Linear Systems (034032)
lecture no. 11

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Outline

State-space representation: continuous-time case

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$, the set of equations

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is known as the state-space representation of a continuous-time $G : u \mapsto y$. Here

- the (internal) signal x is called the state vector of G
- the upper equation (differential) is called the state equation
- the lower one (algebraic) is called the output equation

The quadruple (A, B, C, D) is dubbed a state-space realization of G.

Solution to the state equation: (educated) guess

Consider the function

$$x(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} Bu(s)\mathrm{d}s$$

Using the Leibniz integral rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(s,t)\mathrm{d}s = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(s,t)\mathrm{d}s + \frac{\mathrm{d}b(t)}{\mathrm{d}t} f(b(t),t) - \frac{\mathrm{d}a(t)}{\mathrm{d}t} f(a(t),t),$$

and the relation $\frac{\partial}{\partial t}\mathrm{e}^{A(t-s)} = A\mathrm{e}^{A(t-s)}$, we have that

$$\dot{x}(t) = \int_{-\infty}^{t} A\mathrm{e}^{A(t-s)} Bu(s)\mathrm{d}s + \mathrm{e}^{A(t-t)} Bu(t) = Ax(t) + Bu(t)$$

This is exactly the state equation.

Solution to the state equation and impulse response

Thus, the solution of

$$G:\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)+Du(t)\end{cases}$$

is

$$y(t)=\int_{-\infty}^{t}C\mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s+Du(t).$$

As a matter of fact, this $y=g*u$ under

$$g(t)=D\delta(t)+C\mathrm{e}^{At}B\mathbb{1}(t),$$

which implies that

- the system G is LTI
- g above is its impulse response in terms of that state-space realization
- the system G is causal $\quad$ g has support in $\mathbb{R}_+$

Transfer function via state-space realization

State-space representation in the Laplace domain is

$$G:\begin{cases}sX(s)=AX(s)+BU(s)\\ Y(s)=CX(s)+DU(s)\end{cases}$$

The first (state) equation yields

$$(sI-A)X(s)=BU(s)\quad\Longrightarrow\quad X(s)=(sI-A)^{-1}BU(s)$$

(well defined in $\mathbb{C}_{\max_{\lambda\in\operatorname{spec}(A)}\operatorname{Re}\lambda}$) and the second (output) equation yields

$$Y(s)=(D+C(sI-A)^{-1}B)U(s).$$

Hence, the transfer function of this system is

$$G(s)=D+C(sI-A)^{-1}B.$$

It is indeed the Laplace transform of $g=D\delta+C\exp_A B\mathbb{1}$.

Transfer function via state-space realization (contd)

The transfer function

$$G(s)=D+C(sI-A)^{-1}B=D+\frac{C\operatorname{adj}(sI-A)B}{\det(sI-A)}$$
$$=\frac{D\det(sI-A)+C\operatorname{adj}(sI-A)B}{\det(sI-A)},$$

where $\operatorname{adj}(sI-A)$ is the adjugate matrix. It has *polynomial* entries, so that both the numerator and the denominator of $G(s)$ are polynomials. Thus,

- $G(s)$ is rational
- poles of $G(s)$ are among[1] the eigenvalues of A, i.e. in $\operatorname{spec}(A)$

[1] It's a bit more complicated, but precise relation goes beyond the scope of the course.

Transfer function via state-space realization: properness

Because

$$\lim_{|s|\to\infty}C(sI-A)^{-1}B=\lim_{|s|\to\infty}\tfrac{1}{s}C(I-\tfrac{1}{s}A)^{-1}B=\lim_{\sigma\to0}\sigma C(I-\sigma A)^{-1}B=0$$

we have that

$$\lim_{|s|\to\infty}D+C(sI-A)^{-1}B=D.$$

Compare

$$\lim_{|s|\to\infty}\frac{b_ns^n+b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=b_n.$$

Thus, the transfer function

- $D+C(sI-A)^{-1}B$ is strictly proper iff $D=0$ (or bi-proper iff $D\neq0$).

The parameter D is called the feed-through term of the realization.

Similar realizations

Let $T \in \mathbb{R}^{n\times n}$ be nonsingular. Define $\tilde{x} := Tx$, where x is a state of an LTI system G. Its derivative

$$\dot{\tilde{x}}(t) = T\dot{x}(t) = T(Ax(t) + Bu(t)) = TAT^{-1}\tilde{x}(t) + TBu(t)$$

Because $y(t) = Cx(t) + Du(t) = CT^{-1}\tilde{x}(t) + Du(t)$, we have

$$G : \begin{cases} \dot{\tilde{x}}(t) = TAT^{-1}\tilde{x}(t) + TBu(t) & =: \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) = CT^{-1}\tilde{x}(t) + Du(t) & =: \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}$$

and conclude that $\tilde{x}$ is also a state vector of G, with the realization

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) := (TAT^{-1}, TB, CT^{-1}, D).$$

The realizations (A, B, C, D) and $(TAT^{-1}, TB, CT^{-1}, D)$ are called similar.

Similar realizations (contd)

Not surprisingly,

$$\begin{aligned} \tilde{g}(t) &= D\delta(t) + CT^{-1}e^{TAT^{-1}t}TB\mathbb{1}(t) = D\delta(t) + CT^{-1}Te^{At}T^{-1}TB\mathbb{1}(t) \\ &= D\delta(t) + Ce^{At}B\mathbb{1}(t) \\ &= g(t) \end{aligned}$$

and

$$\begin{aligned} \tilde{G}(s) &= D + CT^{-1}(sI - TAT^{-1})^{-1}TB = D + CT^{-1}(T(sI - A)T^{-1})^{-1}TB \\ &= D + CT^{-1}T(sI - A)^{-1}T^{-1}TB = D + C(sI - A)^{-1}B \\ &= G(s) \end{aligned}$$

In other words,

- similarity transformation does not affect I/O relations,

it only change the internal variable.

Diagonal realization

If an LTI system G has a realization of the form

$$(A, B, C, D) = \left(\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}, 0 \right),$$

then its transfer function

$$\begin{aligned} G(s) &= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \left(sI_n - \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \\ &= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} 1/(s-a_1) & & 0 \\ & \ddots & \\ 0 & & 1/(s-a_n) \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \\ &= \sum_{i=1}^{n} \frac{c_i b_i}{s - a_i} \end{aligned}$$

i.e. diagonalization corresponds to the partial fraction expansion of $G(s)$.

Side remark: time invariance

As saw earlier, state-space representations with constant parameters always correspond to LTI systems. Can we say that realizations with

- time-varying parameters do not correspond to LTI dynamics?

Example: let

$$G : \begin{cases} \dot{x}(t) = -\dfrac{\cos t}{2 - \sin t}x(t) + (2 - \sin t)u(t) \\ y(t) = \dfrac{1}{2 - \sin t}x(t) \end{cases}$$

But

$$\begin{aligned} \dot{y}(t) &= \frac{1}{2 - \sin t}\dot{x}(t) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2 - \sin t}\right)x(t) \\ &= \frac{1}{2 - \sin t}\left(-\frac{\cos t}{2 - \sin t}x(t) + (2 - \sin t)u(t)\right) + \frac{\cos t}{(2 - \sin t)^2}x(t) \\ &= u(t) \end{aligned}$$

In other words, the mapping $u \mapsto y$ is an integrator, which is LTI.

State-space representation: discrete-time case

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$, the set of equations

$$G: \begin{cases} x[t+1] = Ax[t] + Bu[t] \\ y[t] = Cx[t] + Du[t] \end{cases}$$

is refereed to as the state-space representation of a discrete-time LTI system $G: u \mapsto y$. Here

- the (internal) signal x is called the state vector of G
- the upper equation (difference) is called the state equation
- the lower one (algebraic) is called the output equation

The quadruple (A, B, C, D) is dubbed a state-space realization of G.

State-space representation: discrete-time case (contd)

Solution:

$$x[t] = \sum_{s=-\infty}^{t} A^{t-s} Bu[s] \quad \& \quad y[t] = \sum_{s=-\infty}^{t} CA^{t-s} Bu[s] + Du[t]$$

Impulse response:

$$g[t] = D\delta[t] + CA^{k-1}B\mathbb{1}[t-1]$$

Transfer function:

$$G(z) = D + C(zI - A)^{-1}B$$

Similar realizations: still

$$(A, B, C, D) \quad \text{and} \quad (TAT^{-1}, TB, CT^{-1}, D)$$

Outline

From I/O to state space

If (A, B, C, D) is a state-space realization of an LTI G, we know that

$$G(s) = D + C(sI - A)^{-1}B$$

In many situation, we need the other direction, viz.

- given a rational transfer function of a system G, find its realization.

In some applications state equations arise naturally from physics, sometimes they appear quite artificial. Realization are never unique (think of similarity transformations) either. So there are many choices, whose usability depends on a concrete application.

Below we shall see

- a flavor of some common choices.

“Physical” realization

Let $G(s)$ be of the form

$$G(s) = \frac{b}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

(no zeros). It corresponds to the ODE

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0 y(t) = bu(t)$$

w/o derivatives on the input signal. In this case we may always choose

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

as its state vector.

“Physical” realization (contd)

Taking into account that $y^{(n)} = -a_{n-1}y^{(n-1)} - \cdots - a_1\dot{y} - a_0 y + bu$,

$$\dot{x}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \\ y^{(n)}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \\ -a_{n-1}x_n(t) - \cdots - a_0 x_1(t) + bu(t) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u(t)$$

which is a state equation. The output equation reads then

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x(t)$$

“Physical” realization (contd)

Thus, if

$$G(s) = \frac{b}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0},$$

then its possible state-space realization is

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{cccc:c} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -a_0 & -a_1 & \cdots & -a_{n-1} & b \\ \hdashline 1 & 0 & \cdots & 0 & 0 \end{array}\right].$$

This realization is sometimes dubbed *physical*, for its

– state vector comprises the outputs and their first derivatives

(think of position, velocity, acceleration, jerk, et cetera), so is easy to grasp.

What if there are zeros

Let

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

Its output in the Laplace domain

$$Y(s) = G(s)U(s) = \left(\sum_{i=0}^{n-1} b_i s^i\right) \overbrace{\frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}}^{G_v(s)} U(s),$$

Therefore, $y = b_0 v + \cdots + b_{n-1}v^{(n-1)}$, where $v = G_v u$ satisfies

$$v^{(n)}(t) + a_{n-1}v^{(n-1)}(t) + \cdots + a_1\dot{v}(t) + a_0 v(t) = u(t).$$

Now note that G_v admits the “physical” realization with $b = 1$, whose state x_v has $v^{(i-1)}$ as its ith element. Hence,

$$y(t) = \begin{bmatrix} b_0 & \cdots & b_{n-1} \end{bmatrix} x_v(t)$$

and we may just modify the “C” term of the “physical” realization.

Canonical realizations: companion form

Let $G(s)$ be strictly proper, i.e.

$$G(s)=\frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}.$$

The state-space realization discussed above, known as the companion form, is

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]=\left[\begin{array}{c:c} A_{\mathrm{cf}} & B_{\mathrm{cf}} \\ \hdashline C_{\mathrm{cf}} & D_{\mathrm{cf}} \end{array}\right]:=\left[\begin{array}{cccc:c} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -a_0 & -a_1 & \cdots & -a_{n-1} & 1 \\ \hdashline b_0 & b_1 & \cdots & b_{n-1} & 0 \end{array}\right]$$

This realization is

– particularly convenient in feedback control.

But the state vector itself, derivatives of v, is no longer readily interpretable.

Canonical realizations: observer form

Let $G(s)$ be strictly proper, i.e.

$$G(s)=\frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}.$$

Its state-space realization in observer form has

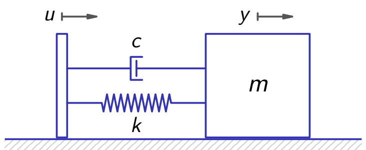
$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]=\left[\begin{array}{c:c} A_{\mathrm{of}} & B_{\mathrm{of}} \\ \hdashline C_{\mathrm{of}} & D_{\mathrm{of}} \end{array}\right]:=\left[\begin{array}{cccc:c} -a_{n-1} & 1 & \cdots & 0 & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 1 & b_1 \\ -a_0 & 0 & \cdots & 0 & b_0 \\ \hdashline 1 & 0 & \cdots & 0 & 0 \end{array}\right].$$

This realization is

– particularly convenient in state estimation

(reconstructing state from partial measurements). Its state is also a mess.

Example



Described by

$$m\ddot{y}(t)+c\dot{y}(t)+ky(t)=c\dot{u}(t)+ku(t)$$

and has no physical realization.

Companion canonical realization:

$$\begin{cases}\dot{x}(t)=\begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}x(t)+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \\ y(t)=\begin{bmatrix} k/m & c/m \end{bmatrix}x(t)\end{cases} \qquad x=\begin{bmatrix} v \\ \dot{v} \end{bmatrix} \qquad \ddot{v}=u-y$$

Observer canonical realization:

$$\begin{cases}\dot{x}(t)=\begin{bmatrix} -c/m & 1 \\ -k/m & 0 \end{bmatrix}x(t)+\begin{bmatrix} c/m \\ k/m \end{bmatrix}u(t) \\ y(t)=\begin{bmatrix} 1 & 0 \end{bmatrix}x(t)\end{cases} \qquad x=\begin{bmatrix} y \\ \dot{y}+(y-u)c/m \end{bmatrix}$$

Canonical realizations: bi-proper case

Let

$$G(s)=\frac{b_ns^n+b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

for $b_n\neq 0$. The trick is to rewrite it as

$$\frac{b_ns^n+b_{n-1}s^{n-1}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_0}=b_n+\overbrace{\frac{(b_{n-1}-b_na_{n-1})s^{n-1}+\cdots+b_0-b_na_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_0}}^{\tilde{G}(s)}$$

Hence, canonical realizations of $G(s)$ are those of the (strictly proper) $\tilde{G}(s)$ complemented by $D=b_n$.

Outline

Solution to state equation

Canonical realizations

Linearization

Nonlinear state-space realization

A class of continuous-time nonlinear systems $G : u \mapsto y$ can be described by

$$G : \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

for some functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, such that

– both they (i.e. f and h) and their derivatives in x and u are continuous.

Such models are frequently derived from first principles. Yet in many cases we prefer to analyze linear models. So an important question is

– how to approximate nonlinear models with linear models ?

Equilibrium

An equilibrium of the system

$$G : \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

is any pair $(x_{\text{eq}}, u_{\text{eq}}) \in \mathbb{R}^n \times \mathbb{R}$ for which the system is at rest, i.e. for which $\dot{x} = f(x, u)$ can be solved for

$$\dot{x} = 0.$$

Hence, an equilibrium should satisfy the algebraic equation

$$f(x_{\text{eq}}, u_{\text{eq}}) = 0.$$

If this equation has no solution in $(x_{\text{eq}}, u_{\text{eq}})$, then no equilibria exist. It may also be the case that there are many (even infinitely many) solutions to this equation, so there are many equilibrium points.

Deviations from equilibrium

Define

$$x_\delta(t) := x(t) - x_{\text{eq}} \quad \text{and} \quad u_\delta(t) := u(t) - u_{\text{eq}},$$

In this case

$$f(x, u) = f(x_{\text{eq}}, u_{\text{eq}}) + \left(\frac{\partial f(x,u)}{\partial x} \bigg|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \right) x_\delta + \left(\frac{\partial f(x,u)}{\partial u} \bigg|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \right) u_\delta + \text{hot}$$

Taking into account that $f(x_{\text{eq}}, u_{\text{eq}}) = 0$ and $\dot{x}_\delta = \dot{x}$, we conclude that

$$\dot{x}_\delta(t) = \left(\frac{\partial f(x,u)}{\partial x} \bigg|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \right) x_\delta(t) + \left(\frac{\partial f(x,u)}{\partial u} \bigg|_{\substack{x = x_{\text{eq}} \\ u = u_{\text{eq}}}} \right) u_\delta(t)$$

is an approximation of $\dot{x} = f(x, u)$ in a "sufficiently small" neighbourhood of $(x, u) = (x_{\text{eq}}, u_{\text{eq}})$. And these dynamics are linear.

Deviations from equilibrium (contd)

The function h in the output equation can be decomposed as

$$h(x,u) = h(x_{\mathrm{eq}},u_{\mathrm{eq}}) + \left(\frac{\partial h(x,u)}{\partial x}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}}\right)x_\delta + \left(\frac{\partial h(x,u)}{\partial u}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}}\right)u_\delta + \text{hot}$$

Defining

$$y_\delta(t) = y(t) - h(x_{\mathrm{eq}},u_{\mathrm{eq}})$$

we conclude that

$$y_\delta(t) = \left(\frac{\partial h(x,u)}{\partial x}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}}\right)x_\delta(t) + \left(\frac{\partial h(x,u)}{\partial u}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}}\right)u_\delta(t)$$

is an approximation of $y = h(x,u)$ in a “sufficiently small” neighbourhood of $(x,u) = (x_{\mathrm{eq}},u_{\mathrm{eq}})$. And this equation is linear.

Linearized dynamics

Thus, we end up with the linear

$$G_\delta : \begin{cases} \dot{x}_\delta(t) = Ax_\delta(t) + Bu_\delta(t) \\ y_\delta(t) = Cx_\delta(t) + Du_\delta(t) \end{cases}$$

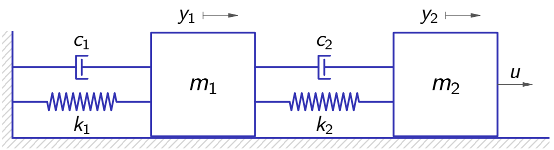
where

$$A := \frac{\partial f(x,u)}{\partial x}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}} \in \mathbb{R}^{n\times n}, \qquad B := \frac{\partial f(x,u)}{\partial u}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}} \in \mathbb{R}^{n},$$

$$C := \frac{\partial h(x,u)}{\partial x}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}} \in \mathbb{R}^{1\times n}, \qquad D := \frac{\partial h(x,u)}{\partial u}\bigg|_{\substack{x=x_{\mathrm{eq}}\\u=u_{\mathrm{eq}}}} \in \mathbb{R}.$$

(this A is known as the Jacobian matrix). The linear system
– G_δ is called the linearization of G around the equilibrium $(x_{\mathrm{eq}},u_{\mathrm{eq}})$
in terms of deviation variables x_δ, u_δ, and y_δ.

Example 1



It is described by $\dot{x} = f(x,u)$ & $y = h(x,u)$ with $x_1 = y_1$, $x_2 = y_2$, $x_3 = \dot{y}_1$, $x_4 = \dot{y}_2$,

$$f(x,u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}^{-1} \begin{bmatrix} x_3 \\ x_4 \\ c_2(x_4 - x_3) - c_1x_3 - k_2(x_1 - x_2 + \xi) - k_1x_1 \\ c_2(x_3 - x_4) + k_2(x_1 - x_2 + \xi) + u \end{bmatrix}$$

and

$$h(x,u) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Example 1: equilibria

Equilibrium equation

$$\begin{bmatrix} x_{\mathrm{eq},3} \\ x_{\mathrm{eq},4} \\ c_2(x_{\mathrm{eq},4} - x_{\mathrm{eq},3}) - c_1x_{\mathrm{eq},3} - k_2(x_{\mathrm{eq},1} - x_{\mathrm{eq},2} + \xi) - k_1x_{\mathrm{eq},1} \\ c_2(x_{\mathrm{eq},3} - x_{\mathrm{eq},4}) + k_2(x_{\mathrm{eq},1} - x_{\mathrm{eq},2} + \xi) + u_{\mathrm{eq}} \end{bmatrix} = 0$$

so that $x_{\mathrm{eq},3} = x_{\mathrm{eq},4} = 0$,

$$k_2(x_{\mathrm{eq},1} - x_{\mathrm{eq},2} + \xi) + k_1x_{\mathrm{eq},1} = 0, \quad \text{and} \quad k_2(x_{\mathrm{eq},1} - x_{\mathrm{eq},2} + \xi) + u_{\mathrm{eq}} = 0$$

(2 equations in 3 variables). Hence,

$$x_{\mathrm{eq}} = \begin{bmatrix} u_{\mathrm{eq}}/k_1 \\ (1/k_1 + 1/k_2)u_{\mathrm{eq}} + \xi \\ 0 \\ 0 \end{bmatrix}$$

for every $u_{\mathrm{eq}} \in \mathbb{R}$.

Example 1: derivatives

Derivatives:

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1-k_2 & k_2 & -c_1-c_2 & c_2 \\ k_2 & -k_2 & c_2 & -c_2 \end{bmatrix}$$

$$\frac{\partial f(x,u)}{\partial u} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix}$$

$$\frac{\partial h(x,u)}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial h(x,u)}{\partial u} = 0$$

are all independent of x and u, meaning that higher derivatives are zero and

- the first-order expansion is accurate.

Example 1: deviation variables and linear model

In terms of

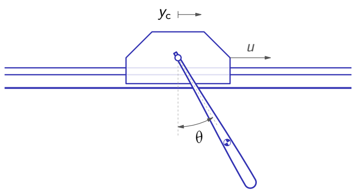
$$x_\delta(t) = x(t) - \begin{bmatrix} (1/k_1)u_{\text{eq}} \\ (1/k_1 + 1/k_2)u_{\text{eq}} + \xi \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad u_\delta(t) = u(t) - u_{\text{eq}}$$

the linear dynamics

$$\begin{cases} \dot{x}_\delta(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} x_\delta(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u_\delta(t) \\ y_\delta(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_\delta(t) \end{cases}$$

is an accurate linear description of the mass-string-damper system.

Example 2



It is described by $\dot{x} = f(x,u)$ & $y = h(x,u)$ with $x_1 = y_c$, $x_2 = \theta$, $x_3 = \dot{y}_c$, $x_4 = \dot{\theta}$,

$$f(x,u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M+m & ml\cos x_2 \\ 0 & 0 & ml\cos x_2 & J+ml^2 \end{bmatrix}^{-1} \begin{bmatrix} x_3 \\ x_4 \\ mlx_4^2 \sin x_2 + u \\ -mlg\sin x_2 \end{bmatrix},$$

and

$$h(x,u) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Example 2: equilibria

Because the inverted matrix is always nonsingular, equilibrium satisfies

$$\begin{bmatrix} x_{\text{eq},3} \\ x_{\text{eq},4} \\ mlx_{\text{eq},4}^2 \sin x_{\text{eq},2} + u_{\text{eq}} \\ -mlg\sin x_{\text{eq},2} \end{bmatrix} = 0$$

so that $x_{\text{eq},3} = x_{\text{eq},4} = u_{\text{eq}} = 0$, $\sin x_{\text{eq},2} = 0$, and any $x_{\text{eq},1}$. Hence,

$$x_{\text{eq}} = \begin{bmatrix} x_{\text{eq},1} \\ \pi k \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad u_{\text{eq}} = 0$$

for all $x_{\text{eq},1} \in \mathbb{R}$, $k \in \mathbb{Z}$ (i.e. the cart may be in any position, the pendulum may be either upright or hung down, and the external force must be zero).

Example 2: derivatives

Note that f is of the form $f = M_f^{-1} N_f$ and take into account that

$$\frac{\partial M_f^{-1} N_f}{\partial v_i} = M_f^{-1} \frac{\partial N_f}{\partial v_i} + \frac{\partial M_f^{-1}}{\partial v_i} N_f = M_f^{-1} \left(\frac{\partial N_f}{\partial v_i} - \frac{\partial M_f}{\partial v_i} f \right)$$

and $f = 0$ at every equilibrium point, by definition. Hence,

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M+m & ml\cos(\pi k) \\ 0 & 0 & ml\cos(\pi k) & J + ml^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -mlg\cos(\pi k) & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f(x,u)}{\partial u} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M+m & ml\cos(\pi k) \\ 0 & 0 & ml\cos(\pi k) & J + ml^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

at every equilibrium point (with obvious derivatives of h).

Example 2: pendulum down

If k is even (pendulum is down), then $\cos(\pi k) = 1$ and

$$\left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2l^2g/\alpha & 0 & 0 \\ 0 & -m(M+m)lg/\alpha & 0 & 0 \end{bmatrix} =: A_{\text{down}}$$

$$\left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \begin{bmatrix} 0 \\ 0 \\ (J+ml^2)/\alpha \\ -ml/\alpha \end{bmatrix} =: B_{\text{down}}$$

where $\alpha := (M+m)J + Mml^2$.

Example 2: pendulum up

If k is odd (inverted pendulum), then $\cos(\pi k) = -1$

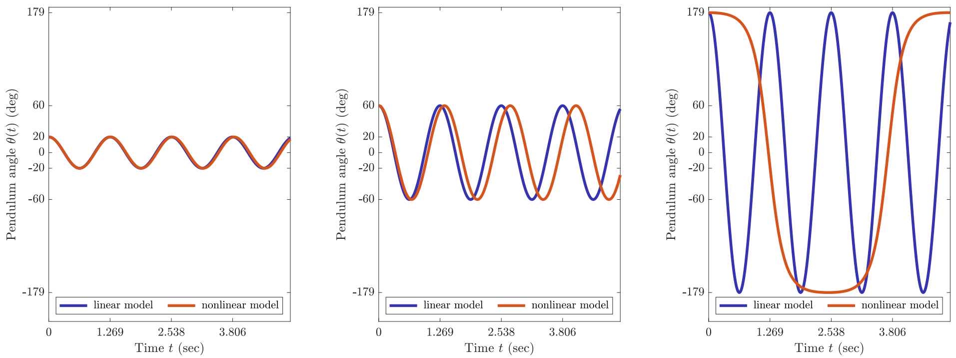
$$\left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2l^2g/\alpha & 0 & 0 \\ 0 & m(M+m)lg/\alpha & 0 & 0 \end{bmatrix} =: A_{\text{up}}$$

$$\left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=x_{\text{eq}}\\u=u_{\text{eq}}}} = \begin{bmatrix} 0 \\ 0 \\ (J+ml^2)/\alpha \\ ml/\alpha \end{bmatrix} =: B_{\text{up}}$$

where $\alpha := (M+m)J + Mml^2$.

Example 2: simulations

Nonlinear and linearized ($k = 0$) responses,



are close for small initial angles, but become quite different for large ones.

Example 3: SIR epidemic spread model

Let

- s be the number of susceptibles in the population
- i be the number of infectives in the population
- r be the number of removed in the population

The SIR model:

$$\begin{cases} \frac{d}{dt}s(t) = -b(t)s(t)i(t) \\ \frac{d}{dt}i(t) = b(t)s(t)i(t) - ai(t) \\ \frac{d}{dt}r(t) = ai(t) \end{cases} \quad \text{or} \quad \dot{x} = \begin{bmatrix} -x_1x_2u \\ x_1x_2u - ax_2 \\ ax_2 \end{bmatrix}$$

with the state and control signals

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ i \\ r \end{bmatrix} \quad \text{and} \quad u = b,$$

respectively (u affects it via lockdowns / vaccination).

Example 3: equilibrium and derivatives

Equilibrium:

$$\begin{bmatrix} -x_{\text{eq},1}x_{\text{eq},2}u_{\text{eq}} \\ x_{\text{eq},1}x_{\text{eq},2}u_{\text{eq}} - ax_{\text{eq},2} \\ ax_{\text{eq},2} \end{bmatrix} = 0 \quad \Longrightarrow \quad x_{\text{eq}} = \begin{bmatrix} x_{\text{eq},1} \\ 0 \\ x_{\text{eq},3} \end{bmatrix}$$

for arbitrary $x_{\text{eq},1}, x_{\text{eq},3}, u_{\text{eq}} \in \mathbb{R}$. Hence,

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \left.\begin{bmatrix} -x_2u & -x_1u & 0 \\ x_2u & x_1u - a & 0 \\ 0 & a & 0 \end{bmatrix}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \begin{bmatrix} 0 & -x_{\text{eq},1}u_{\text{eq}} & 0 \\ 0 & x_{\text{eq},1}u_{\text{eq}} - a & 0 \\ 0 & a & 0 \end{bmatrix}$$

$$\left.\frac{\partial f(x)}{\partial u}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \left.\begin{bmatrix} -x_1x_2 \\ x_1x_2 \\ 0 \end{bmatrix}\right|_{\substack{x=x_{\text{eq}} \\ u=u_{\text{eq}}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 3: deviation variables and linear equations

In terms of

$$x_\delta(t) = \begin{bmatrix} s(t) \\ i(t) \\ r(t) \end{bmatrix} - \begin{bmatrix} x_{\text{eq},1} \\ 0 \\ x_{\text{eq},3} \end{bmatrix} \quad \text{and} \quad u_\delta(t) = u(t) - u_{\text{eq}}$$

the linearized state equation is

$$\dot{x}_\delta(t) = \begin{bmatrix} 0 & -x_{\text{eq},1}u_{\text{eq}} & 0 \\ 0 & x_{\text{eq},1}u_{\text{eq}} - a & 0 \\ 0 & a & 0 \end{bmatrix} x_\delta(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u_\delta(t)$$

is a description of the SIR model. But

- this model is useless, it is not affected by the input signal at all.

This shows that

- linearized model might make no sense, even if it can be derived.

Discrete case

A class of discrete-time nonlinear systems $G : u \mapsto y$ can be described by

$$G : \begin{cases} x[t+1] = f(x[t], u[t]) \\ y[t] = h(x[t], u[t]) \end{cases}$$

for some functions f and h. Its linearization is almost identical to that of its continuous-time counterpart, modulo the definition of equilibria, which now must satisfy $x[t+1] = x[t]$, so that the algebraic equation to check is

$$f(x_{\text{eq}}, u_{\text{eq}}) = x_{\text{eq}}$$

rather than $f(x_{\text{eq}}, u_{\text{eq}}) = 0$. The rest is mechanical...