הטכניון - מכון טכנולוגי לישראל, הפקולטה להנדסת מכונות
TECHNION — Israel Institute of Technology, Faculty of Mechanical Engineering

Linear Systems (034032)

Tutorial 3

1 Topics

Laplace transform, Z transform, Region of Convergence, partial fraction expansion.

2 Definitions

2.1 Fourier series

If $x : \mathbb{R} \to \mathbb{R}$ is T-periodic and continuous, then we can decompose it into a Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k] \mathrm{e}^{\mathrm{j}\omega_0 k t}$$

where

$$X[k] = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \mathrm{e}^{-\mathrm{j}\omega_0 k t} \mathrm{d}t$$

and

$$\omega_0 = \frac{2\pi}{T}$$

2.2 Fourier transform

The Fourier transform is defined as

$$X(\mathrm{j}\omega) = (\mathfrak{F}\{x\})(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} x(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t$$

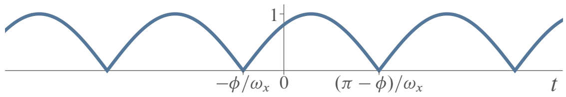
Under some mild conditions (see slides), the inverse Fourier transform is

$$x(t) = (\mathfrak{F}^{-1}\{X\})(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\mathrm{j}\omega) \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$

3 Problems

Question 1. Consider the signal x defined as

$$x(t) = |\sin(\omega_x t + \phi)| =$$



1. Identify the period T and the fundamental frequency ω_0.
2. Decompose this signal into its Fourier series.
3. Apply the Fourier transform to this signal.

Question 2. Match the signals in the time domain (figure 1) to their corresponding magnitude Fourier spectrum (figure 2).

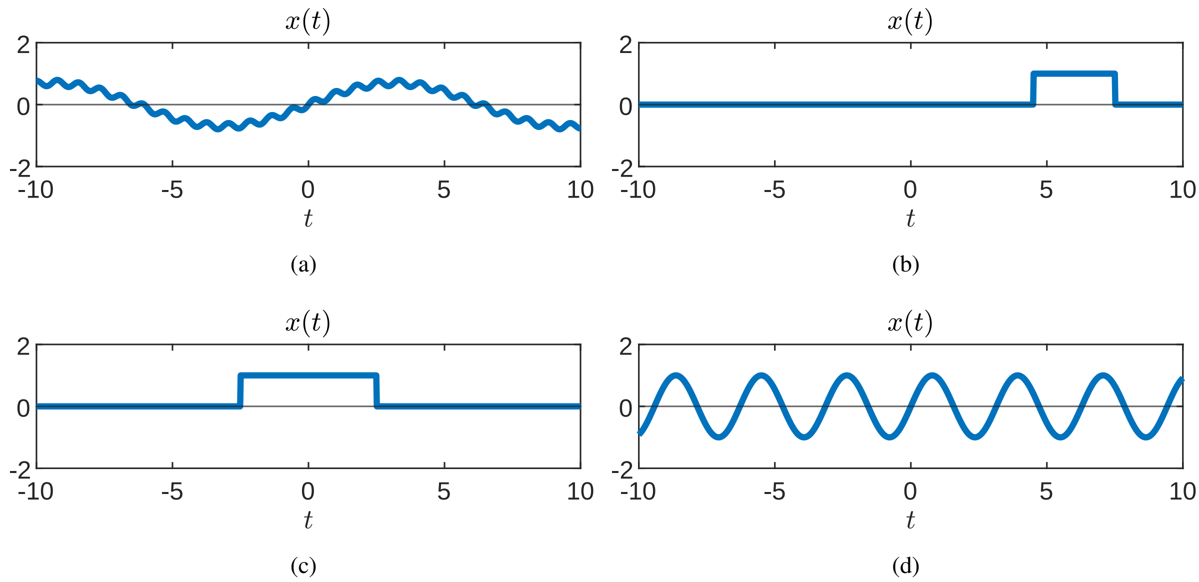


Fig. 1: Signals in the time domain.

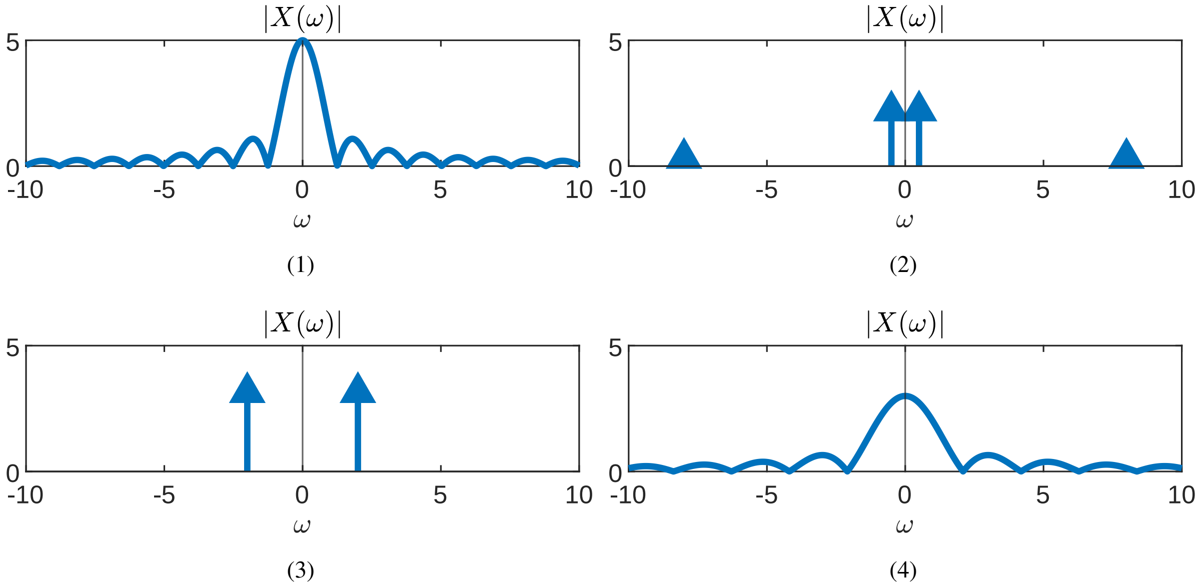


Fig. 2: Signals in the frequency domain.

Question 3. Consider the signal y shown in figure 3 defined as

$$y = \mathbb{S}_{1/2}\,\mathrm{rect} - \mathbb{S}_{-1/2}\,\mathrm{rect} \qquad \Longrightarrow \qquad y(t) = \mathrm{rect}\big(t + \tfrac{1}{2}\big) - \mathrm{rect}\big(t - \tfrac{1}{2}\big),$$

with

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}.$$

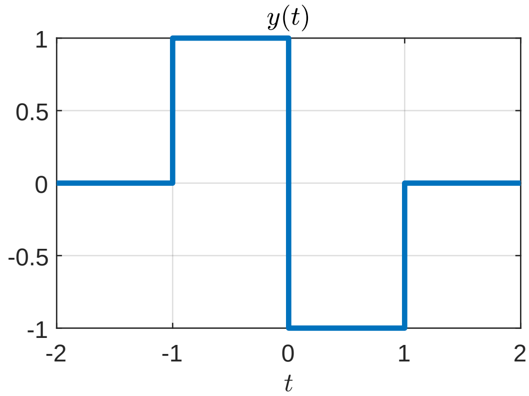


Fig. 3: Signal used in question 3.

1. Show that $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tent}(t)$.
2. Find the Fourier transform of y.
3. Find the Fourier transform of the following signal by using the previous result.

$$y(t) = \mathrm{rect}\left(\frac{t}{a} + \frac{1}{2}\right) - \mathrm{rect}\left(\frac{t}{a} - \frac{1}{2}\right), \text{ with } a > 0$$

Question 4. Consider $y = \mathrm{sinc}$, where

$$\mathrm{sinc}(t) := \frac{\sin(t)}{t}.$$

1. Find the Fourier transform of this signal.

2. Use Parseval's theorem,

$$\|y\|_2^2 = \frac{1}{2\pi}\|Y\|_2^2,$$

   where $\|y\|_2^2 = \int_{-\infty}^{+\infty}|y(t)|^2\mathrm{d}t$ and $\|Y\|_2^2 = \int_{-\infty}^{+\infty}|Y(\mathrm{j}\omega)|^2\mathrm{d}\omega$, to calculate the 2-norm squared of y.

4 Homework problems

Question 5. Consider the signal x defined as

$$x(t) = 5\sin(2t) + 0.1\cos(10t)$$

1. Our signal contains a "slow" and a "fast" part. Identify them.
2. Identify the period T and the fundamental frequency ω_0.
3. Decompose this signal into its Fourier series.
4. Apply the Fourier transform to this signal.

Question 6. Show that $X(\mathrm{j}\omega) = \overline{X(-\mathrm{j}\omega)}$ if $x(t) \in \mathbb{R}$.

Question 7. Consider the signal x defined as

$$x(t) = |\sin(\omega_x t)|$$

1. Identify the period T and the fundamental frequency ω_0.
2. Decompose this signal into its Fourier series.
3. Derive the time shift property of Fourier series. If $y = \mathbb{S}_\tau x$, then $Y[k] = \mathrm{e}^{\mathrm{j}\omega_0 \tau k} X[k]$
4. Use the previous results and the time shift property to derive the Fourier series of the signal in question 1, i.e. $x(t) = |\sin(\omega_x t + \phi)|$.

Question 8. Show that for any $\varsigma \in \mathbb{R}$, the following is true.

$$y = \mathbb{P}_\varsigma x = x(\varsigma t) \Rightarrow Y(\mathrm{j}\omega) = \frac{1}{|\varsigma|} X\left(\frac{1}{\varsigma}\mathrm{j}\omega\right)$$

Question 9. Show that $\mathfrak{F}\{\mathrm{rect}\} = \mathrm{sinc}(\omega/2)$.

Question 10. Show that $\mathfrak{F}\{\mathrm{tent}\} = \mathrm{sinc}^2(\omega/2)$.

Question 11 (Signals and Transforms, exercise 3.10)**.** Let $x(t)$ be the 2π-periodic signal such that

$$x(t) = t^2, \quad -\pi \leq t \leq \pi$$

Determine the Fourier coefficients $X[k]$ and write down the Fourier series of x.

Question 12 (Signals and Transforms, exercise 4.4(f))**.**

1. Determine the Fourier transform of the following signal.

$$x(t) = \mathrm{e}^{-|t|}$$

2. Use the above result to calculate the Fourier transform of this signal.

$$x(t) = \frac{1}{1+t^2}$$