Linear Systems (034032)

TUTORIAL 2

1 Topics

Signal norms, sampling, reconstruction, standard signals, complex number review.

2 Problems

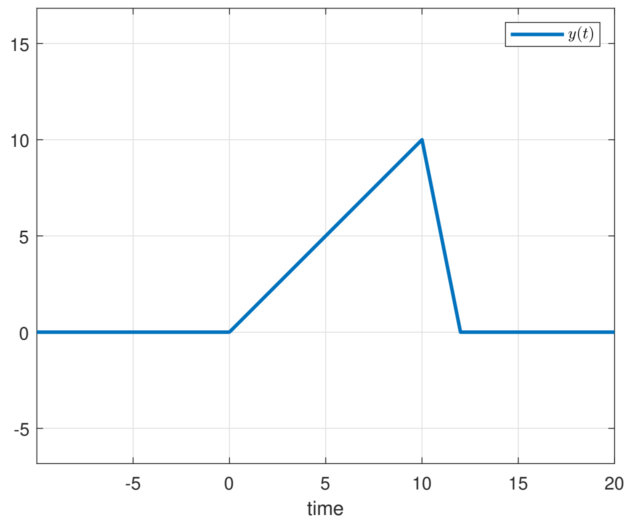


Fig. 1: Triangle signal

Question 1. Consider the continuous time signal

$$y(t) = \begin{cases} t & 0 \le t < 10 \\ 60 - 5t & 10 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$$

shown in Fig. 1. Construct y using the step function, $\mathbb{1}$, and the ramp signal, ramp.

Question 2. Let y be the signal from Question 1. Let $f := f_1 + f_2 + f_3$, where $f_1 = \delta$, $f_2 = 3\mathbb{S}_{-2}\delta$, and $f_3 = 0.5\mathbb{S}_{-3}\delta$ i.e.

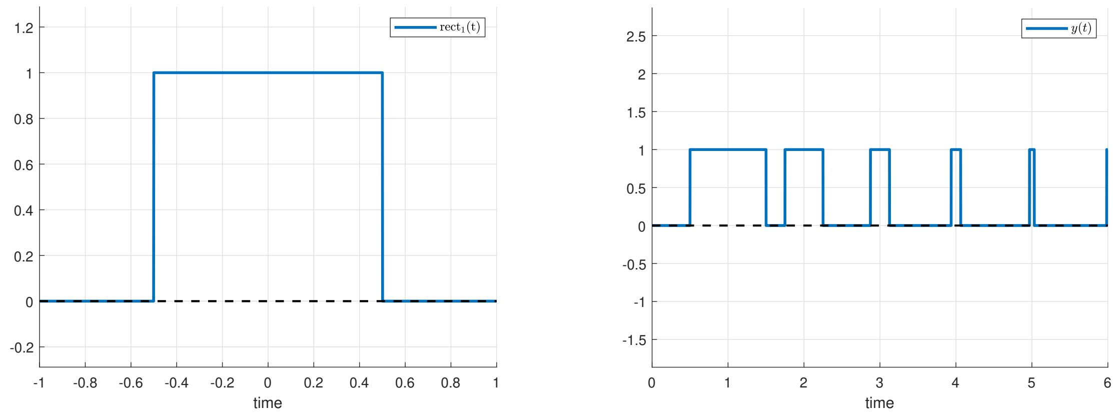
$$f(t) = \delta(t) + 3\delta(t-2) + 0.5\delta(t-3)$$

Find the convolution $y * f$.

Question 3. Let y be the signal from Question 1.

1. Compute the L_∞ and L_2 norms of y.
2. Find the ideally sampled signal $\bar{y}$ with the sampling period $h = 2$.

3. Compute the ℓ_∞ and ℓ_2 norms of $\bar{y}$.
4. Convert the sampled signal $\bar{y}$ back the the continuous time using the ZOH to produce an analog y_{ZOH}.
5. Compute the error, $e := y - y_{\text{ZOH}}$, and its L_∞ and L_2 norms.



(a) rectangular pulse, rect

(b) signal y

Fig. 2: Plots for the solution of Question 4.

Question 4. Reminder that the rectangular pulse signal is the following

$$\text{rect}(t) = \begin{cases} 1 & |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Consider the continuous time signal

$$y = \sum_{i=1}^{\infty} \mathbb{S}_{-i}(\mathbb{P}_{2^{i-1}}\,\text{rect}) \qquad \Longrightarrow \qquad y(t) = \sum_{i=1}^{\infty} \text{rect}(2^{i-1}(t-i))$$

Fig. 2 shows rect and y.

1. Compute the L_2 norms of y.
2. Find the ideally sampled signal $\bar{y}$ with $h = 1$.
3. Compute the ℓ_2 norms of $\bar{y}$.

3 Complex Numbers Review

3.1 Imaginary Unit

The imaginary unit is j

$$\mathrm{j} := \sqrt{-1} \quad \Longrightarrow \quad \mathrm{j}^2 = -1$$

3.2 Representations

Let z be a complex number. Two representations are

- canonical, $(a, b) : z = a + \mathrm{j}b$
- polar, $(r, \theta) : z = r \cos\theta + \mathrm{j}r \sin\theta$

Euler's formula is

$$\mathrm{e}^{\mathrm{j}x} = \cos x + \mathrm{j} \sin x$$

The polar can also written using Euler formula

$$z = r\cos\theta + \mathrm{j}r\sin\theta = r\mathrm{e}^{\mathrm{j}\theta}$$

3.3 Complex plane

The x axis is the real axis, and the y axis is the imaginary axis. Plotting the point $z = 2 + \mathrm{j}$.

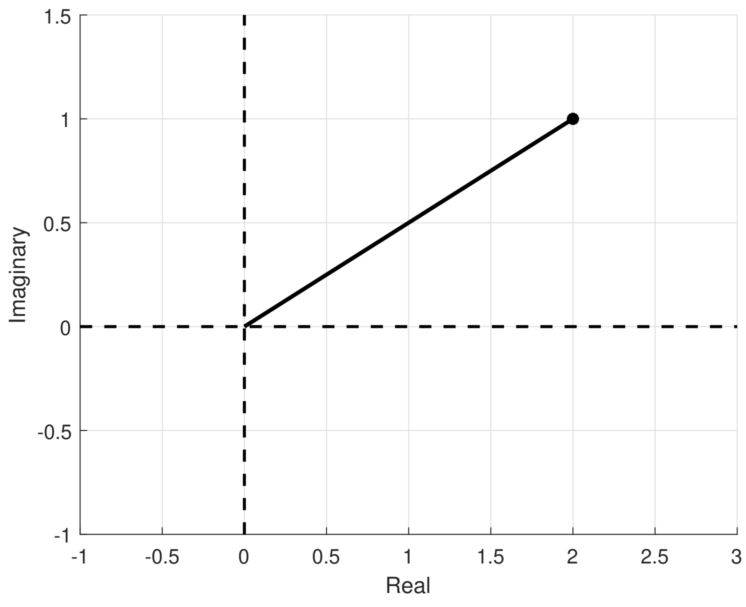


Fig. 3: $z = 2 + \mathrm{j}$ on the complex plane

3.4 Basic functions

Conjugate

$$\bar{z} = a - \mathrm{j}b = r\mathrm{e}^{-\mathrm{j}\theta}$$

Real / imaginary parts

$$\operatorname{Re} z = a \quad \text{and} \quad \operatorname{Im} z = b.$$

Absolute value / modulus with canonical and polar representations

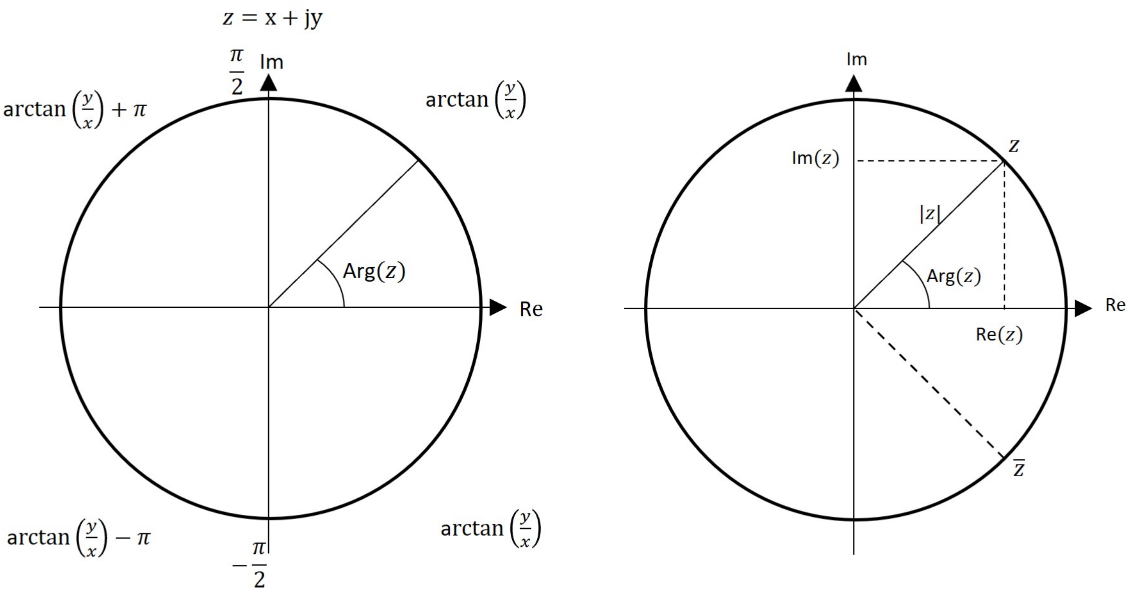
$$|z| = \sqrt{a^2 + b^2}$$
$$|z| = |r\mathrm{e}^{-\mathrm{j}\theta}| = |r| \cdot |\mathrm{e}^{-\mathrm{j}\theta}| = |r|\,|\cos\theta + \mathrm{j}\sin\theta| = |r|\sqrt{\cos^2\theta + \sin^2\theta} = |r|\sqrt{1} = |r|$$

The absolute value squared can also be expressed using the conjugate.

$$|z|^2 = z \cdot \bar{z}$$

Argument is the arctangent of the imaginary part divided by the real part. The codomain of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The arg$(z)$ functions, has shifting of the angle for the codomain to be $(-\pi, \pi]$. Here $z = x + \mathrm{j}y$

$$\arg(x + \mathrm{j}y) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \arctan(\frac{y}{x}) + \pi & x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & x < 0, y < 0 \\ +\frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ \text{undefined} & x = 0, y = 0 \end{cases}$$



(a) complex plane for arg function

(b) complex plane for various characteristics of $z \in \mathbb{C}$

Fig. 4: Complex plane diagrams.

Various characteristics of a complex number can be shown on the complex plane, see Fig. 4(b).

3.5 Connecting Representations

Given a polar representation (r, θ), its canonical representation (a, b) has

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

Given a canonical representation (a, b), its polar representation (r, θ) has

$$r = |a + \mathrm{j}b| \quad \text{and} \quad \theta = \arg(a + \mathrm{j}b)$$

3.6 Operations

Let z and w be two complex numbers with representations

$$z = a + \mathrm{j}b = r\mathrm{e}^{\mathrm{j}t} \quad \text{and} \quad w = c + \mathrm{j}d = \rho\mathrm{e}^{\mathrm{j}\theta}$$

3.6.1 Canonical Form

Addition, it is element wise.

$$z + w = (a + \mathrm{j}b) + (c + \mathrm{j}d) = a + c + \mathrm{j}b + \mathrm{j}d = (a + c) + \mathrm{j}(b + d)$$

Subtraction is done similarly.

Multiplication is done by multiplying each with the other. In canonical form

$$z \cdot w = (a + \mathrm{j}b) \cdot (c + \mathrm{j}d) = ac + \mathrm{j}ad + \mathrm{j}bc - bd = (ac - bd) + \mathrm{j}(ad + bc)$$

Division requires multiplying by the conjugate of the denominator

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z \cdot \bar{w}}{|w|^2} = \frac{(ac + bd) + \mathrm{j}(-ad + bc)}{c^2 + d^2}$$

3.6.2 Polar Form

Addition, also element wise.

$$\begin{aligned} z + w &= (r\cos t + \mathrm{j}r\sin t) + (\rho\cos\theta + \mathrm{j}\rho\sin\theta) \\ &= (r\cos t + \rho\cos\theta) + \mathrm{j}(r\sin t + \rho\sin\theta) = \ldots \\ &= |z + w|\mathrm{e}^{\mathrm{j}\arg z + w} \end{aligned}$$

The result is less clear in the polar representation. Subtraction is done similarly.

Multiplication, here it is cleaner

$$z \cdot w = r\mathrm{e}^{\mathrm{j}t} \cdot \rho\mathrm{e}^{\mathrm{j}\theta} = (r \cdot \rho)\mathrm{e}^{\mathrm{j}t}\mathrm{e}^{\mathrm{j}\theta} = (r\rho)\mathrm{e}^{\mathrm{j}(t+\theta)}$$

Division,

$$\frac{z}{w} = \frac{r\mathrm{e}^{\mathrm{j}t}}{\rho\mathrm{e}^{\mathrm{j}\theta}} = \frac{r}{\rho}\mathrm{e}^{\mathrm{j}t}\mathrm{e}^{-i\theta} = \frac{r}{\rho}\mathrm{e}^{\mathrm{j}(t-\theta)}$$

Question 5. Calculate the absolute value and argument of z.

$$z = \frac{2 - \mathrm{j}2\sqrt{3}}{-5 - \mathrm{j}5} \cdot \frac{-\sqrt{3} + \mathrm{j}}{\mathrm{j}} \cdot 2$$