Linear Systems (034032)

tutorial 3

1 Topics

Laplace transform, Z transform, Region of Convergence, partial fraction expansion.

2 Definitions

2.1 Fourier series

If $x : \mathbb{R} \to \mathbb{R}$ is T-periodic and continuous, then we can decompose it into a Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k] \mathrm{e}^{\mathrm{j}\omega_0 k t}$$

where

$$X[k] = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \mathrm{e}^{-\mathrm{j}\omega_0 k t} \mathrm{d}t$$

and

$$\omega_0 = \frac{2\pi}{T}$$

2.2 Fourier transform

The Fourier transform is defined as

$$X(\mathrm{j}\omega) = (\mathfrak{F}\{x\})(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} x(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t$$

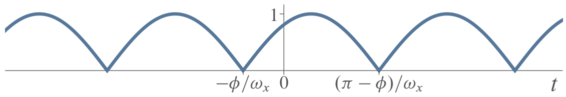
Under some mild conditions (see slides), the inverse Fourier transform is

$$x(t) = (\mathfrak{F}^{-1}\{X\})(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\mathrm{j}\omega) \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$

3 Problems

Question 1. Consider the signal x defined as

$$x(t) = |\sin(\omega_x t + \phi)| =$$



1. Identify the period T and the fundamental frequency ω_0.
2. Decompose this signal into its Fourier series.
3. Apply the Fourier transform to this signal.

Solution.

1. The period of the sine wave is $2\pi/\omega_x$, but the period of our signal is half that.

$$T = \frac{\pi}{\omega_x} \qquad \Longrightarrow \qquad \omega_0 = \frac{2\pi}{T} = 2\omega_x$$

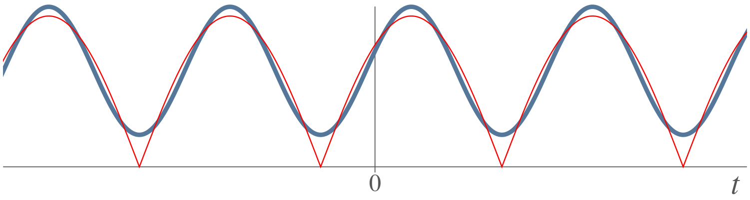
2. By the definition of the Fourier coefficient,

$$\begin{aligned}
X[k] &= \frac{1}{T}\int_{-T/2}^{T/2} |\sin(\omega_x t + \phi)| e^{-j\omega_0 k t} dt = \frac{1}{T}\int_{-\phi/\omega_x}^{T-\phi/\omega_x} \sin(\omega_x t + \phi) e^{-j\omega_0 k t} dt \\
&= \frac{1}{T}\int_{-\phi/\omega_x}^{T-\phi/\omega_x} \frac{e^{j(\omega_x t+\phi)} - e^{-j(\omega_x t+\phi)}}{2j} e^{-j2\omega_x k t} dt \\
&= \frac{1}{2jT}\int_{-\phi/\omega_x}^{T-\phi/\omega_x} \left(e^{j((\omega_x - 2\omega_x k)t+\phi)} - e^{-j((\omega_x + 2\omega_x k)t+\phi)}\right) dt \\
&= \frac{1}{2jT} e^{j\phi}\int_{-\phi/\omega_x}^{T-\phi/\omega_x} e^{j(1-2k)\omega_x t} dt - \frac{1}{2jT} e^{-j\phi}\int_{-\phi/\omega_x}^{T-\phi/\omega_x} e^{-j(1+2k)\omega_x t} dt \\
&= \frac{1}{2jT} e^{j\phi} \frac{1}{j(1-2k)\omega_x}\left[e^{j(1-2k)\omega_x t}\right]_{-\phi/\omega_x}^{T-\phi/\omega_x} + \frac{1}{2jT} e^{-j\phi} \frac{1}{j(1+2k)\omega_x}\left[e^{-j(1+2k)\omega_x t}\right]_{-\phi/\omega_x}^{T-\phi/\omega_x} \\
&= -\frac{1}{2\pi}\left(\frac{e^{j\phi}}{1-2k} e^{-j(1-2k)\phi}\left(e^{j(1-2k)\pi} - 1\right) + \frac{e^{-j\phi}}{1+2k} e^{j(1+2k)\phi}\left(e^{-j(1+2k)\pi} - 1\right)\right) \\
&= \frac{1}{\pi}\left(\frac{e^{j\phi}}{1-2k} e^{-j(1-2k)\phi} + \frac{e^{-j\phi}}{1+2k} e^{j(1+2k)\phi}\right) = \frac{1}{\pi}\left(\frac{e^{j2\phi k}}{1-2k} + \frac{e^{j2\phi k}}{1+2k}\right) \\
&= \frac{e^{j2\phi k}}{\pi}\left(\frac{1+2k+1-2k}{1-4k^2}\right) = \frac{2e^{j2\phi k}}{(1-4k^2)\pi},
\end{aligned}$$

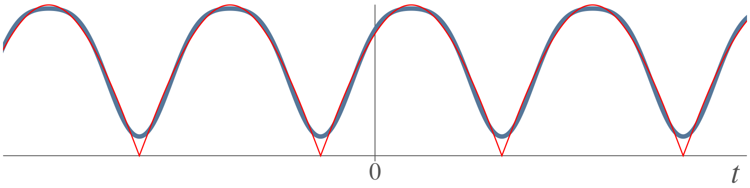
where the fact that $|\sin(\omega t + \phi)|e^{-j\omega_0 k t}$ is T-periodic for all $k \in \mathbb{Z}$ and is used, in addition to $T\omega_x = \pi$.

In figure 1, we show the partial Fourier series.

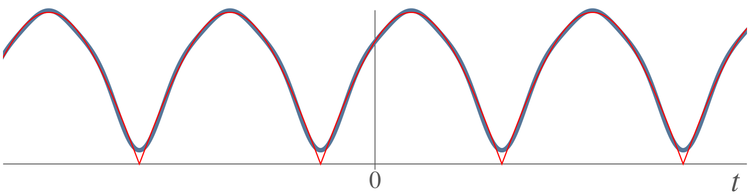
$$x_N(t) = \sum_{k=-N}^{N} X[k] e^{j\omega_0 k t}$$



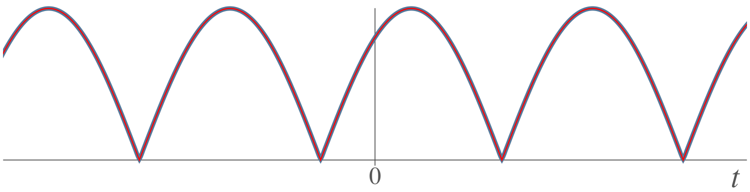
(a) $N = 1$



(b) $N = 2$



(c) $N = 3$



(d) $N = 51$

Fig. 1: Partial Fourier series.

3. A T-periodic and continuous signal x can easily be transformed to the Fourier domain. We shall denote the Fourier coefficients by $\mathcal{X}[k]$ in order to avoid confusion with $X(\mathrm{j}\omega)$.

$$x(t) = \sum_{k=-\infty}^{+\infty} \mathcal{X}[k]\mathrm{e}^{\mathrm{j}\omega_0 k t}$$

$$\Rightarrow X(\mathrm{j}\omega) = \left(\mathfrak{F}\left\{\sum_{k=-\infty}^{+\infty} \mathcal{X}[k]\mathrm{e}^{\mathrm{j}\omega_0 k t}\right\}\right)(\omega) = \sum_{k=-\infty}^{+\infty} \mathcal{X}[k]\mathfrak{F}\left\{\mathrm{e}^{\mathrm{j}\omega_0 k t}\right\}$$

$$\Rightarrow X(\mathrm{j}\omega) = 2\pi \sum_{k=-\infty}^{+\infty} \mathcal{X}[k]\delta(\omega - k\omega_0),$$

where we used the fact that $\mathfrak{F}\left\{\mathrm{e}^{\mathrm{j}\omega_0 k t}\right\} = 2\pi\delta(\omega - k\omega_0)$ as was seen in the lecture. We can now use this result to calculate the Fourier spectrum knowing that $\mathcal{X}[k] = \frac{2\mathrm{e}^{\mathrm{j}2\phi k}}{(1-4k^2)\pi}$.

$$X(\mathrm{j}\omega) = 2\pi \sum_{k=-\infty}^{+\infty} \frac{2\mathrm{e}^{\mathrm{j}2\phi k}}{(1-4k^2)\pi}\delta(\omega - k\omega_0) = 4 \sum_{k=-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{j}2\phi k}}{1-4k^2}\delta(\omega - k\omega_0)$$

The magnitude of the Fourier spectrum is shown in figure 2.

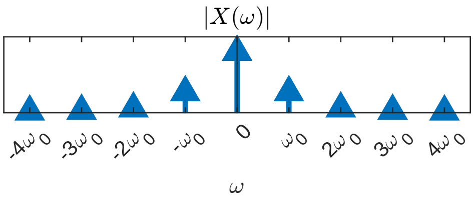


Fig. 2: Fourier spectrum.

▽

Question 2. Match the signals in the time domain (figure 3) to their corresponding magnitude Fourier spectrum (figure 4).

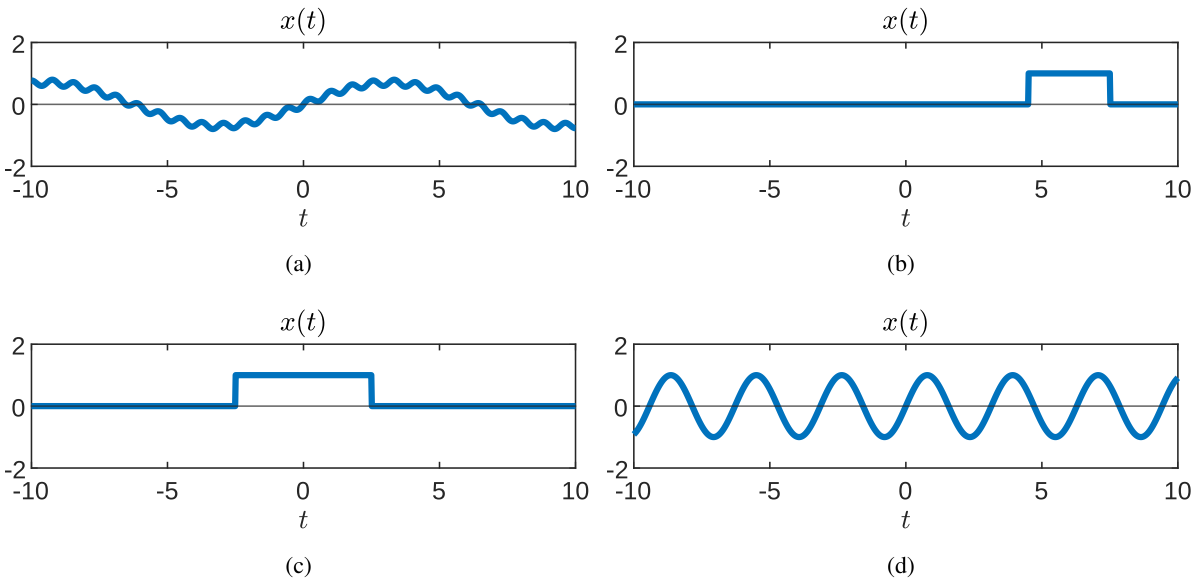


Fig. 3: Signals in the time domain.

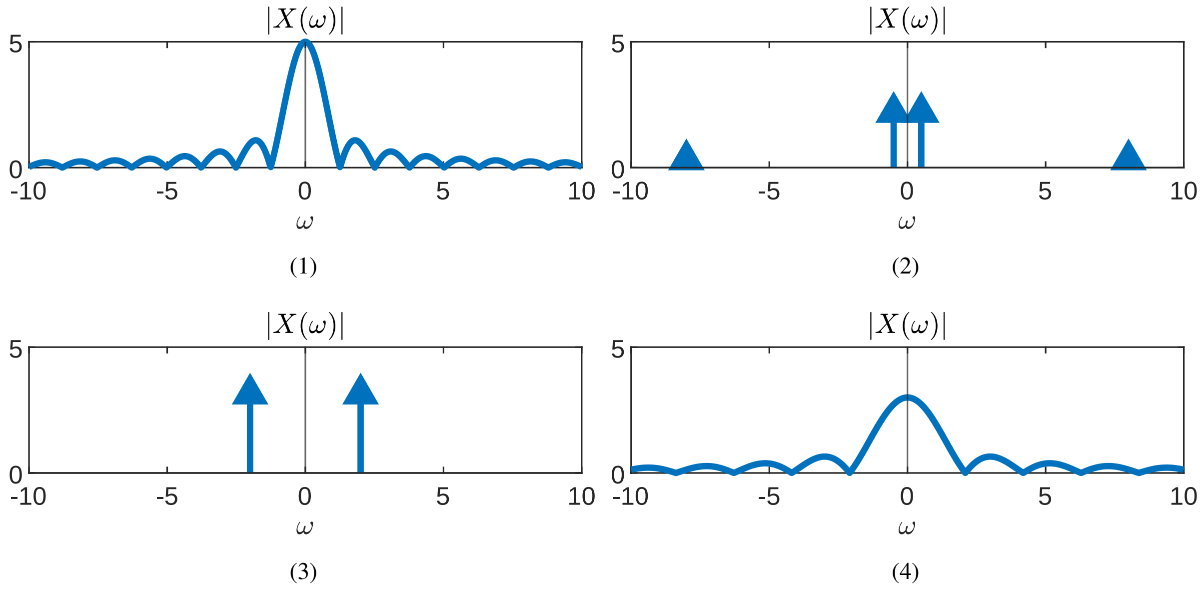


Fig. 4: Signals in the frequency domain.

Solution.

$x(t)$	$\lvert X(\mathrm{j}\omega)\rvert$
(a)	(2)
(b)	(4)
(c)	(1)
(d)	(3)

Periodic signals are always characterized by Dirac pulses in the Fourier domain. Signals (a) and (d) only have sinusoidal components and therefore fall into this category. The main contribution of signal (a) is slower than signal (d), but signal (a) also has an extra high frequency component.

Signals (b) and (c) are rectangular pulses and are therefore aperiodic. The shorter pulse is more stretched in the Fourier domain and vice versa. This is known as the time scaling property. Take $\varsigma > 0$ (see question 8 for $\varsigma \in \mathbb{R}$). If $y(t) = x(\varsigma t)$, then

$$
\begin{aligned}
Y(\mathrm{j}\omega) &= \int_{-\infty}^{+\infty} x(\varsigma t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{\varsigma}\int_{-\infty}^{+\infty} x(\varsigma t)\mathrm{e}^{-\mathrm{j}\frac{\omega}{\varsigma}(\varsigma t)}\mathrm{d}(\varsigma t) \\
&= \frac{1}{\varsigma}\int_{-\infty}^{+\infty} x(v)\mathrm{e}^{-\mathrm{j}\frac{\omega}{\varsigma}v}\mathrm{d}v = \frac{1}{\varsigma}X\left(\frac{1}{\varsigma}\mathrm{j}\omega\right),
\end{aligned}
$$

with $v = \varsigma t$. Therefore, stretching the signal in the time domain will cause the signal to contract in the Fourier domain.

What about a shift in time as is the case in signal (b)? If $y(t) = x(t+\tau)$, then

$$
\begin{aligned}
Y(\mathrm{j}\omega) &= \int_{-\infty}^{+\infty} x(t+\tau)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \mathrm{e}^{\mathrm{j}\omega\tau}\int_{-\infty}^{+\infty} x(t+\tau)\mathrm{e}^{-\mathrm{j}\omega(t+\tau)}\mathrm{d}(t+\tau) \\
&= \mathrm{e}^{\mathrm{j}\omega\tau}\int_{-\infty}^{+\infty} x(v)\mathrm{e}^{-\mathrm{j}\omega v}\mathrm{d}v = \mathrm{e}^{\mathrm{j}\omega\tau}X(\mathrm{j}\omega),
\end{aligned}
$$

with $v = x + \tau$. Therefore, the shift in time only influences the phase of the signal and therefore has no effect on the the magnitude of the spectrum $|X(\mathrm{j}\omega)|$, which is what we plot in figure 4. ∇

Question 3. Consider the signal y shown in figure 5 defined as

$$y = \mathbb{S}_{1/2}\,\mathrm{rect} - \mathbb{S}_{-1/2}\,\mathrm{rect} \qquad \Longrightarrow \qquad y(t) = \mathrm{rect}\big(t + \tfrac{1}{2}\big) - \mathrm{rect}\big(t - \tfrac{1}{2}\big),$$

with

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}.$$

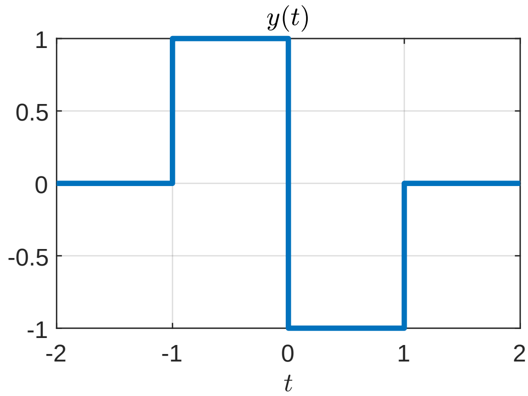


Fig. 5: Signal used in question 3.

1. Show that $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tent}(t)$.
2. Find the Fourier transform of y.
3. Find the Fourier transform of the following signal by using the previous result.

$$y(t) = \mathrm{rect}\left(\frac{t}{a} + \frac{1}{2}\right) - \mathrm{rect}\left(\frac{t}{a} - \frac{1}{2}\right), \text{with } a > 0$$

Solution.

1. The tent function is given by

$$\mathrm{tent}(t) = \begin{cases} 1 - |t| & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases} = \begin{cases} 1 - t & \text{if } 0 \le t \le 1 \\ 1 + t & \text{if } -1 \le t \le 0 \\ 0 & \text{if } |t| > 1 \end{cases}.$$

   Deriving this gives

$$\frac{\mathrm{d}\,\mathrm{tent}(t)}{\mathrm{d}t} = \begin{cases} -1 & \text{if } 0 < t < 1 \\ 1 & \text{if } -1 < t < 0 \\ 0 & \text{if } |t| > 1 \end{cases},$$

   which is equal to our signal y.

2. We will use the following two things you've seen in the lecture to find the Fourier transform of x.

$$(\mathfrak{F}\{\mathrm{tent}\})(\mathrm{j}\omega) = \mathrm{sinc}^2(\omega/2)$$

$$y = \mathrm{d}x/\mathrm{d}t \quad \text{and} \quad \lim_{t \to \pm\infty} x(t) = 0 \quad \Longrightarrow \quad Y(\mathrm{j}\omega) = \mathrm{j}\omega X(\mathrm{j}\omega)$$

The first statement is left as an exercise to the reader (see question 10). The second statement is the differentiation property and will be proven here.

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int_{-\infty}^{+\infty} y(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}x(t) \\ &= \left[\mathrm{e}^{-\mathrm{j}\omega t}x(t)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} x(t)\mathrm{d}(\mathrm{e}^{-\mathrm{j}\omega t}) = \mathrm{j}\omega \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \mathrm{j}\omega X(\mathrm{j}\omega), \end{aligned}$$

where we used integration by parts. Combining both of these results, we get

$$Y(\mathrm{j}\omega) = \mathrm{j}\omega(\mathfrak{F}\{\text{tent}\})(\mathrm{j}\omega) = \mathrm{j}\omega\,\mathrm{sinc}^2(\omega/2)$$

3. We will use the time scaling property:

$$y(t) = x(\varsigma t) \Rightarrow Y(\mathrm{j}\omega) = \frac{1}{|\varsigma|} X\left(\frac{1}{\varsigma}\mathrm{j}\omega\right)$$

In our case $\varsigma = 1/a > 0$. Therefore,

$$Y(\mathrm{j}\omega) = \frac{1}{1/a}\mathrm{j}\frac{\omega}{1/a}\,\mathrm{sinc}^2\left(\frac{\omega}{1/a}/2\right) = \mathrm{j}\omega a^2\,\mathrm{sinc}^2(a\omega/2)$$

∇

Question 4. Consider $y = \mathrm{sinc}$, where

$$\mathrm{sinc}(t) := \frac{\sin(t)}{t}.$$

1. Find the Fourier transform of this signal.
2. Use Parseval's theorem,

$$\|y\|_2^2 = \frac{1}{2\pi}\|Y\|_2^2,$$

where $\|y\|_2^2 = \int_{-\infty}^{+\infty}|y(t)|^2\mathrm{d}t$ and $\|Y\|_2^2 = \int_{-\infty}^{+\infty}|Y(\mathrm{j}\omega)|^2\mathrm{d}\omega$, to calculate the 2-norm squared of y.

Solution.

1. We will use the following result you've seen in the lecture (see also question 9).

$$(\mathfrak{F}\{\text{rect}\})(\mathrm{j}\omega) = \mathrm{sinc}(\omega/2)$$

We will also use the duality property, which we will prove here once again. If $y = X|_{\omega=t}$, i.e. $y(t) = X(\mathrm{j}t)$, where $X(\mathrm{j}\omega) = (\mathfrak{F}\{x\})(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$, then

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{+\infty} y(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{-\infty}^{+\infty} X(\mathrm{j}t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \\ &= \frac{2\pi}{2\pi}\int_{-\infty}^{+\infty} X(\mathrm{j}\tilde{\omega})\mathrm{e}^{\mathrm{j}(-\omega)\tilde{\omega}}\mathrm{d}\tilde{\omega} = 2\pi x(-\omega) \end{aligned}$$

where we used $t = \tilde{\omega}$ in addition to the definition of the inverse Fourier transform.

Now we can apply these things to our signal. $y(t) = \text{sinc}(t) = X(\text{j}t)$. By using the scaling property and linearity, we know that

$$\begin{gathered}
(\mathfrak{F}\{\text{rect}\})(\text{j}\omega) = \text{sinc}(\omega/2) \\
\Downarrow \\
(\mathfrak{F}\{\mathbb{P}_{1/2}\,\text{rect}\})(\text{j}\omega) = 2\,\text{sinc}(\omega), \qquad \text{where} \quad (\mathbb{P}_{1/2}\,\text{rect})(t) = \text{rect}(t/2) \\
\Downarrow \\
\left(\mathfrak{F}\left\{\frac{1}{2}\mathbb{P}_{1/2}\,\text{rect}\right\}\right)(\text{j}\omega) = \text{sinc}(\omega)
\end{gathered}$$

From this we can deduce that the left-hand-side is $x(t)$, i.e. $x(t) = \text{rect}(t/2)/2$. Therefore,

$$Y(\text{j}\omega) = 2\pi x(-\omega) = 2\pi\frac{1}{2}\text{rect}(-\omega/2) = \pi\,\text{rect}(-\omega/2) = \pi\,\text{rect}(\omega/2),$$

where the fact that $\text{rect}(-t) = \text{rect}(t)$ was used.

2. It is not trivial at all to calculate the integral of the sinc^2. Luckily, we can use Parseval's theorem.

$$\begin{aligned}
\|y\|_2^2 &= \int_{-\infty}^{+\infty} \text{sinc}^2(t)\text{d}t = \frac{1}{2\pi}\|Y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\pi\,\text{rect}(\omega/2)|^2\text{d}\omega \\
&= \frac{\pi}{2}\int_{-\infty}^{+\infty}\text{rect}(\omega/2)\text{d}\omega\,\Big|_{\tilde{\omega}=\omega/2} = \pi\int_{-\infty}^{+\infty}\text{rect}(\tilde{\omega})\text{d}\tilde{\omega} = \pi\int_{-0.5}^{0.5}\text{d}\tilde{\omega} = \pi,
\end{aligned}$$

where we used the fact that $\text{rect}^2 = \text{rect}$.

$\triangledown$

4 Homework problems

Question 5. Consider the signal x defined as

$$x(t) = 5\sin(2t) + 0.1\cos(10t)$$

1. Our signal contains a "slow" and a "fast" part. Identify them.
2. Identify the period T and the fundamental frequency ω_0.
3. Decompose this signal into its Fourier series.
4. Apply the Fourier transform to this signal.

Solution.

1. $5\sin(2t)$ is the slow part, while $0.1\cos(10t)$ is faster, because $2 < 10$.
2. The period is the least common multiple of the periods of both parts. In our case it's simple because 10 is a multiple of 2.

$$\omega_0 = 2$$
$$T = \frac{2\pi}{\omega_0} = \pi$$

3. We can use a trick here to make our lives easier.

$$\sin(2t) = \frac{\mathrm{e}^{\mathrm{j}2t} - \mathrm{e}^{-\mathrm{j}2t}}{2\mathrm{j}} = \frac{\mathrm{e}^{\mathrm{j}\omega_0 t} - \mathrm{e}^{-\mathrm{j}\omega_0 t}}{2\mathrm{j}}$$
$$\cos(10t) = \frac{\mathrm{e}^{\mathrm{j}10t} + \mathrm{e}^{-\mathrm{j}10t}}{2} = \frac{\mathrm{e}^{\mathrm{j}5\omega_0 t} + \mathrm{e}^{-\mathrm{j}5\omega_0 t}}{2}$$

$$\begin{aligned}\Rightarrow x(t) &= 5\frac{\mathrm{e}^{\mathrm{j}\omega_0 t} - \mathrm{e}^{-\mathrm{j}\omega_0 t}}{2\mathrm{j}} + 0.1\frac{\mathrm{e}^{\mathrm{j}5\omega_0 t} + \mathrm{e}^{-\mathrm{j}5\omega_0 t}}{2} \\ &= -2.5\mathrm{j}\mathrm{e}^{\mathrm{j}\omega_0 t} + 2.5\mathrm{j}\mathrm{e}^{-\mathrm{j}\omega_0 t} + 0.05\mathrm{e}^{\mathrm{j}5\omega_0 t} + 0.05\mathrm{e}^{-\mathrm{j}5\omega_0 t}\end{aligned}$$

Now, we can easily identify the Fourier coefficients.

$$X[k] = \begin{cases} \mp 2.5\mathrm{j} & k = \pm 1 \\ 0.05 & k = \pm 5 \\ 0 & \text{otherwise} \end{cases}$$

4. We can use the result from question 1.

$$\begin{aligned}X(\mathrm{j}\omega) &= 2\pi \sum_{k=-\infty}^{+\infty} \mathcal{X}[k]\delta(\omega - k\omega_0) \\ &= 2\pi\left(-2.5\mathrm{j}\delta(\omega - \omega_0) + 2.5\mathrm{j}\delta(\omega + \omega_0) + 0.05\delta(\omega - 5\omega_0) + 0.05\delta(\omega + 5\omega_0)\right)\end{aligned}$$

∇

Question 6. Show that $X(\mathrm{j}\omega) = \overline{X(-\mathrm{j}\omega)}$ if $x(t) \in \mathbb{R}$.

Solution.

$$\overline{X(-\mathrm{j}\omega)} = \overline{\int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{j}(-\omega)t}\mathrm{d}t} = \int_{-\infty}^{+\infty} \overline{x(t)}\,\overline{\mathrm{e}^{\mathrm{j}\omega t}}\mathrm{d}t = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = X(\mathrm{j}\omega),$$

where we used the fact that $\overline{x(t)} = x(t)$. An important consequence of this is that $|X(\mathrm{j}\omega)| = |X(-\mathrm{j}\omega)|$ and $\arg X(\mathrm{j}\omega) = -\arg X(-\mathrm{j}\omega)$.

We can also prove this by using the conjugation property, i.e. if $y = \overline{x}$, then $Y(\mathrm{j}\omega) = \overline{X(-\mathrm{j}\omega)}$. Applying this to our real signal $x(t) \in \mathbb{R}$, we get

$$x = \overline{x} \Rightarrow X(\mathrm{j}\omega) = \overline{X(-\mathrm{j}\omega)}$$

▽

Question 7. Consider the signal x defined as

$$x(t) = |\sin(\omega_x t)|$$

1. Identify the period T and the fundamental frequency ω_0.
2. Decompose this signal into its Fourier series.
3. Derive the time shift property of Fourier series. If $y = \mathbb{S}_\tau x$, then $Y[k] = \mathrm{e}^{\mathrm{j}\omega_0\tau k}X[k]$
4. Use the previous results and the time shift property to derive the Fourier series of the signal in question 1, i.e. $x(t) = |\sin(\omega_x t + \phi)|$.

Solution.

1. $T = \frac{\pi}{\omega_x}$, $\omega_0 = \frac{2\pi}{T} = 2\omega_x$.
2. $X[k] = \frac{2}{(1-4k^2)\pi}$.
3. If $y(t) = x(t+\tau)$, then
$$\begin{aligned} Y[k] &= \int_{-T/2}^{+T/2} y(t)\mathrm{e}^{-\mathrm{j}\omega_0 kt}\mathrm{d}t = \int_{-T/2}^{+T/2} x(t+\tau)\mathrm{e}^{-\mathrm{j}\omega_0 kt}\mathrm{d}t \\ &= \mathrm{e}^{\mathrm{j}\omega_0\tau k}\int_{-T/2}^{+T/2} x(t+\tau)\mathrm{e}^{-\mathrm{j}\omega_0 k(t+\tau)}\mathrm{d}(t+\tau)\Big|_{s=t+\tau} = \mathrm{e}^{\mathrm{j}\omega_0\tau k}\int_{-T/2+\tau}^{+T/2+\tau} x(s)\mathrm{e}^{-\mathrm{j}\omega_0 ks}\mathrm{d}s \\ &= \mathrm{e}^{\mathrm{j}\omega_0\tau k}\int_{-T/2}^{+T/2} x(s)\mathrm{e}^{-\mathrm{j}\omega_0 ks}\mathrm{d}s = \mathrm{e}^{\mathrm{j}\omega_0\tau k}X[k], \end{aligned}$$
   where we used the fact that $x(s)\mathrm{e}^{-\mathrm{j}\omega_0 ks}$ is T-periodic.
4. The time shift is equal to $\tau = \phi/\omega_x = 2\phi/\omega_0$. Therefore, we get $\mathrm{e}^{\mathrm{j}\omega_0\tau k} = \mathrm{e}^{\mathrm{j}2\phi k}$. Thus, $X[k] = \frac{2\mathrm{e}^{\mathrm{j}2\phi k}}{(1-4k^2)\pi}$.

▽

Question 8. Show that for any $\varsigma \in \mathbb{R}$, the following is true.

$$y = \mathbb{P}_\varsigma x = x(\varsigma t) \Rightarrow Y(\mathrm{j}\omega) = \frac{1}{|\varsigma|}X\left(\frac{1}{\varsigma}\mathrm{j}\omega\right)$$

Solution. See the lecture slides. ▽

Question 9. Show that $\mathfrak{F}\{\text{rect}\} = \text{sinc}(\omega/2)$.

Solution. See the lecture slides. ▽

Question 10. Show that $\mathfrak{F}\{\text{tent}\} = \text{sinc}^2(\omega/2)$.

Solution. See the lecture slides. ▽

Question 11 (Signals and Transforms, exercise 3.10)**.** Let $x(t)$ be the 2π-periodic signal such that

$$x(t) = t^2, \quad -\pi \le t \le \pi$$

Determine the Fourier coefficients $X[k]$ and write down the Fourier series of x.

Solution.

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]e^{jkt}$$

where $\omega_0 = 1$ and

$$X[k] = \begin{cases} \frac{\pi^2}{3} & k = 0 \\ (-1)^k \frac{2}{k^2} & k \neq 0 \end{cases}$$

▽

Question 12 (Signals and Transforms, exercise 4.4(f))**.**

1. Determine the Fourier transform of the following signal.

$$x(t) = e^{-|t|}$$

2. Use the above result to calculate the Fourier transform of this signal.

$$x(t) = \frac{1}{1+t^2}$$

Solution.

1. By integration, we get $(\mathfrak{F}\{e^{-|t|}\})(\omega) = \frac{2}{1+\omega^2}$.
2. By using linearity, we know that

$$\left(\mathfrak{F}\left\{\frac{1}{2}e^{-|t|}\right\}\right)(\omega) = \frac{1}{1+\omega^2}.$$

By duality, we have

$$\left(\mathfrak{F}\left\{\frac{1}{1+t^2}\right\}\right)(\omega) = 2\pi x(-\omega) = 2\pi\frac{1}{2}e^{-|-\omega|} = \pi e^{-|\omega|}.$$

▽