Linear Systems (034032)
lecture no. 12

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Previously on Linear Systems . . .

An input / output (I/O) system $G : u \mapsto y$ is a *mapping* between input and output signals, whose domain is $\mathbb{R}$ (or $\mathbb{Z}$, in the discrete case). To analyze them at time instances $t \geq 0$ we need to

- either know the whole input history in $t < 0$
- or assume that all inputs have support in $\mathbb{R}_+$

The LTI system

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is solved via its state, as

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$$x(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} Bu(s) \mathrm{d}s \implies y(t) = \int_{-\infty}^{t} C\mathrm{e}^{A(t-s)} Bu(s) \mathrm{d}s + Du(t)$$

the output equation is static, so readily handleable

Outline

Outline

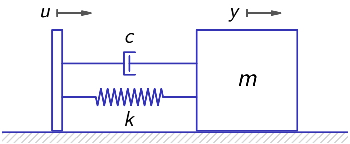
Initial conditions

Stability of unforced motion (Lyapunov stability)

Modal behavior

Coda

Example



whose dynamics are

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = c\dot{u}(t) + ku(t)$$

Suppose that we do not know $u(t)$ in $t \leq 0$ and that $u(t) = 0$ for all $t > 0$. The question of interest (a variation on the initial value problem) is

- can we determine $y(t)$ for $t > 0$ from the knowledge of $y(0)$ and $\dot{y}(0)$?

To answer this question, select $m = 1$, $c = 2$, and $k = 1 + \pi^2 \approx 10.870$ (so that $\omega_n \approx 3.3$, $\zeta \approx 0.3$, and $\omega_d = \pi$) and consider input signals of the form $u = \alpha \mathbb{S}_2 \mathbb{1} + (\beta - \alpha)\mathbb{S}_\tau \mathbb{1} - \beta \mathbb{1}$, i.e.

$$u(t) = \alpha \mathbb{1}(t + 2) + (\beta - \alpha)\mathbb{1}(t + \tau) - \beta \mathbb{1}(t) =$$

for some $\tau \in (0, 2)$ and $\alpha, \beta \in \mathbb{R}$.

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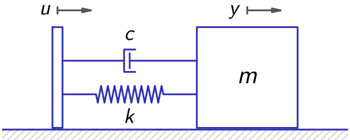

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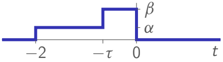

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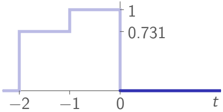
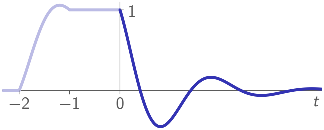
$u(t) =$ 

and $y(t) =$ 

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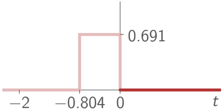
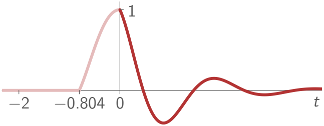
$u(t) =$ 

and $y(t) =$ 

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– what information does enable us to disregard past inputs?

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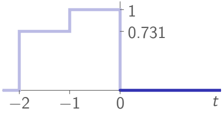
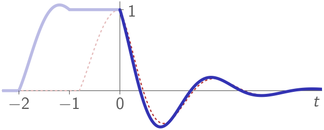
$u(t) =$ 

and $y(t) =$ 

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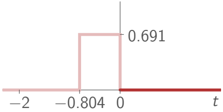
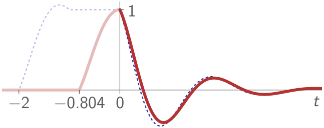
$u(t) =$ 

and $y(t) =$ 

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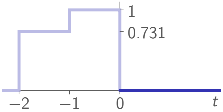
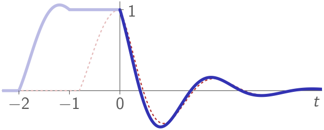
$u(t) =$ 

and $y(t) =$ 

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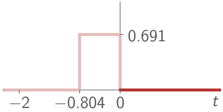
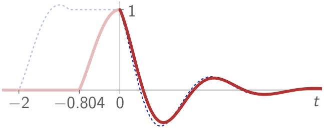
$u(t) =$ 

and $y(t) =$ 

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State as history accumulator

Given t_0, let us take another look at the state solution at $t_0 + t > t_0$

$$\begin{aligned} x(t_0+t) &= \int_{-\infty}^{t_0+t} \mathrm{e}^{A(t_0+t-s)} B u(s) \mathrm{d}s \\ &= \int_{-\infty}^{t_0} \mathrm{e}^{A(t_0+t-s)} B u(s) \mathrm{d}s + \int_{t_0}^{t_0+t} \mathrm{e}^{A(t_0+t-s)} B u(s) \mathrm{d}s \end{aligned}$$

and because $\mathrm{e}^{A(t_0+t-s)} = \mathrm{e}^{At}\mathrm{e}^{A(t_0-s)}$ and e^{At} does not depend on s,

$$= \mathrm{e}^{At} \underbrace{\int_{-\infty}^{t_0} \mathrm{e}^{A(t_0-s)} B u(s) \mathrm{d}s}_{x(t_0)} + \underbrace{\int_{t_0}^{t_0+t} \mathrm{e}^{A(t_0+t-s)} B u(s) \mathrm{d}s}_{u(s) \text{ for } s > t_0}$$

Therefore, the knowledge of $x(t_0)$ and $u(t)$ for $t > t_0$ is enough to compute $x(t)$ for all $t > t_0$. In other words,

- $x(t)$ at a given t accumulates the effect of the input history up to t.

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Returning to our mass-spring-damper example, what we need to determine $y(t)$ for $t > 0$ is a

- state at $t = 0$.

This is a realization-dependent issue. Pick the observer form, viz.

$$(A, B, C, D) = \left(\begin{bmatrix} -c/m & 1 \\ -k/m & 0 \end{bmatrix}, \begin{bmatrix} c/m \\ k/m \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, 0 \right)$$

for which (see Lect 11, Slide 23)

$$x = \begin{bmatrix} y \\ \dot{y} + \frac{c}{m}(y - u) \end{bmatrix}.$$

Hence, the information about the system at $t = 0$ required to start up is

$$y(0) \quad \text{and} \quad \dot{y}(0) - \frac{c}{m}u(0) \quad (\text{effectively, } \dot{y}(0) \text{ and } u(0))$$

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Starting point

The reasoning above implies that

– systems can be analyzed from any time point $t = t_0$ if $x(t_0)$ is known.

In the time-invariant case this starting point can be always chosen as $t = 0$. Consequently, $\mathbb{R}_+$ is taken as the domain of all involved signals and systems are considered in the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with an initial condition $x_0 \in \mathbb{R}^n$. In other words,

– systems may be treated as mappings $(x_0, u) \mapsto y$ operating on $\mathbb{R}_+$

in the state space.

Remark: It shall be emphasized that "operating on $\mathbb{R}_+$" is not the same as "having support in $\mathbb{R}_+$" assumed when systems were studied from pure I/O perspectives. Unless $x_0 = 0$, the presence of initial conditions implies that involved signals acted in $\mathbb{R}_-$ as well. The notion of the state just enables us to ignore their particular forms. All we need to know is the state at $t = 0$.

Finite-dimensional systems

Since *dynamical* systems are those with *memory*, the property of a state to hold the whole history suggests that

- the dimension of $x(t)$ may be regarded as a measure of the complexity

of corresponding dynamics. Systems admitting a finite vector as their state are then called finite dimensional.

We know that

- an LTI system with a state realization has a proper & rational transfer function
- an LTI system with a proper & rational transfer function always admits a state-space realization (cf. canonical realizations)

Hence, an LTI system (with a proper transfer function) is

- finite dimensional iff its transfer function is rational.

Systems with irrational transfer functions, such as the delay element ($e^{-\tau s}$) or finite-memory integrator ($(1 - e^{-\tau s})/s$), are infinite dimensional.

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Solution with initial conditions

If

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

then

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s$$

(we already saw that) and

$$y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s + Du(t)$$

As a matter of fact,

- effect of initial conditions on the state is $\mathrm{e}^{At}x_0$
- effect of $u = \delta$ on the state is $\mathrm{e}^{At}B$ (if $t > 0$)

Thus, response to nonzero initial condition and the Dirac delta at the input are closely related, although the former is richer (x_0 is arbitrary, B is fixed).

Initial conditions and similarity transformations

Nothing really special, still the transformation $x \to \tilde{x} = Tx$. Just remember that in the new coordinates

$$\begin{cases} \dot{\tilde{x}}(t) = (TAT^{-1})\tilde{x}(t) + (TB)u(t), & \tilde{x}(0) = Tx_0 \\ y(t) = (CT^{-1})\tilde{x}(t) + Du(t) \end{cases}$$

i.e. initial conditions are also affected by T.

Systems with initial conditions in the Laplace domain

In

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

we

- no longer assume that the state x has support in $\mathbb{R}_+$
- do not know how x behaved on $\mathbb{R}_-$

As such,

- it is natural to switch from the bilateral to unilateral Laplace analysis.

The main consequence of that is the differentiation rule, which is now

$$y(t) = \dot{x}(t) \implies Y(s) = sX(s) - x_0$$

Thus, the state equation reads

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

Unforced (autonomous) motion

Given

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

the equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

i.e. the state equation with zero inputs, is known as the unforced motion (or autonomous motion) of the system, where the state responds only to initial conditions. Because the

- initial conditions response is a richer form of the impulse response,

the unforced motion fully represents properties of the system G, while being easier to analyze.

Also worth emphasizing is that for every $t_0 \geq 0$,

- the behavior of $x(t)$ in $t > t_0$ is *completely* determined by $x(t_0)$,

no matter how this $x(t_0)$ was reached.

Discrete version

The state is still the history accumulator, with the state representation

$$G : \begin{cases} x[t+1] = Ax[t] + Bu[t], & x[0] = x_0 \\ y[t] = Cx[t] + Du[t] \end{cases}$$

and its solution

$$x[t] = A^t x_0 + \sum_{s=0}^{t} A^{t-s} Bu[s] \quad \& \quad y[t] = CA^t x_0 + \sum_{s=0}^{t} CA^{t-s} Bu[s] + Du[t]$$

In the z-domain the state variable is

$$X(z) = (zI - A)^{-1} x_0 + (zI - A)^{-1} BU(z)$$

because, again, the need to switch to the unilateral z transform.

Outline

Lyapunov stability: nonlinear case

An equilibrium $x_{eq} \in \mathbb{R}^n$ of autonomous dynamics $\dot{x} = f(x)$ is said to be

- stable if for every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0) - x_{eq}\| < \delta \implies \|x(t) - x_{eq}\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- asymptotically stable if it is stable and there is $\delta > 0$ such that

$$\|x(0) - x_{eq}\| < \delta \implies \lim_{t\to\infty} \|x(t) - x_{eq}\| = 0$$

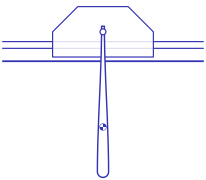
The region of attraction of an asymptotically stable equilibrium is the set of initial conditions $x(0)$ that generate states x converging to x_{eq}. If the region of attraction is the whole set $\mathbb{R}^n$, then the equilibrium is said to be globally asymptotically stable.

Remark 1: In principle, we may assume w.l.o.g. that $x_{eq} = 0$ in the definition. Otherwise, just rewrite the dynamics in terms of deviations $x_\delta = x - x_{eq}$.

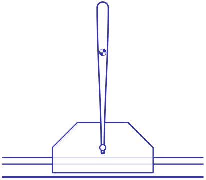
Remark 2: $\delta(\epsilon)$ is clearly a monotonically increasing function of ϵ. But unless the stability property is global, it is not strictly monotonic and often saturates.

Examples

Pendulum has essentially 2 nontrivial equilibria (assuming no dry friction):



stable



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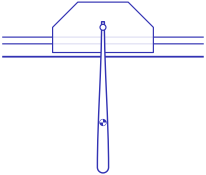

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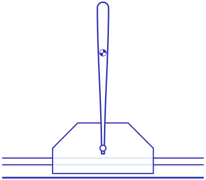

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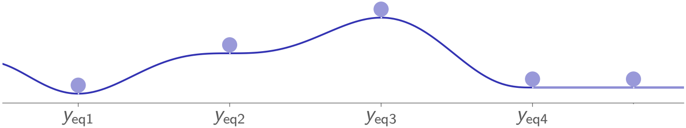

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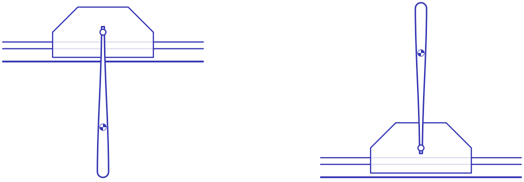

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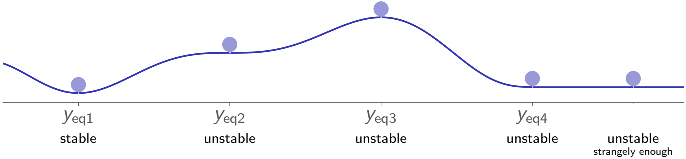

Lyapunov stability: linear case

If the system is linear, i.e. $f(x) = Ax$ for some $A \in \mathbb{R}^{n \times n}$, then the stability analysis is greatly simplified.

Theorem

An equilibrium of the system $\dot{x} = Ax$ *is*

- *stable iff* $\operatorname{spec}(A) \in \{s \in \mathbb{C} \mid \operatorname{Re} s \leq 0\}$ *and the geometric multiplicity of every pure imaginary eigenvalue equals its algebraic multiplicity*
- *asymptotically stable iff* $\operatorname{spec}(A) \in \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$

and those properties are global.

Remark: Because poles of the transfer function belong to $\operatorname{spec}(A)$, a system is I/O stable whenever its state-space realization is asymptotically stable by Lyapunov.

An equilibrium of a linear system must satisfy $Ax_{eq} = 0$. Hence,

- if $\det(A) \neq 0$, then the only equilibrium is $x_{eq} = 0$
- if $\det(A) = 0$, then there is an infinite number of equilibrium points spanned by all right eigenvectors associated with the eigenvalues of A at the origin

Lyapunov stability: linear case

If the system is linear, i.e. $f(x) = Ax$ for some $A \in \mathbb{R}^{n\times n}$, then the stability analysis is greatly simplified.

Theorem

An equilibrium of the system $\dot{x} = Ax$ is

- *stable iff* $\operatorname{spec}(A) \in \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ *and the geometric multiplicity of every pure imaginary eigenvalue equals its algebraic multiplicity*
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Connecting linear and nonlinear

Theorem (Lyapunov's indirect method)

Let $\dot{x} = f(x)$ for a continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}^n$, $x_{\text{eq}} \in \mathbb{R}^n$ be its equilibrium, and

$$A = \frac{\partial f(x)}{\partial x}\bigg|_{x=x_{\text{eq}}} \in \mathbb{R}^{n\times n}$$

be the corresponding Jacobian.

- *If $\operatorname{spec}(A) \in \mathbb{C} \setminus \bar{\mathbb{C}}_0$, then x_{eq} is asymptotically stable.*
- *If A has at least one eigenvalue in $\mathbb{C}_0$, then x_{eq} is unstable.*

Remark: It is worth emphasizing that the equivalence in the asymptotic stability conditions above is normally only *local*, even though linear properties are global. In other words, if the linearized dynamics $\dot{x} = Ax$ are (globally) asymptotically stable, its nonlinear original might still be asymptotically stable only locally.

Connecting linear and nonlinear (contd)

If the rightmost eigenvalue of the Jacobian matrix is on the imaginary axis, then the stability conclusion is no longer unambiguous.

Example

Let $\dot{x} = \alpha x^3$ for $\alpha \neq 0$. Its only equilibrium is at $x_{\mathsf{eq}} = 0$ and the Jacobian $A = 0$, for any α. The linearized dynamics, $\dot{x}_\delta = 0$, are stable then. But the solution to the original equation is

$$x(t) = \frac{x(0)}{\sqrt{1 - 2\alpha x^2(0)t}} = \begin{cases} & \text{if } \alpha < 0 \\ & \text{if } \alpha > 0 \end{cases}$$

(finite escape point at $t = t_e := 1/(2\alpha x^2(0))$ if $\alpha > 0$). Thus, the system is

- asymptotically stable if $\alpha < 0$ $\qquad \delta = \epsilon$ for all $\epsilon > 0$
- unstable if $\alpha > 0$

Connecting linear and nonlinear (contd)

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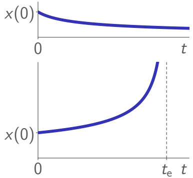

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Outline

Initial conditions

Stability of unforced motion (Lyapunov stability)

Modal behavior

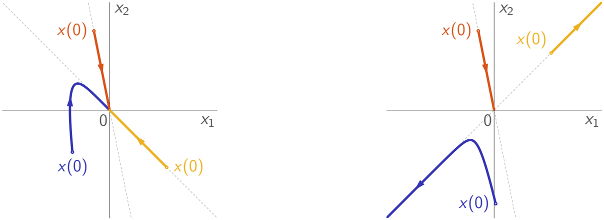
Coda

Examples 1 and 2

Consider autonomous linear dynamics

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} x(t) \quad \text{and} \quad \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} x(t)$$

whose characteristic polynomials are $(s+5)(s+1)$ and $(s+5)(s-1)$, i.e. one of them is asymptotically stable and another one is unstable. The initial condition responses can be visualized in the state (phase) plane by plotting $x_2(t)$ vs. $x_1(t)$ as t grows from 0 to ∞:



Responses are qualitatively different for different initial conditions. Why?

Background: diagonalization

If $A \in \mathbb{R}^{n\times n}$ is not defective, then all its n (right) eigenvectors η_i are linearly independent and the matrix having them as columns is nonsingular. Define

$$T := \begin{bmatrix} \eta_1 & \cdots & \eta_n \end{bmatrix} \quad \text{and} \quad T^{-1} =: \begin{bmatrix} \upsilon_1' \\ \vdots \\ \upsilon_n' \end{bmatrix} \quad \Longrightarrow \quad T^{-1}T = [\upsilon_i'\eta_j]$$

(i.e. υ_i is the transpose complex conjugate of the ith row of T^{-1}) such that $\upsilon_i'\eta_j = \delta_{ij}$ (Kronecker delta). In this case

$$A = T\Lambda_A T^{-1} = \begin{bmatrix} \eta_1 & \cdots & \eta_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \upsilon_1' \\ \vdots \\ \upsilon_n' \end{bmatrix}$$

If $\lambda_j = \overline{\lambda_i}$, then[1] $\eta_j = \overline{\eta_i}$ and $\upsilon_j = \overline{\upsilon_i}$ as well.

[1] Just conjugate $A\eta_i = \eta_i\lambda_i$ and remember that A is real valued to convince yourselves.

State solution: real eigenvalues

Consider the autonomous state equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

and assume that A is not defective with real eigenvalues. The solution

$$\begin{aligned} x(t) &= e^{At}x_0 = e^{T\Lambda_A T^{-1}t}x_0 = T e^{\Lambda_A t} T^{-1}x_0 \\ &= \begin{bmatrix} \eta_1 & \cdots & \eta_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \upsilon_1' x_0 \\ \vdots \\ \upsilon_n' x_0 \end{bmatrix} = \sum_{i=1}^{n} \eta_i e^{\lambda_i t}(\upsilon_i' x_0). \end{aligned}$$

Defining $\mu_i(x_0) := \upsilon_i' x_0 := \langle x_0, \upsilon_i \rangle \in \mathbb{R}$, we get

$$x(t) = \sum_{i=1}^{n} \eta_i e^{\lambda_i t} \mu_i(x_0)$$

where the initial conditions only affect the constant scalar coefficients μ_i.

Modes

The form

$$x(t) = \sum_{i=1}^{n} \eta_i e^{\lambda_i t} \mu_i(x_0)$$

means that solutions to unforced systems are a superposition of elementary exponential signals $\exp_{\lambda_i}$, each one of which is shaped by the corresponding eigenvectors, $\eta_i \in \mathbb{R}^n$. The signal

- $\eta_i \exp_{\lambda_i}$ is known as the ith mode of the system

and the scalar

- $\mu_i(x_0) = \upsilon_i' x_0$ is called the degree of excitation of the ith mode by x_0.

In fact, $\mu_i(x_0)$ is the ith coordinate of x_0 with respect to the eigenbasis $\{\eta_i\}$ of $\mathbb{R}^n$, cf.

$$x_0 = \sum_{i=1}^{n} \alpha_i \eta_i \quad \Longrightarrow \quad \alpha_i = \upsilon_i' x_0.$$

and then

- $e^{\lambda_i t} \mu_i(x_0)$ is the corresponding coordinate of $x(t)$.

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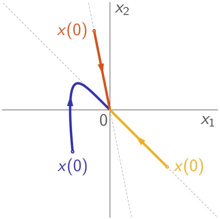
- $e^{\lambda_i t} \mu_i(x_0)$ is the corresponding coordinate of $x(t)$.

Modal response in the phase plane: Example 1

Return to the stable 2-order system with $\lambda_1 = -5$ and $\lambda_2 = -1$.



The initial conditions $x(0) = \eta_1$ and $x(0) = \eta_2$ generate the traverse of $x(t)$ along the corresponding eigenvectors, while $x(0) = -1.25\eta_1 - \eta_2$ results in a superposition of two modes. Some observations:

- trajectories can never cross the eigenvector lines if do not start there
- asymptotic stability $\Longrightarrow$ every trajectory converges to the origin
- trajectories eventually approach the eigenvector of the slowest mode

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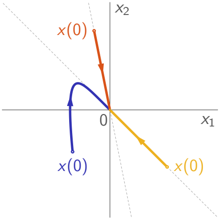

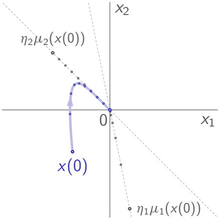

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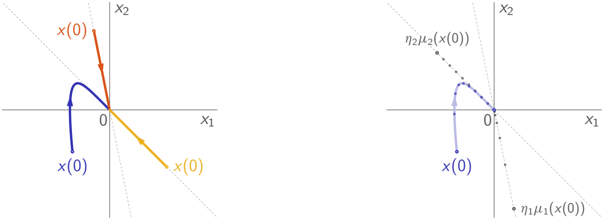

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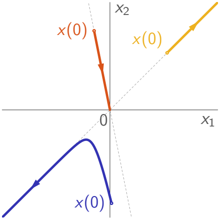
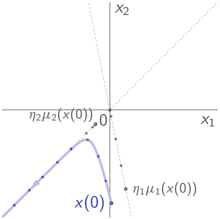
Modal response in the phase plane: Example 2

Return to the unstable 2-order system with $\lambda_1 = -5$ and $\lambda_2 = 1$.



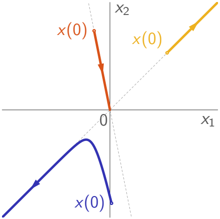


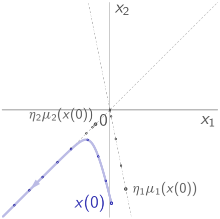
The initial conditions $x(0) = \eta_1$ and $x(0) = \eta_2$ generate the traverse of $x(t)$ along the corresponding eigenvectors, while $x(0) = -\eta_1 - 0.25\eta_2$ results in a superposition of two modes. Additional observations:

- instability $\implies$ every trajectory with $\mu_2(x(0)) \neq 0$ diverges
- trajectories eventually approach the eigenvector of the unstable mode

Modal response in the phase plane: Example 2

Return to the unstable 2-order system with $\lambda_1 = -5$ and $\lambda_2 = 1$.





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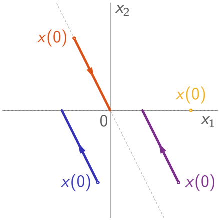
- trajectories eventually approach the eigenvector of the unstable mode

Modal response in the phase plane: Example 3

Consider

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x(t)$$

with $(\lambda_1, \eta_1) = (-2, \left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right]/\sqrt{5})$ and $(\lambda_2, \eta_2) = (0, \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right])$. Its second mode is stable, but not asymptotically (aka neutrally stable). The phase plane is



Starting on $x(0) = \eta_2$ results in a point $(x = x(0))$, other initial conditions generate the traverse along a line parallel to η_1 and shifted by $\mu_2(x(0))$, so all converge to $\mu_2(x(0))\eta_2$, like those from $x(0) = \eta_1$, $x(0) = -\eta_1 - 0.6\eta_2$, and $x(0) = -\eta_1 + 0.4\eta_2$.

Second-order complex eigenvalues

If $A \in \mathbb{R}^{2\times 2}$ has eigenvalues at $\sigma \pm j\omega$, then $\exists \eta_r, \eta_i, \upsilon_r, \upsilon_i \in \mathbb{R}^2$ such that

$$A = \begin{bmatrix} \eta_r + j\eta_i & \eta_r - j\eta_i \end{bmatrix} \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix} \begin{bmatrix} \upsilon_r' + j\upsilon_i' \\ \upsilon_r' - j\upsilon_i' \end{bmatrix}$$

Hence,

$$\begin{aligned} x(t) = e^{At}x_0 &= \begin{bmatrix} \eta_r + j\eta_i & \eta_r - j\eta_i \end{bmatrix} \begin{bmatrix} e^{(\sigma + j\omega)t} & 0 \\ 0 & e^{(\sigma - j\omega)t} \end{bmatrix} \begin{bmatrix} \upsilon_r' + j\upsilon_i' \\ \upsilon_r' - j\upsilon_i' \end{bmatrix} x_0 \\ &= e^{\sigma t} \begin{bmatrix} \eta_r + j\eta_i & \eta_r - j\eta_i \end{bmatrix} \begin{bmatrix} e^{j\omega t} & 0 \\ 0 & e^{-j\omega t} \end{bmatrix} \begin{bmatrix} \upsilon_r' + j\upsilon_i' \\ \upsilon_r' - j\upsilon_i' \end{bmatrix} x_0 \\ &= \big(\eta_r \cos(\omega t) - \eta_i \sin(\omega t)\big) e^{\sigma t} \mu_r(x_0) \\ &\qquad + \big(\eta_i \cos(\omega t) + \eta_r \sin(\omega t)\big) e^{\sigma t} \mu_i(x_0) \end{aligned}$$

where

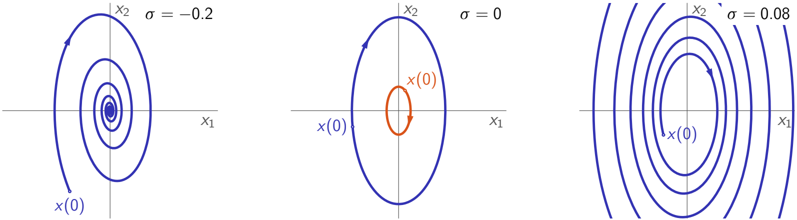
- $\mu_r(x_0) := 2\upsilon_r' x_0$ and $\mu_i(x_0) := -2\upsilon_i' x_0$ are degrees of excitation
- $(\eta_r \cos(\omega t) - \eta_i \sin(\omega t))e^{\sigma t}$ and $(\eta_i \cos(\omega t) + \eta_r \sin(\omega t))e^{\sigma t}$ are modes

Modal response in the phase plane: Example 4

Consider

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\sigma^2 - \omega^2 & \sigma \end{bmatrix} x(t)$$

whose eigenvalues are at $\sigma \pm j\omega$. The phase plane plots for $\omega = 2$ are



These pictures are representative, namely

- if $\sigma < 0$ (asymptotically stable), then spirals converging to the origin
- if $\sigma = 0$ (neutrally stable), then ellipses
- if $\sigma > 0$ (unstable), then diverging spirals

Outline

Math literacy

- complex numbers
  - the imaginary unit j
  - canonical ($\operatorname{Re} c + \mathrm{j} \operatorname{Im} c$) and polar ($|c| e^{\mathrm{j} \arg c}$) representations
  - operations with complex numbers
  - complex plane
- complex functions
  - poles and removable singularities
  - rational functions, their poles, zeros, properness
  - partial fraction expansion of rational functions, residues
- linear algebra
  - eigenvalues and eigenvectors
  - similarity transformations and diagonalization
  - Cayley–Hamilton and its meaning
  - matrix power, functions of matrices (especially, matrix exponential)
  - computing functions of matrices
  - matrix calculus

Notions to understand

- signals in time and transformed domains
  - main properties, viz. support, decay, convergence, periodicity
  - standard signals, viz. step, ramp, sinc, exponential, harmonic, Dirac delta
  - oprations on signals, viz. scaling (also of time $\mathbb{P}_\varsigma$), shift ($\mathbb{S}_\tau$), convolution
  - signal norms, viz. L_1, L_2, L_∞, as well as ℓ_1, ℓ_2, ℓ_∞; energy and power
  - Fourier series and transforms, their meaning in terms of harmonic content
  - sampling in time and frequency domains, frequency aliasing, ω_s and ω_N
  - Laplace and z transforms, with RoC and use in solving diff [...] equations
- systems and their classifications
  - systems as constraints on interrelated signals, I/O systems as mappings
  - dynamic vs. static, SISO vs. MIMO, time invariant vs. time varying, linear vs. nonlinear, finite dimensional vs. infinite dimensional, causality
  - block-diagrams; series, parallel, and feedback interconnections of systems

Notions to understand (contd)

- LTI systems, I/O viewpoint
  - LTI systems and convolution, impulse response
  - transfer functions and system properties via them (stability and causality)
  - BIBO and L_2 / ℓ_2 stability conditions, also in the rational case
  - stability tests (checking if a polynomial is Hurwitz or Schur)
  - step responses, steady state and transients
  - transients characteristics: over- and undershoot, raise time, settling time
  - step responses of 1- and 2-order systems (including effects of zeros)
  - frequency response, magnitude and phase, processing harmonic inputs
  - frequency response plots: Bode (construct and read) and polar (read)
  - systems as filters (low-pass, high-pass, band-pass, band-stop, notch)
- LTI systems, state-space viewpoint
  - state-space realizations
  - from state space to transfer functions and back again
  - linearization technique (equilibrium, Jacobian, et alii)
  - state as history accumulator
  - initial conditions and unforced motion
  - Lyapunov stability and its visualization on the phase plane

Formulae to memorize

- convolution of signals, $x * y$, in both time and transformed domains
- the sifting property of the Dirac delta,

$$\int_{\mathbb{R}} f(t)\delta(t - t_0)\mathrm{d}t = f(t_0)$$

- frequency response of sampled signals,

$$\bar{X}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{h}\sum_{i\in\mathbb{Z}} X(\mathrm{j}\tfrac{\theta+2\pi i}{h}) \quad \text{or} \quad \bar{X}(\mathrm{e}^{\mathrm{j}\omega h}) = \frac{1}{h}\sum_{i\in\mathbb{Z}} X\bigl(\mathrm{j}(\omega + 2\omega_{\mathrm{N}} i)\bigr)$$

  and the Nyquist frequency $\omega_{\mathrm{N}} := \pi/h$
- time shift property $(\mathbb{S}_\tau x)$ of Fourier, Laplace, and z transforms,

$$\mathrm{e}^{\mathrm{j}\omega\tau}X(\mathrm{j}\omega), \quad \mathrm{e}^{\mathrm{j}\theta\tau}X(\mathrm{e}^{\mathrm{j}\theta}), \quad \mathrm{e}^{\tau s}X(s), \quad z^{\tau}X(z)$$

- inverse Laplace transforms of $\frac{1}{s}$, $\frac{1}{s+a}$, $\frac{s\sin\phi+\omega\cos\phi}{s^2+\omega^2}$ when RoC $= \mathbb{C}_\alpha$

Formulae to memorize (contd)

- condition for $a_2 s^2 + a_1 s + a_0$ and $a_3 s^3 + a_2 s^2 + a_1 s + a_0$ being Hurwitz
- standard forms of 1- and 2-order transfer functions, i.e.

$$\frac{k_{\mathrm{st}}}{\tau s + 1} \quad \text{and} \quad \frac{k_{\mathrm{st}} \omega_{\mathrm{n}}^2}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2}$$

- values in dB of 0.01, 0.1, $1/\sqrt{2}$, 1, $\sqrt{2}$, 10, 100
- Bode plots of basic blocks (gain, 1- and 2-order), quantitatively
- bandwidth of the frequency response of a system with $G(s) = \frac{1}{\tau s + 1}$
- solution to the state equation, $x(t) = \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A(t-s)} B u(s) \mathrm{d}s$
- canonical companion and observer state-space realizations, viz.

$$\left[\begin{array}{cccc|c} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -a_0 & -a_1 & \cdots & -a_{n-1} & 1 \\ \hline b_0 & b_1 & \cdots & b_{n-1} & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cccc|c} -a_{n-1} & 1 & \cdots & 0 & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 1 & b_1 \\ -a_0 & 0 & \cdots & 0 & b_0 \\ \hline 1 & 0 & \cdots & 0 & 0 \end{array}\right]$$