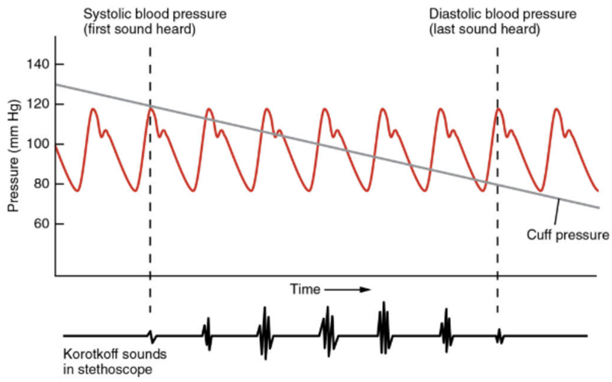
הטכניון – מכון טכנולוגי לישראל, הפקולטה להנדסת מכונות

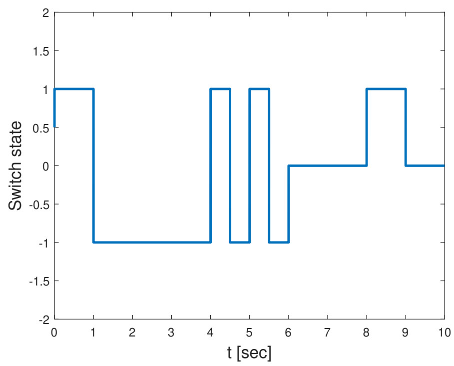
TECHNION — Israel Institute of Technology, Faculty of Mechanical Engineering

Linear Systems (034032)

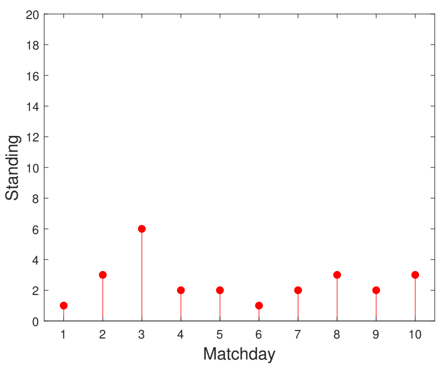
TUTORIAL 1



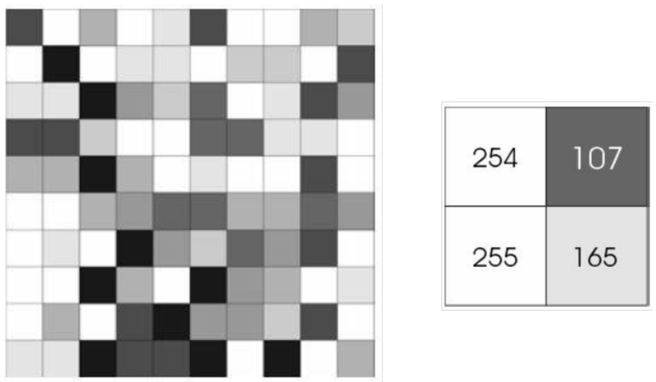
(a) Blood pressure cycle



(b) A 3-state switch value



(c) Liverpool FC weekly Premier League standings in 10 first matchdays of season 2021-2022



(d) A 8 bit grayscale image 10×10 pixel

Fig. 1: Signals for Question 1

Question 1. Describe properties (domains and codomains) of the signals in Fig. 1.

Solution.

1(a) $p : \mathbb{R}_+ \to \mathbb{R}_+$, i.e. it assigns an element of $\mathbb{R}_+$ (codomain, pressure) to every element of $\mathbb{R}_+$ (domain, time).

1(b) $u : \mathbb{R}_+ \to \{-1, 0, 1\}$, i.e. it assigns an element in the 3-elements set $\{-1, 0, 1\}$ (codomain, switch values) to every element of $\mathbb{R}_+$ (domain, time).

1(c) $u : \mathbb{Z}_{1..10} \to \mathbb{Z}_{1..20}$, i.e. it assigns an element of $\mathbb{Z}_{1..20}$ (codomain, standing) to every element of $\mathbb{Z}_{1..10}$ (domain, matchday).

1(d) $g : \mathbb{Z}_{1..10} \times \mathbb{Z}_{1..10} \to \mathbb{Z}_{0..255}$, i.e. it assigns an element of $\mathbb{Z}_{0..255}$ (codomain, 8-bit grayscale brightness) to every element of $\mathbb{Z}_{1..10} \times \mathbb{Z}_{1..10}$ (domain, (x, y) pixel coordinate).

$\triangledown$

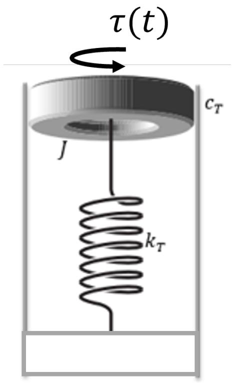


Fig. 2: Spring mass damper system.

Question 2. Consider a mass rotating inside a cylinder depicted in Fig. 2. The mass, whose moment of inertia is J, is attached to a torsion spring, whose torsion coefficient is k_T. An external torque τ acts on the mass and friction between the mass and the cylinder is assumed to generate a viscous friction torque $\tau_c = -c_T\dot{\theta}$. Find the relation between input signal τ and output signal θ.

Solution. The Newtonian motion equation of the mass is $J\ddot{\theta} = \tau_{\text{net}}$, where τ_{net} is the net torque applied to it. In our case,

$$\tau_{\text{net}} = \tau - k_T\theta - c_T\dot{\theta}.$$

Hence, we end up with

$$J\ddot{\theta}(t) + c_T\dot{\theta}(t) + k_T\theta(t) = \tau(t),$$

which is the relation describing the system $\tau \mapsto \theta$. ▽

Question 3. Consider the following system described in Fig. 3.

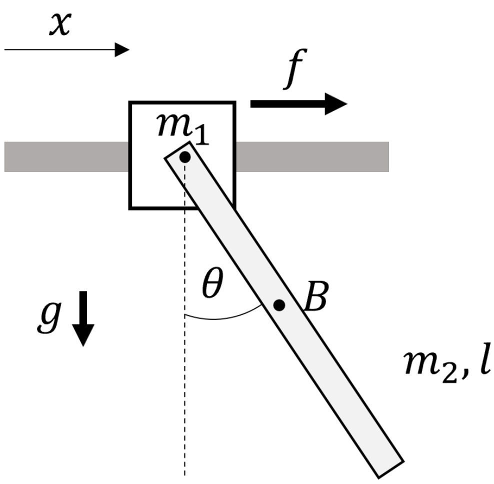


Fig. 3: Cart and pendulum

The input is a force f acting on the cart m_1, which is constrained to slide without friction in the horizontal direction. A pendulum, mass m_2, length l, is attached to the cart and is free to rotate around its axis. The outputs are the position of the cart x and the angle of the pendulum θ. Write the equations of motion.

Solution. The free-body diagram (FBD) is shown in Fig.4.

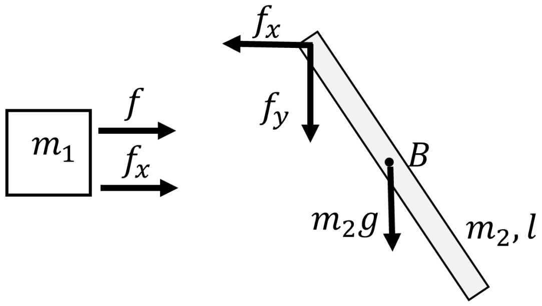


Fig. 4: FBD

Equations of motion on the cart

$$\sum F_x = m_1\ddot{x}(t) = f(t) + f_x(t) \tag{1}$$

The position, velocity, and acceleration of the pendulum's center of mass B is given by

$$p_B = (x_B, y_B) = (x + \frac{l}{2}\sin\theta, -\frac{l}{2}\cos\theta)$$
$$v_B = (\dot{x}_B, \dot{y}_B) = (\dot{x} + \frac{l}{2}\cos\theta\dot{\theta}, \frac{l}{2}\sin\theta\dot{\theta})$$
$$a_B = (\ddot{x}_B, \ddot{y}_B) = (\ddot{x} + \frac{l}{2}[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta], \frac{l}{2}[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta])$$

Equations of motion on the pendulum are

$$\Sigma F_x = m_2\ddot{x}_B(t) = -f_x$$
$$\Sigma F_y = m_2\ddot{y}_B(t) = -f_y - m_2 g$$
$$I_B\ddot{\theta} = f_x\frac{l}{2}\cos\theta + f_y\frac{l}{2}\sin\theta \tag{2}$$

where $I_B = \frac{1}{12}m_2 l^2$ is the moment of inertia. From the first two equations, we get

$$f_x = -m_2(\ddot{x} + \frac{l}{2}[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta])$$

$$f_y = -m_2(\frac{l}{2}[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta]) - m_2 g$$

Substitute f_x and f_y into the third equation (2) and the cart equation of motion (1) we get

$$\frac{1}{3}m_2 l^2\ddot{\theta} + \frac{1}{2}m_2 l\ddot{x}\cos\theta + \frac{1}{2}m_2 gl\sin\theta = 0$$

$$(m_1 + m_2)\ddot{x} + \frac{1}{2}m_2 l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = f$$

Let's assume the pendulum rotates around $\theta = 0$ with small angles hence

$$\sin\theta \approx \theta \quad \cos\theta \approx 1 \quad \sin\theta\dot{\theta}^2 \approx 0$$

the equations take the form

$$\frac{1}{3}m_2 l^2\ddot{\theta} + \frac{1}{2}m_2 l\ddot{x} + \frac{1}{2}m_2 g l\theta = 0$$

$$(m_1 + m_2)\ddot{x} + \frac{1}{2}m_2 l\ddot{\theta} = f$$

▽

Norms

Norms are a class of functions that enable us to quantify the size of a vector by assigning a nonnegative scalar to each vector.
Properties of a Norm:

1. Positive Definiteness: It should always be nonnegative. It is zero if and only if the vector is zero, i.e., zero vector. $\|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = 0$
2. Homogeneity: Multiplying a vector by a scalar multiplies the vector's norm by the scalar's absolute value. $\|\alpha v\| = |\alpha|\|v\|$
3. Triangle inequality: The norm of a sum of two vectors is no more than the sum of their norms. $\|v\| + \|u\| \geq \|v + u\|$

Two useful norms are

1. Norm-2 : $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$
2. Norm-infinity: $\|x\|_\infty = \max |x_i|$

Question 4. Calculate $\|x\|_2$ and $\|x\|_\infty$ where

1. $x = [x_1, x_2, x_3]$ and $|x_2| \geq |x_1| \geq |x_3|$
2. $x = [1, 0, 0]$
3. $x = [1, 1, 1]$

Solution.

1. $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{\sum_{i=1}^{3} |x_i|^2} = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2}$
   $\|x\|_\infty = \max |x_i| = |x_2|$
2. $\|x\|_2 = \sqrt{|1|^2 + |0|^2 + |0|^2} = 1$
   $\|x\|_\infty = 1$
3. $\|x\|_2 = \sqrt{|1|^2 + |1|^2 + |1|^2} = \sqrt{3}$
   $\|x\|_\infty = 1$

▽

Question 5. Find the sets of a 2D vector x such that $\|x\|_2 = 1$ and $\|x\|_\infty = 1$

Solution.

▽

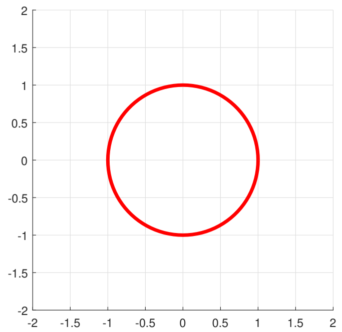


Fig. 5: $\|x\|_2$

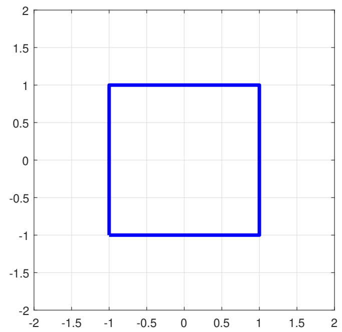


Fig. 6: $\|x\|_\infty$