Linear Systems (034032)

TUTORIAL 2

1 Topics

Signal norms, sampling, reconstruction, standard signals, complex number review.

2 Problems

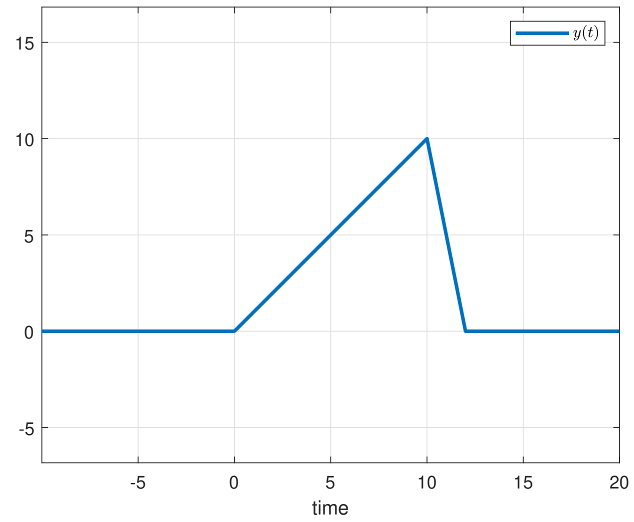


Fig. 1: Triangle signal

Question 1. Consider the continuous time signal

$$y(t) = \begin{cases} t & 0 \le t < 10 \\ 60 - 5t & 10 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$$

shown in Fig. 1. Construct y using the step function, $\mathbb{1}$, and the ramp signal, ramp.

Solution.

$$y = \text{ramp} - 6\mathbb{S}_{-10}\,\text{ramp} + 5\mathbb{S}_{-12}\,\text{ramp} \quad \implies \quad y(t) = t\mathbb{1}(t) - 6(t-10)\mathbb{1}(t-10) + 5(t-12)\mathbb{1}(t-12)$$

which is what we need. ▽

Question 2. Let y be the signal from Question 1. Let $f := f_1 + f_2 + f_3$, where $f_1 = \delta$, $f_2 = 3\mathbb{S}_{-2}\delta$, and $f_3 = 0.5\mathbb{S}_{-3}\delta$ i.e.

$$f(t) = \delta(t) + 3\delta(t-2) + 0.5\delta(t-3)$$

Find the convolution $y * f$.

Solution. Remember the following definition / properties:

- $(y * f)(t) = \int_{-\infty}^{\infty} y(s) f(t-s) \mathrm{d}s = \int_{-\infty}^{\infty} y(t-s) f(s) \mathrm{d}s$
- $y * (f_1 + f_2 + f_3) = y * f_1 + y * f_2 + y * f_3$
- $\int_{-\infty}^{\infty} y(t)\delta(t - t_0)\mathrm{d}t = y(t_0)$ (sifting property)

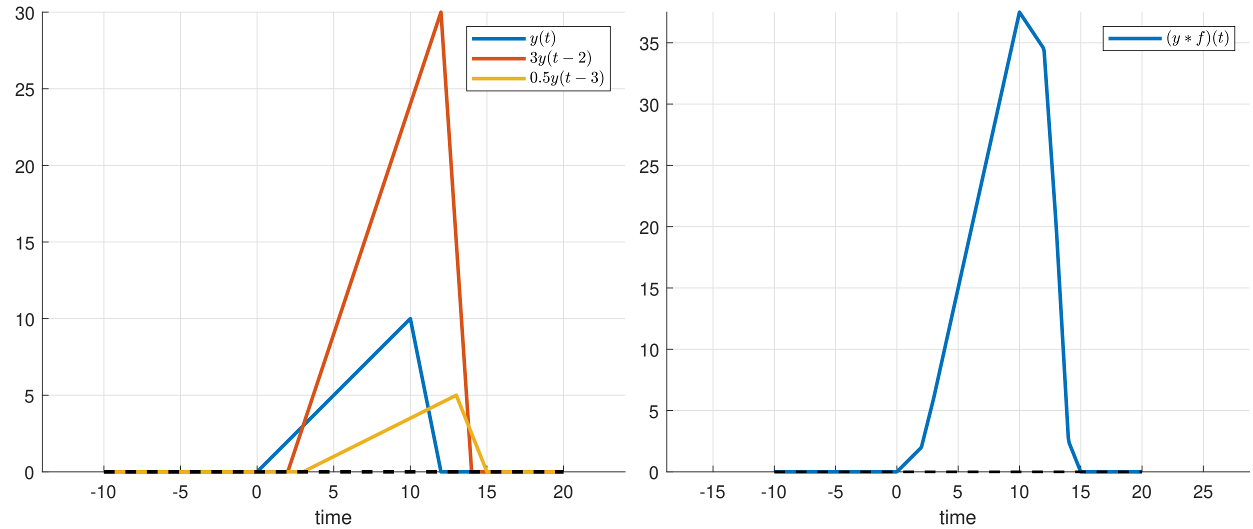
Hence,

$$
\begin{aligned}
(y * f_1)(t) &= \int_{-\infty}^{\infty} y(t-s)\delta(s)\mathrm{d}s = y(t) \\
(y * f_2)(t) &= \int_{-\infty}^{\infty} y(t-s)3\delta(s-2)\mathrm{d}s = 3y(t-2) \\
(y * f_3)(t) &= \int_{-\infty}^{\infty} y(t-s)0.5\delta(s-3)\mathrm{d}s = 0.5y(t-3)
\end{aligned}
$$

and we end up with

$$
y * f = y + 3\mathbb{S}_{-2}y + 0.5\mathbb{S}_{-3}y \quad \Longrightarrow \quad (y * f)(t) = y(t) + 3y(t-2) + 0.5y(t-3),
$$

see Fig. 2(a).



(a) Shifted and scaled triangles

(b) Convolution $y * f$

Fig. 2: Plots for question 2 solution.

Fig. 2(b) shows the plot of the resulting $y * f$. ▽

Question 3. Let y be the signal from Question 1.

1. Compute the L_∞ and L_2 norms of y.
2. Find the ideally sampled signal $\bar{y}$ with the sampling period $h = 2$.
3. Compute the ℓ_∞ and ℓ_2 norms of $\bar{y}$.

4. Convert the sampled signal $\bar{y}$ back the the continuous time using the ZOH to produce an analog y_{ZOH}.
5. Compute the error, $e := y - y_{\text{ZOH}}$, and its L_∞ and L_2 norms.

Solution. Remember that the L_2 and L_∞ norms of continuous-time signals are defined as

$$\|f\|_2 := \left(\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t\right)^{1/2} \quad \text{and} \quad \|f\|_\infty := \sup_t |f(t)|$$

and the ℓ_2 and ℓ_∞ norms of discrete-time signals are defined as

$$\|f\|_2 := \left(\sum_{t=-\infty}^{\infty} |f[t]|^2\right)^{1/2} \quad \text{and} \quad \|f\|_\infty := \sup_t |f[t]|$$

1. From definitions,

$$\|y\|_\infty = \sup_t |y(t)| = 10$$

and

$$\begin{aligned}\|y\|_2^2 = \int_{-\infty}^{\infty} |y(t)|^2 \mathrm{d}t &= \int_0^{10} t^2 \mathrm{d}t + \int_{10}^{12} (60-5t)^2 \mathrm{d}t \\ &= \frac{t^3}{3}\Big|_0^{10} + \frac{(60-5t)^3}{-15}\Big|_{10}^{12} = \frac{10^3-0}{3} + \frac{0-10^3}{-15} = 400,\end{aligned}$$

so that

$$\|y(t)\|_2 = \sqrt{400} = 20.$$

2. The ideal sampler assigns $\bar{y}[i] = y(ih)$. Hence,

$$\bar{y}[i] = \begin{cases} ih & 0 \le i \le 5 \\ 0 & \text{otherwise} \end{cases} = \{\ldots, 0, 2, 4, 6, 8, 10, 0, \ldots\}$$

see Fig. 3(a).

3. From definitions,

$$\|\bar{y}\|_\infty = \sup_i |\bar{y}[i]| = \sup\{\ldots, 0, 2, 4, 6, 8, 10, 0, \ldots\} = 10$$

and

$$\|\bar{y}\|_2^2 = \sum_{i=-\infty}^{\infty} |\bar{y}[i]|^2 = \sum_{i=1}^{5} |\bar{y}[i]|^2 = 4 + 16 + 36 + 64 + 100 = 220$$

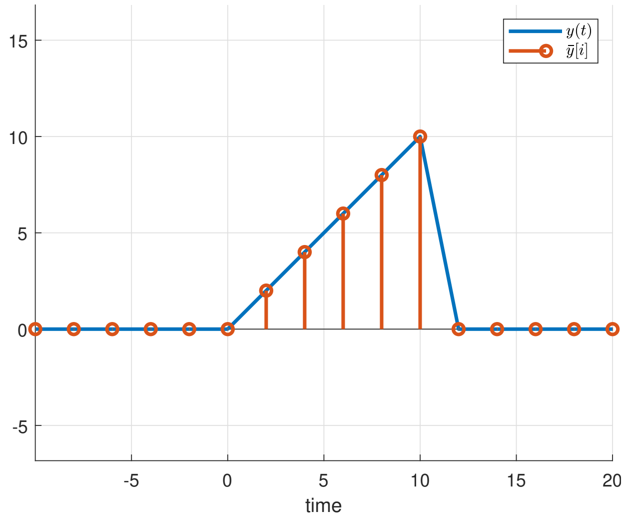
so that

$$\|\bar{y}\|_2 = \sqrt{220} \approx 14.8324.$$

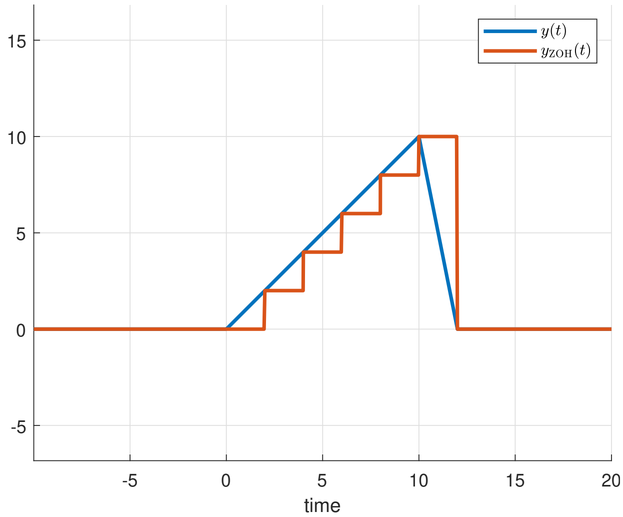
4. Fig. 3(b) presents the reconstructed y by the zero-order hold. This function can be written as:

$$\begin{aligned} y_{\text{ZOH}}(t) &= \sum_{i=-\infty}^{\infty} (\bar{y}[i] - \bar{y}[i-1]) \cdot \mathbb{1}(t - ih) = \sum_{i=1}^{6} (\bar{y}[i] - \bar{y}[i-1]) \cdot \mathbb{1}(t - ih) \\ &= 2\mathbb{1}(t-2) + 2\mathbb{1}(t-4) + 2\mathbb{1}(t-6) + 2\mathbb{1}(t-8) + 2\mathbb{1}(t-10) - 10\mathbb{1}(t-12) \\ &= \begin{cases} 2\left\lfloor \frac{t}{2} \right\rfloor & 0 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

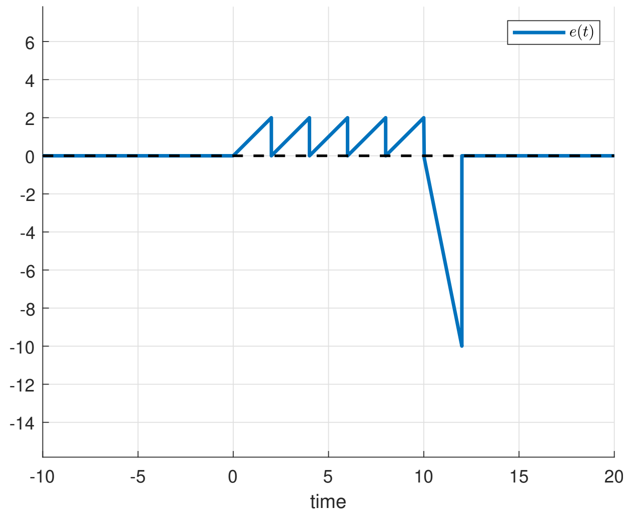
The second equality removes the zero values of the sum. The last equality expresses it more concisely using the floor function, $\lfloor x \rfloor$, which is the greatest integer less than or equal to x.



(a) Sampled triangle signal



(b) ZOH of sampled triangle signal



(c) Error signal, $e(t)$

Fig. 3: Plots for the solution of Question 3.

5. Substituting the expression for y_{ZOH} into the definition for the error, we have that

$$e(t) = y(t) - y_{\text{ZOH}}(t) = \begin{cases} t - 2\left\lfloor \frac{t}{2} \right\rfloor & 0 \le t < 10 \\ 60 - 5t - 2\left\lfloor \frac{t}{2} \right\rfloor & 10 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$$

This e is shown in Fig. 3(c). Its norms are

$$\|e\|_\infty = \sup_t |e(t)| = 10$$

and

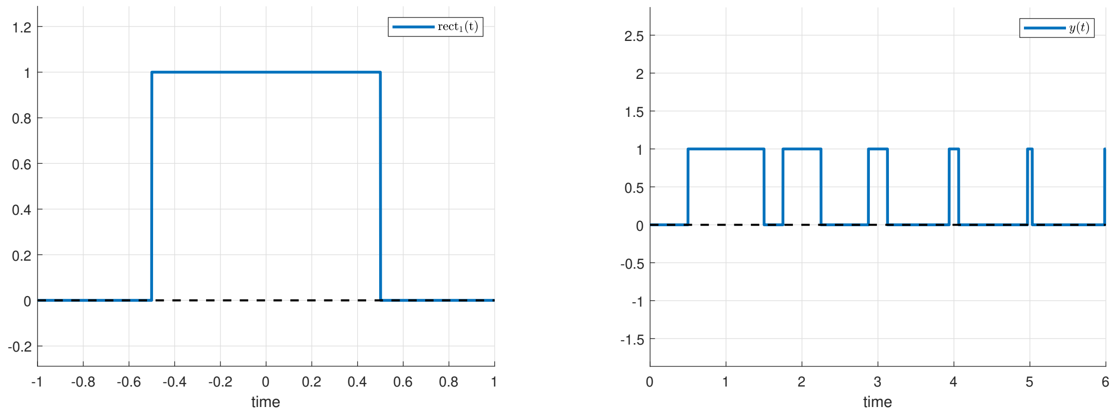
$$\|e\|_2^2 = \int_{-\infty}^{\infty} |e(t)|^2 \mathrm{d}t = \int_0^{12} |e(t)|^2\, \mathrm{d}t = \int_{10}^{12} \left|60 - 5t - 2\left\lfloor \frac{t}{2} \right\rfloor\right|^2 \mathrm{d}t + \int_0^{10} \left|t - 2\left\lfloor \frac{t}{2} \right\rfloor\right|^2 \mathrm{d}t$$

$$= \int_{10}^{12} |60 - 5t - 10|^2\, \mathrm{d}t + 5\int_0^2 t^2 \mathrm{d}t = \frac{(50-5t)^3}{-15}\Big|_{10}^{12} + 5\cdot\frac{1}{3}2^3 = \frac{200}{3} + \frac{40}{3} = 80$$

from which

$$\|e\|_2 = \sqrt{80} \approx 8.9443.$$

That's all … ▽



(a) rectangular pulse, rect

(b) signal y

Fig. 4: Plots for the solution of Question 4.

Question 4. Reminder that the rectangular pulse signal is the following

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Consider the continuous time signal

$$y = \sum_{i=1}^{\infty} \mathbb{S}_{-i}(\mathbb{P}_{2^{i-1}}\, \mathrm{rect}) \quad\Longrightarrow\quad y(t) = \sum_{i=1}^{\infty} \mathrm{rect}(2^{i-1}(t-i))$$

Fig. 4 shows rect and y.

1. Compute the L_2 norms of y.
2. Find the ideally sampled signal $\bar{y}$ with $h = 1$.
3. Compute the ℓ_2 norms of $\bar{y}$.

Solution.

1. Reminder: time shift, time scale

$$(\mathbb{S}_\tau)(t) = x(t+\tau) \quad \text{and} \quad (\mathbb{P}_\alpha x)(t) = x(\alpha t)$$

Applying the time shift and scaling

$$x := \mathbb{P}_{2^{i-1}}\,\mathrm{rect} \qquad \Longrightarrow \qquad x(t) = \mathrm{rect}(2^{i-1}t)$$

and

$$y = \mathbb{S}_{-i}(\mathbb{P}_{2^{i-1}}\,\mathrm{rect}) = \mathbb{S}_{-i}x \qquad \Longrightarrow \qquad y(t) = x(t-i) = \mathrm{rect}\big(2^{i-1}(t-i)\big)$$

By the definition of the L_2 norm,

$$\begin{aligned}
\|y\|_2^2 &= \int_{-\infty}^{\infty} |y(t)|^2\,\mathrm{d}t = \int_0^{\infty}\Big|\sum_{i=1}^{\infty}\mathrm{rect}\big(2^{i-1}(t-i)\big)\Big|^2\mathrm{d}t \\
&= \int_0^{\infty}\sum_{i=1}^{\infty}\mathrm{rect}\big(2^{i-1}(t-i)\big)\mathrm{d}t = \int_{0.5}^{1.5} 1\mathrm{d}t + \int_{1.75}^{2.75} 1\mathrm{d}t + \cdots + \int_{i-1/2^i}^{i+1/2^i} 1\mathrm{d}t + \cdots \\
&= \sum_{i=1}^{\infty}\int_{i-1/2^i}^{i+1/2^i} 1\mathrm{d}t = \sum_{i=1}^{\infty}\frac{1}{2^{i-1}} = \sum_{i=0}^{\infty}\frac{1}{2^i} = 2
\end{aligned}$$

The second line first equality is from the fact that each section of the sum is not overlapping, so the signal is 0 or 1. The second line second equality is the integrals divided into the nonzero sections. The last equality uses the solution to a geometric sum, $\sum_{k=0}^{\infty} r^k = 1/(1-r)$ whenever $0 < r < 1$. Thus,

$$\|y\|_2 = \sqrt{2}.$$

2. The sampling is such that it lines up with the nonzero values of the signal. Hence, $\bar{y} = \mathbb{S}_{-1}\mathbb{1}$.

3. Calculating

$$\|\bar{y}\|_2^2 = \sum_{i=-\infty}^{\infty} |\bar{y}[i]|^2 = \sum_{i=1}^{\infty} 1 = \infty.$$

Therefore, the norm is infinite.

This is an example of a signal whose L_2 norm is finite, but the ℓ_2 norm is not.

That's all … ▽

3 Complex Numbers Review

3.1 Imaginary Unit

The imaginary unit is j

$$\mathrm{j} := \sqrt{-1} \quad \implies \quad \mathrm{j}^2 = -1$$

3.2 Representations

Let z be a complex number. Two representations are

- canonical, $(a, b) : z = a + \mathrm{j}b$
- polar, $(r, \theta) : z = r\cos\theta + \mathrm{j}r\sin\theta$

Euler's formula is

$$\mathrm{e}^{\mathrm{j}x} = \cos x + \mathrm{j}\sin x$$

The polar can also written using Euler formula

$$z = r\cos\theta + \mathrm{j}r\sin\theta = r\mathrm{e}^{\mathrm{j}\theta}$$

3.3 Complex plane

The x axis is the real axis, and the y axis is the imaginary axis. Plotting the point $z = 2 + \mathrm{j}$.

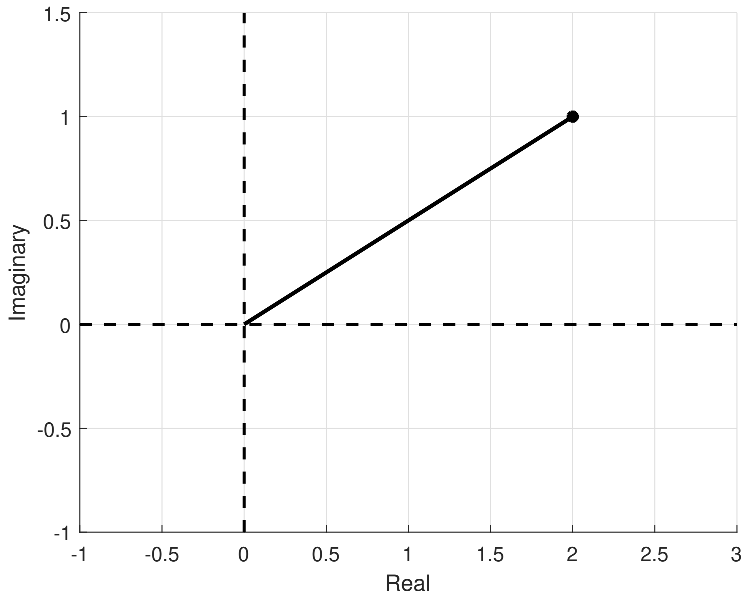


Fig. 5: $z = 2 + \mathrm{j}$ on the complex plane

3.4 Basic functions

Conjugate

$$\bar{z} = a - \mathrm{j}b = r\mathrm{e}^{-\mathrm{j}\theta}$$

Real / imaginary parts

$$\operatorname{Re} z = a \quad \text{and} \quad \operatorname{Im} z = b.$$

Absolute value / modulus with canonical and polar representations

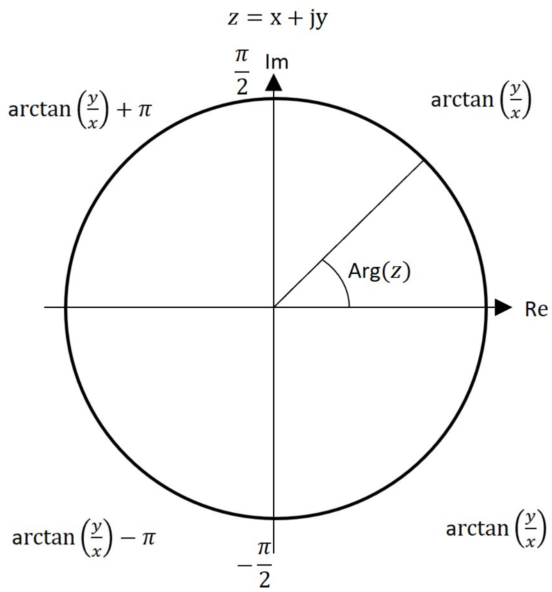
$$|z| = \sqrt{a^2 + b^2}$$
$$|z| = |re^{-j\theta}| = |r| \cdot |e^{-j\theta}| = |r|\,|\cos\theta + j\sin\theta| = |r|\sqrt{\cos^2\theta + \sin^2\theta} = |r|\sqrt{1} = |r|$$

The absolute value squared can also be expressed using the conjugate.

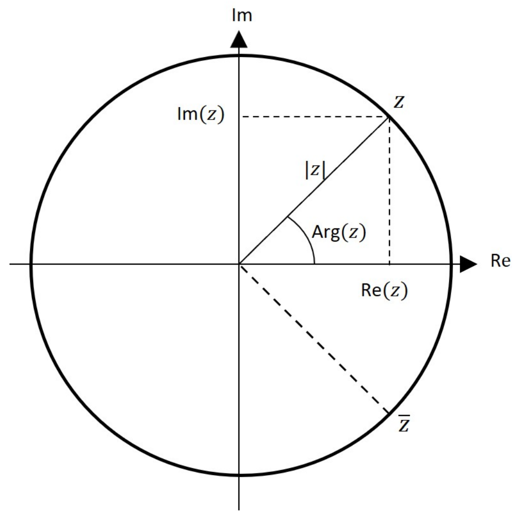
$$|z|^2 = z \cdot \bar{z}$$

Argument is the arctangent of the imaginary part divided by the real part. The codomain of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The $\arg(z)$ functions, has shifting of the angle for the codomain to be $(-\pi, \pi]$. Here $z = x + jy$

$$\arg(x + jy) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \arctan(\frac{y}{x}) + \pi & x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & x < 0, y < 0 \\ +\frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ \text{undefined} & x = 0, y = 0 \end{cases}$$



(a) complex plane for arg function



(b) complex plane for various characteristics of $z \in \mathbb{C}$

Fig. 6: Complex plane diagrams.

Various characteristics of a complex number can be shown on the complex plane, see Fig. 6(b).

3.5 Connecting Representations

Given a polar representation (r, θ), its canonical representation (a, b) has

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

Given a canonical representation (a, b), its polar representation (r, θ) has

$$r = |a + \mathrm{j}b| \quad \text{and} \quad \theta = \arg(a + \mathrm{j}b)$$

3.6 Operations

Let z and w be two complex numbers with representations

$$z = a + \mathrm{j}b = r\mathrm{e}^{\mathrm{j}t} \quad \text{and} \quad w = c + \mathrm{j}d = \rho\mathrm{e}^{\mathrm{j}\theta}$$

3.6.1 Canonical Form

Addition, it is element wise.

$$z + w = (a + \mathrm{j}b) + (c + \mathrm{j}d) = a + c + \mathrm{j}b + \mathrm{j}d = (a + c) + \mathrm{j}(b + d)$$

Subtraction is done similarly.

Multiplication is done by multiplying each with the other. In canonical form

$$z \cdot w = (a + \mathrm{j}b) \cdot (c + \mathrm{j}d) = ac + \mathrm{j}ad + \mathrm{j}bc - bd = (ac - bd) + \mathrm{j}(ad + bc)$$

Division requires multiplying by the conjugate of the denominator

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z \cdot \bar{w}}{|w|^2} = \frac{(ac + bd) + \mathrm{j}(-ad + bc)}{c^2 + d^2}$$

3.6.2 Polar Form

Addition, also element wise.

$$\begin{aligned} z + w &= (r\cos t + \mathrm{j}r\sin t) + (\rho\cos\theta + \mathrm{j}\rho\sin\theta) \\ &= (r\cos t + \rho\cos\theta) + \mathrm{j}(r\sin t + \rho\sin\theta) = \ldots \\ &= |z + w|\mathrm{e}^{\mathrm{j}\arg z + w} \end{aligned}$$

The result is less clear in the polar representation. Subtraction is done similarly.

Multiplication, here it is cleaner

$$z \cdot w = r\mathrm{e}^{\mathrm{j}t} \cdot \rho\mathrm{e}^{\mathrm{j}\theta} = (r \cdot \rho)\mathrm{e}^{\mathrm{j}t}\mathrm{e}^{\mathrm{j}\theta} = (r\rho)\mathrm{e}^{\mathrm{j}(t+\theta)}$$

Division,

$$\frac{z}{w} = \frac{r\mathrm{e}^{\mathrm{j}t}}{\rho\mathrm{e}^{\mathrm{j}\theta}} = \frac{r}{\rho}\mathrm{e}^{\mathrm{j}t}\mathrm{e}^{-i\theta} = \frac{r}{\rho}\mathrm{e}^{\mathrm{j}(t-\theta)}$$

Question 5. Calculate the absolute value and argument of z.

$$z = \frac{2 - \mathrm{j}2\sqrt{3}}{-5 - \mathrm{j}5} \cdot \frac{-\sqrt{3} + \mathrm{j}}{\mathrm{j}} \cdot 2$$

x	$-\infty$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$+\infty$
arctan x [deg]	$-90°$	$-60°$	$-30°$	$0°$	$30°$	$60°$	$90°$
arctan x [rad]	$-\frac{1}{2}\pi$	$-\frac{1}{3}\pi$	$-\frac{1}{6}\pi$	0	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$

Fig. 7: Arctangent characteristic angles.

Solution. First, change each term to its polar representation,

$$\begin{aligned} 2 - \mathrm{j}2\sqrt{3} &= 4\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}} \\ -5 - \mathrm{j}5 &= 5\sqrt{2}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{4}-\pi\right)} \\ -\sqrt{3} + \mathrm{j}1 &= 2\mathrm{e}^{\mathrm{j}\left(-\frac{\pi}{6}+\pi\right)} \\ \mathrm{j} &= 1\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} \\ 2 &= 2\mathrm{e}^{\mathrm{j}0} \end{aligned}$$

Now, calculate the overall absolute value and argument based using multiplication and division of the polar representation

$$z = \frac{4\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}}{5\sqrt{2}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{4}-\pi\right)}} \frac{2\mathrm{e}^{\mathrm{j}\left(-\frac{\pi}{6}+\pi\right)}}{1\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}} \cdot 2\mathrm{e}^{\mathrm{j}0},$$

from which

$$|z| = \frac{4\cdot 2}{5\sqrt{2}\cdot 1}\cdot 2 = \frac{8\sqrt{2}}{5} \quad \text{and} \quad \arg(z) = -\frac{\pi}{3} - \left(\frac{\pi}{4} - \pi\right) + \left(-\frac{\pi}{6} + \pi\right) - \frac{\pi}{2} + 0 = \frac{3}{4}\pi.$$

That's all . . . ▽