Linear Systems (034032)

TUTORIAL 4

1 Topics

Discrete-time Fourier transform (DTFT). Sampling in the frequency domain. Zero-order hold (ZOH).

2 Background results

The discrete-time Fourier transform (DTFT) is defined as

$$X(\mathrm{e}^{\mathrm{j}\theta}) = (\mathfrak{F}\{x\})(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{t=-\infty}^{+\infty} x[t]\mathrm{e}^{-\mathrm{j}\theta t}.$$

Under some technical conditions (see slides), the inverse discrete-time Fourier transform results in

$$x[t] = (\mathfrak{F}^{-1}\{X\})[t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{j}\theta})\mathrm{e}^{\mathrm{j}\theta t}\mathrm{d}\theta.$$

The periodic summation of a signal x with period T is

$$x_T := \sum_{i=-\infty}^{\infty} \mathbb{S}_{iT}x \qquad \Longrightarrow \qquad x_T(t) = \sum_{i=-\infty}^{\infty} x(t+iT).$$

The spectrum of the sampled version of x, i.e. the discrete signal $\bar{x}$ such that $\bar{x}[i] = x(ih)$, is

$$\bar{X} = \frac{1}{h}\mathbb{P}_{1/h}X_{2\pi/h} \qquad \Longrightarrow \qquad \bar{X}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{h}X_{2\pi/h}(\mathrm{j}\theta/h).$$

Let $\bar{x}$ be a discrete signal and x_{ZOH} be a continuous-signal obtained from $\bar{x}$ via the zero-order hold, i.e.

$$x_{\text{ZOH}}(t) = \bar{x}[i], \quad \forall t \in (ih, (i+1)h)$$

for a given sampling period $h > 0$. The spectrum of x is then

$$X(\mathrm{j}\omega) = h\,\mathrm{sinc}\Big(\frac{\omega h}{2}\Big)\mathrm{e}^{-\mathrm{j}\omega h/2}\bar{X}(\mathrm{e}^{\mathrm{j}\omega h}).$$

Theorem 1 (the Sampling Theorem)**.** *If* $\mathrm{supp}(X) \subset [-\omega_{\text{N}}, \omega_{\text{N}}]$*, then* x *can be perfectly recovered from its sampled measurements as*

$$x = \sum_{i=-\infty}^{\infty} x(ih)\mathbb{S}_{-ih}\mathbb{P}_{\omega_{\text{N}}}\,\mathrm{sinc} \qquad \Longrightarrow \qquad x(t) = \sum_{i=-\infty}^{\infty} x(ih)\,\mathrm{sinc}((t-ih)\omega_{\text{N}})$$

known as the sinc-interpolator (sinc hold).

3 Problems

Question 1. Let $x = \text{rect}_{2N}$, i.e.

$$x[t] = \begin{cases} 1 & \text{if } |t| \leq N \\ 0 & \text{otherwise} \end{cases}$$

see Fig. 1. Find the DTFT of x.

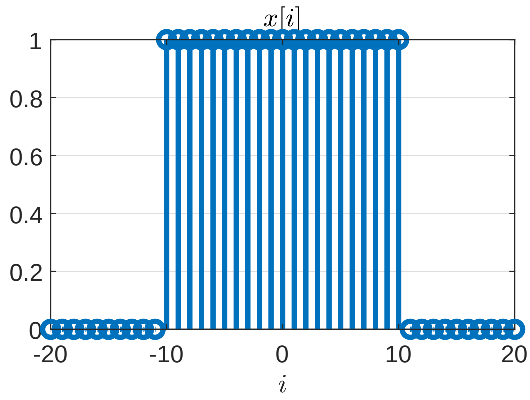

Fig. 1: Discrete-time signal $x[i]$

Question 2. Let y be such that

$$y(t) = \frac{1}{2\pi} \text{sinc}^2(t/2)$$

1. What is the minimal sampling frequency, ω_s, to fulfil the sampling theorem? What is the corresponding sampling period, h?
2. What is the DTFT of sampled signal $\tilde{y}$ sampled at $\tilde{\omega}_s = 3/4\omega_s$ of the frequency in section 1?
3. What is the DTFT of sampled signal $\bar{y}$ sampled at the frequency in section 1, ω_s?
4. Let $\hat{y}$ be the signal resulting from performing zero-order hold on $\bar{y}$. What is the Fourier transform of $\hat{y}$?

Question 3. Let y be a signal with the Fourier transform

$$Y(\text{j}\omega) = \text{e}^{-\frac{1}{2}\omega^2}$$

Its spectrum is shown in Fig. 2. What is the minimal sampling frequency that fulfils the Sampling Theorem?

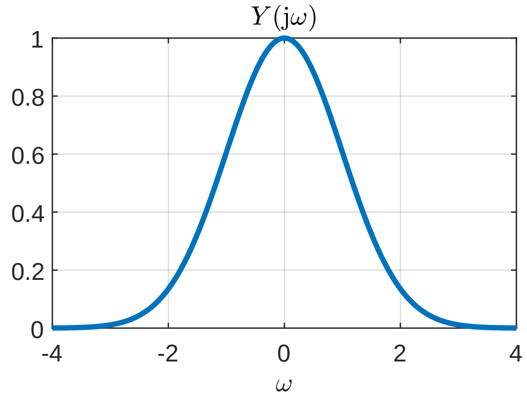

Fig. 2: Fourier transform, $Y(\mathrm{j}\omega)$

Question 4. Let x be a signal with the Fourier transform in Fig. 3.

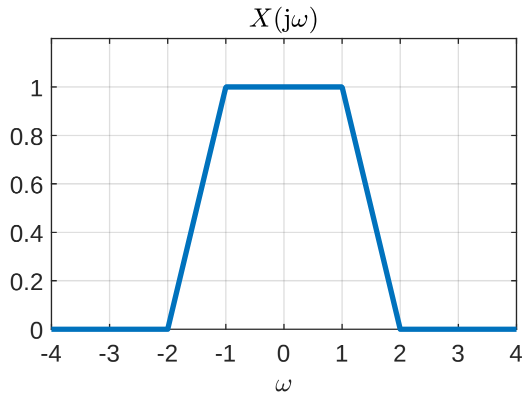

Fig. 3: Fourier transform, $X(\mathrm{j}\omega)$

Draw the spectrum of the sampled signal, sampled with a sampling period of $h = \pi$

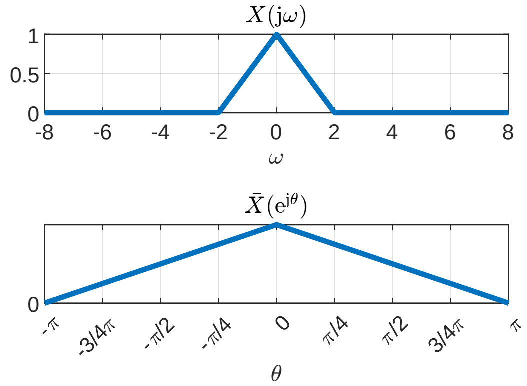

Fig. 5: Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$

4 Homework problems

Question 5. Consider the continuous time signal x and its sampled signal $\bar{x}$. The Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$ are shown in Fig. 4.

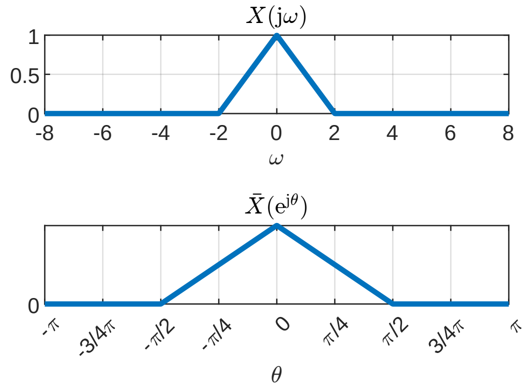

Fig. 4: Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$

1. What is the minimal sampling frequency needed to have perfect reconstruction?
2. Has the signal been distorted in the frequency domain?
3. What is the sampling period h?
4. Calculate the sampling frequency ω_s. Does this conform with the sampling theorem?
5. What is the height of the triangle $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$?

Question 6. Consider a continuous time signal x and its sampled version $\bar{x}$. The Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$ are shown in Fig. 5.

1. Has the signal been distorted in the frequency domain?
2. Calculate the sampling frequency ω_s and sampling period h.
3. What is the height of the triangle $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$?

Question 7. Consider a continuous time signal x and its sampled version $\bar{x}$. The Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$ are shown in Fig. 6.

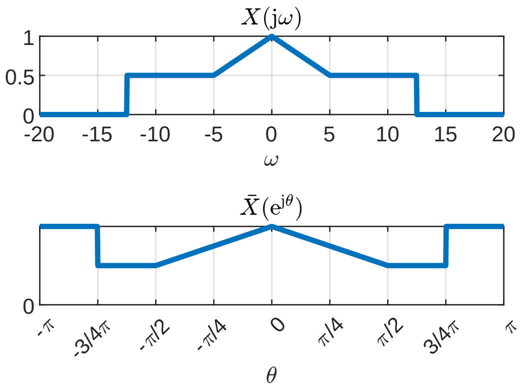

Fig. 6: Fourier transform $X(\mathrm{j}\omega)$ and DTFT $\bar{X}(\mathrm{e}^{\mathrm{j}\theta})$

1. Has the signal been distorted in the frequency domain?
2. Calculate the sampling frequency ω_s and sampling period h.
3. Does this conform with the sampling theorem.

Question 8. Let x be a signal with the Fourier transform in Fig. 7.

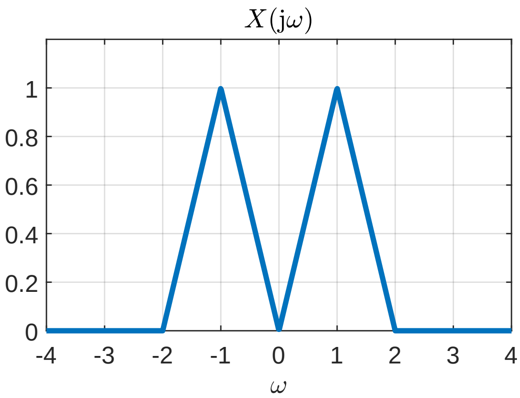

Fig. 7: Fourier transform, $X(\mathrm{j}\omega)$

Draw the spectrum of the sampled signal, sampled with a sampling frequency of $\omega_N = 1.5$