hat{Y}(\mathrm{e}^{\mathrm{j}\theta})$, and it components.

The spectrum and its individual components are shown in Fig. 7.

▽

Question 3. Let y be a signal with the Fourier transform

$$Y(\mathrm{j}\omega) = \mathrm{e}^{-\frac{1}{2}\omega^2}$$

Its spectrum is shown in Fig. 8. What is the minimal sampling frequency that fulfils the Sampling Theorem?

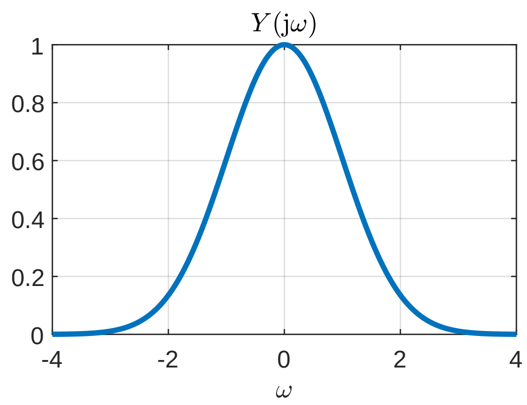


Fig. 8: Fourier transform, $Y(\mathrm{j}\omega)$

Solution. There is no sampling frequency that satisfies the sampling theorem. The spectrum of y is nonzero for all ω. ▽

Question 4. Let x be a signal with the Fourier transform in Fig. 9.

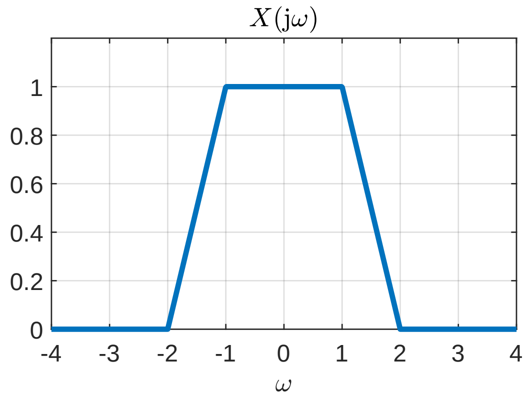


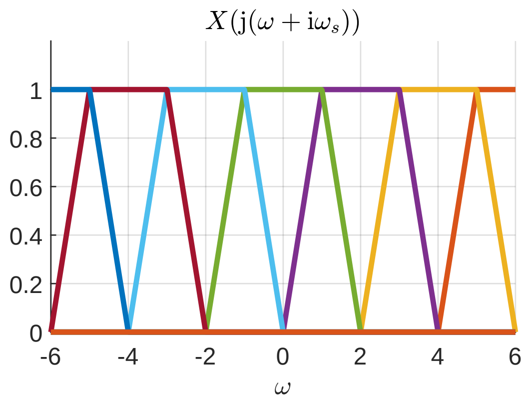
Fig. 9: Fourier transform, $X(\mathrm{j}\omega)$

Draw the spectrum of the sampled signal, sampled with a sampling period of $h = \pi$

Solution. The sampling frequency and Nyquist frequency are

$$h = \pi \quad \Longrightarrow \quad \omega_{\mathrm{S}} = \frac{2\pi}{h} = 2 \quad \omega_{\mathrm{N}} = \frac{\omega_{\mathrm{S}}}{2} = 1.$$

First the periodic summation is taken of X and we consider one period, $\omega \in [-\omega_{\mathrm{N}}, \omega_{\mathrm{N}}]$, this is done in Fig. 10. The the periodic summation is frequency-scaled and amplitude scaled by $1/h$, the result in Fig. 11.



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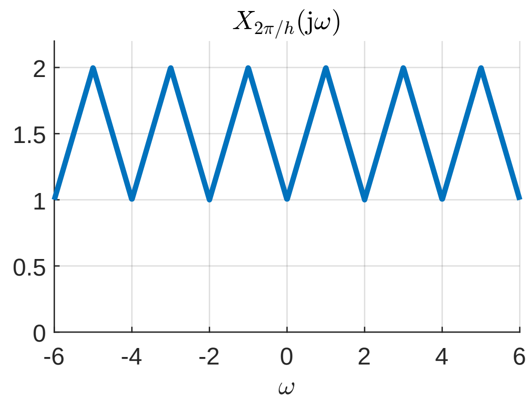

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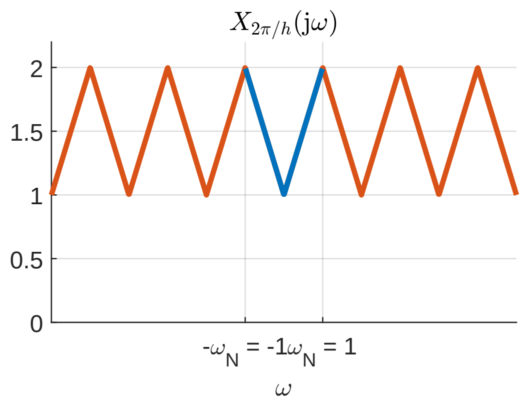

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Fig. 10: Shifted signals and periodic summation of $X(\mathrm{j}\omega)$

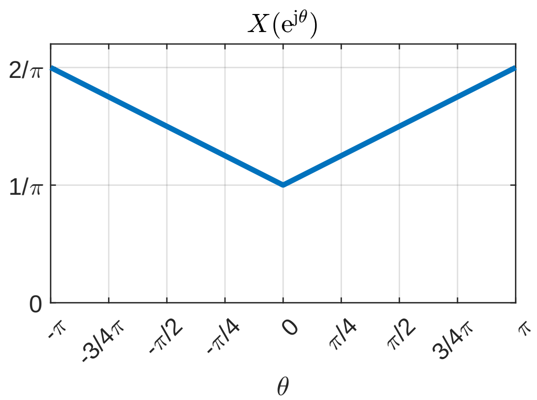


Fig. 11: Discrete-time Fourier transform, $X(\mathrm{e}^{\mathrm{j}\theta})$

▽

4 Homework problems

Question 5. Consider the continuous time signal x and its sampled signal $\bar{x}$. The Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$ are shown in Fig. 12.

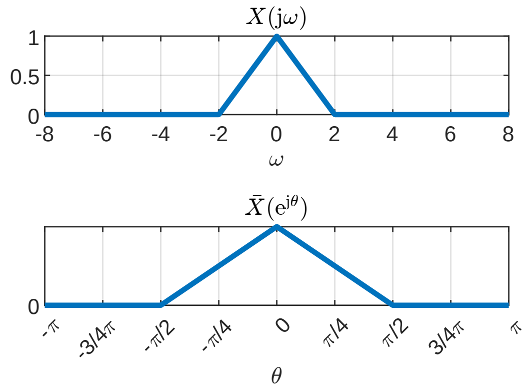


Fig. 12: Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$

1. What is the minimal sampling frequency needed to have perfect reconstruction?
2. Has the signal been distorted in the frequency domain?
3. What is the sampling period h?
4. Calculate the sampling frequency ω_s. Does this conform with the sampling theorem?
5. What is the height of the triangle $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$?

Solution.

1. The signal only has frequencies in support $\omega \in [-2, 2]$. Therefore, the minimal Nyquist frequency is given by $\omega_N = 2$. This means that the minimal sampling frequency is twice that, i.e. $\omega_s = 4$.
2. The DTFT is also triangular and therefore there is no distortion.
3. We have
$$\bar{X}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{h} X_{\frac{2\pi}{h}}(\mathrm{j}\theta/h) = \frac{1}{h} \sum_{i=-\infty}^{+\infty} X\left(\mathrm{j}\frac{\theta}{h} + i\frac{2\pi}{h}\right).$$
Because we have no aliasing, for $\theta \in [-\pi, \pi]$, we only have to consider $i = 0$.
$$\bar{X}(\mathrm{e}^{\mathrm{j}\theta})\big|_{\theta\in[-\pi,\pi]} = \frac{1}{h} X\left(\mathrm{j}\frac{\theta}{h}\right)$$
We can see in the figure that $\omega = \pm 2$ corresponds to $\theta = \pm\pi/2$. In other words
$$0 = X\left(\pm\mathrm{j}\frac{\pi/2}{h}\right) = X(\pm\mathrm{j}2) \implies \frac{\pi}{2h} = 2 \implies \boxed{h = \frac{\pi}{4}}$$
4. The sampling frequency is given by $\omega_s = 2\pi/h = 8$. This is larger than $\omega_s = 4$ and therefore conforms with the sampling theorem.

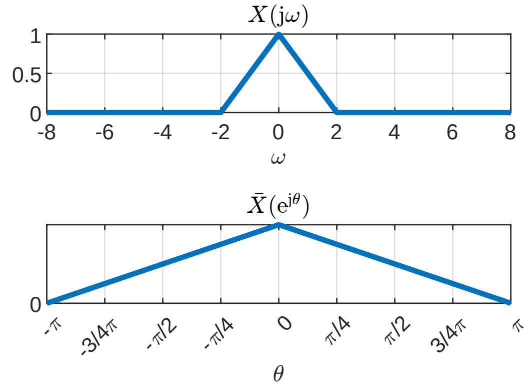


Fig. 13: Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$

5. The height of the triangle is given by $\bar{X}(\mathrm{e}^{\mathrm{j}0}) = \bar{X}(1)$.

$$\bar{X}(1) = \frac{1}{h} X(0)$$

   Given that $X(0) = 1$ (see figure) and $h = \pi/4$, we get

$$\bar{X}(1) = \frac{4}{\pi}$$

▽

Question 6. Consider a continuous time signal x and its sampled version $\bar{x}$. The Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$ are shown in Fig. 13.

1. Has the signal been distorted in the frequency domain?
2. Calculate the sampling frequency ω_s and sampling period h.
3. What is the height of the triangle $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$?

Solution.

1. The DTFT is also triangular and therefore there is no distortion.
2. The edge of the triangle corresponds to $\theta = \pm\pi$ and we are therefore sampling at exactly the minimal required frequency for perfect reconstruction.

$$\omega_\mathrm{s} = 4$$

   Therefore $h = 2\pi/\omega_\mathrm{s} = \pi/2$.

3. Given that $X(0) = 1$ (see figure) and $h = \frac{\pi}{2}$, we get

$$\bar{X}(1) = \frac{2}{\pi}$$

▽

Question 7. Consider a continuous time signal x and its sampled version $\bar{x}$. The Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$ are shown in Fig. 14.

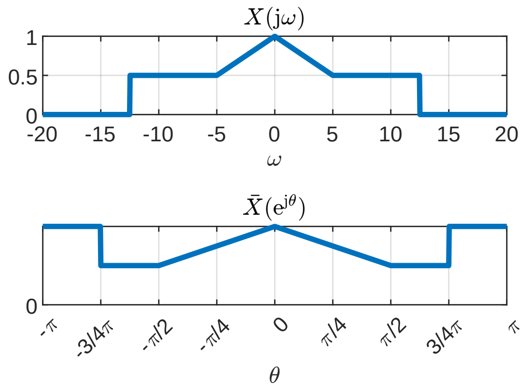


Fig. 14: Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$

1. Has the signal been distorted in the frequency domain?
2. Calculate the sampling frequency ω_s and sampling period h.
3. Does this conform with the sampling theorem.

Solution.

1. The Fourier transform and DTFT have different shapes and therefore are signal has indeed been distorted.
2. Both the Fourier transform and the DTFT have a clear triangle without aliasing and therefore we can use this to our advantage.

$$\omega = \pm 5 \leftrightarrow \theta = \pm\frac{\pi}{2}$$
$$\Rightarrow \omega = \frac{\theta}{h} \Rightarrow 5 = \frac{\pi/2}{h}$$
$$\Rightarrow \boxed{h = \frac{\pi}{10}}$$

   Therefore $\omega_s = 2\pi/h = 20$.

3. The signal must have support in $[-\omega_N, \omega_N]$ for perfect reconstruction. In our case, $\omega_N = \omega_s/2 = 10$. In our case $\mathrm{supp}(X) \notin [-\omega_N, \omega_N]$, which is why we have aliasing in our discretized signal.

$\triangledown$

Question 8. Let x be a signal with the Fourier transform in Fig. 15.

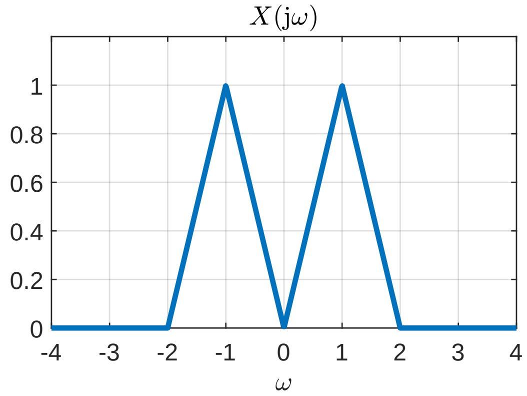


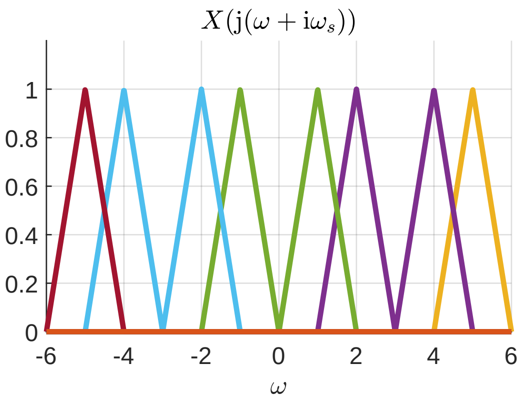
Fig. 15: Fourier transform, $X(j\omega)$

Draw the spectrum of the sampled signal, sampled with a sampling frequency of $\omega_N = 1.5$

Solution. The sampling frequency and Nyquist frequency are

$$\omega_N = 1.5 \quad \Longrightarrow \quad \omega_s = 2\omega_N = 3 \quad h = \frac{2\pi}{\omega_s} = 2\pi/3.$$

First the periodic summation is taken of X and we consider one period, $\omega \in [-\omega_N, \omega_N]$, this is done in Fig. 16. The the periodic summation is frequency-scaled and amplitude scaled by $1/h$, the result in Fig. 17.



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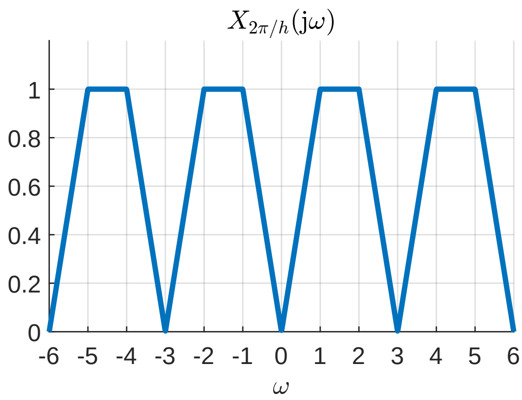

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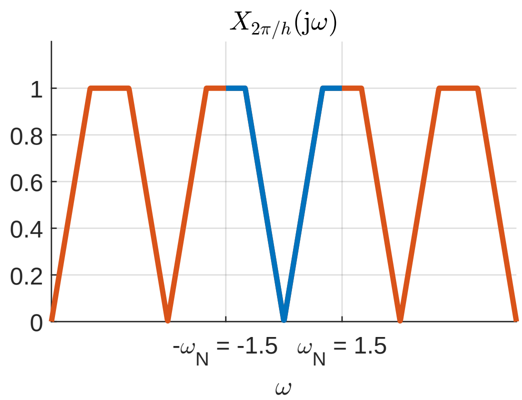

(c)

Fig. 16: Shifted signals and periodic summation of $X(j\omega)$

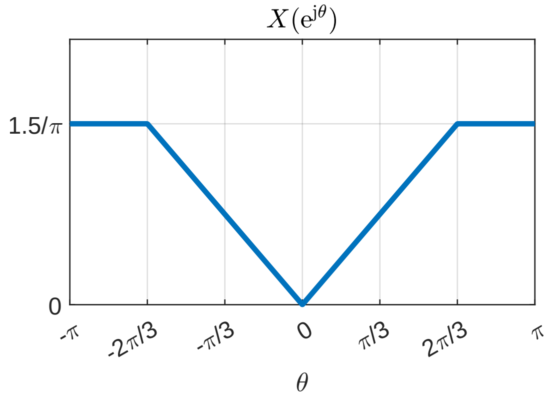

Fig. 17: Discrete-time Fourier transform, $X(e^{j\theta})$

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