Linear Systems (034032)
lecture no. 9

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Previously on Linear Systems . . .

Let G be a stable continuous-time LTI system with the impulse response g. Its frequency response $G(\mathrm{j}\omega) = (\mathfrak{F}\{g\})(\mathrm{j}\omega)$ or

$$G(\mathrm{j}\omega) = G(s)|_{s=\mathrm{j}\omega} = (\mathfrak{L}\{g\})(\mathrm{j}\omega).$$

In the discrete-time case, the frequency response $G(\mathrm{e}^{\mathrm{j}\theta}) = (\mathfrak{F}\{g\})(\mathrm{e}^{\mathrm{j}\theta})$ or

$$G(\mathrm{e}^{\mathrm{j}\theta}) = G(z)|_{z=\mathrm{e}^{\mathrm{j}\theta}} = (\mathfrak{Z}\{g\})(\mathrm{e}^{\mathrm{j}\theta}).$$

The frequency response shapes

- the response to harmonic inputs
- the response to periodic inputs
- the steady-state response to test sine wave inpute

Outline

Systems as filters I

Frequency response plots

Systems as filters II

Frequency-domain response of LTI systems

By the convolution property of the Fourier transform,

$$y(t) = (Gu)(t) \quad \Longleftrightarrow \quad Y(\mathrm{j}\omega) = G(\mathrm{j}\omega)U(\mathrm{j}\omega)$$

whenever the corresponding Fourier transforms exist. Hence, the frequency response of G just scales every harmonic component of the input, so that

- harmonics with frequencies ω at which $|G(\mathrm{j}\omega)| > 1$ are amplified
- harmonics with frequencies ω at which $|G(\mathrm{j}\omega)| < 1$ are attenuated
- harmonics with frequencies ω at which $|G(\mathrm{j}\omega)| < 1/\sqrt{2}$ do not pass
  this is a convention, facilitating categorical conclusions; take it with a grain of salt

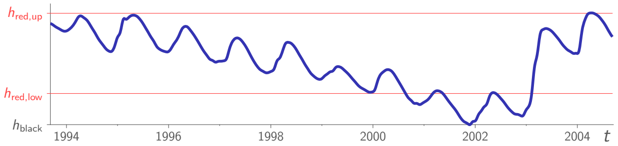
This multiplication property

- facilitates the use of LTI system as filters,

whose task is to shape the spectrum of signals of interest (to pass "desired" components and to block "unwanted" ones).

Example

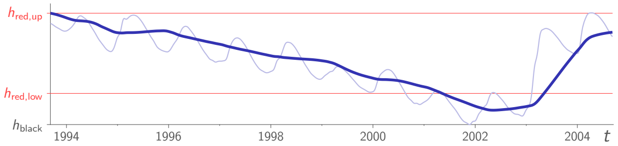
Kinneret water level h from Sep 1993 to Sep 2004



If processed by the finite-memory integrator, so that $h_a = G_{\text{fmi},1} h$ or

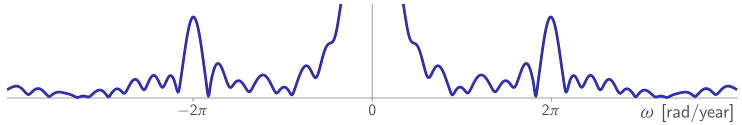
$$h_a(t) = \int_{t-1}^{t} h(s)\,ds \iff H_a(s) = \frac{1 - e^{-s}}{s} H(s)$$

(average over the last year), we end up with



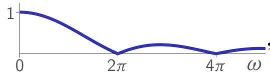
Example: frequency-domain insight

The amplitude spectrum of h (with $h_{\text{red,up}}$ taken as zero) is

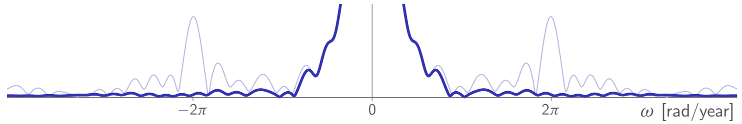


The frequency response magnitude of $G_{\text{fmi},1}$,

$$|G_{\text{fmi},1}(j\omega)| = \left|\frac{1 - e^{-j\omega}}{j\omega}\right| = \left|\operatorname{sinc}\left(\frac{\omega}{2}\right)\right| =$$



is zero at $\omega = 2\pi$, so the peaks at $\omega = \pm 2\pi$ [rad/year] are filtered out,



eliminating the effect of annual cycles.

Classification

Depending on their purpose, filters may be categorized as

- low-pass filters allow only harmonics with $\omega \le \omega_b$ to pass
  i.e. $|G(j\omega)| \ge \frac{1}{\sqrt{2}} \iff \omega \le \omega_b$, which is known as the bandwidth of G
- high-pass filters allow only harmonics with $\omega \ge \omega_c$ to pass
  i.e. $|G(j\omega)| \ge \frac{1}{\sqrt{2}} \iff \omega \ge \omega_c$, which is known as the cutoff frequency of G
- band-pass filters allow only harmonics with $\omega_1 \le \omega \le \omega_2$ to pass
  i.e. $|G(j\omega)| \ge \frac{1}{\sqrt{2}} \iff \omega \in [\omega_1, \omega_2]$
- band-stop filters allow only harmonics with $\omega \le \omega_1$ and $\omega \ge \omega_2$ to pass
  i.e. $|G(j\omega)| \ge \frac{1}{\sqrt{2}} \iff \omega \notin (\omega_1, \omega_2)$, which is known as the stopband of G

The question is

- what are systems, whose frequency responses are such filters?

Ideal filters

- ideal low-pass has

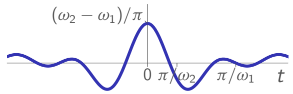
$$F_{\text{ilp},\omega_b}(j\omega) = \operatorname{rect}_{2\omega_b}(\omega) \iff f_{\text{ilp},\omega_b}(t) = \frac{\omega_b}{\pi} \operatorname{sinc}(\omega_b t)$$

  (see Lect. 3, Slide 37) and is non-causal and even BIBO *un*stable[1], for $f_{\text{ilp},\omega_b} \notin L_1$. Hence, not quite practical.
- ideal high-pass $F_{\text{ihp},\omega_c} = 1 - F_{\text{ilp},\omega_c}$ and $f_{\text{ihp},\omega_b}(t) = \delta(t) - \frac{\omega_b}{\pi}\operatorname{sinc}(\omega_b t)$
- ideal band-pass has

$$F_{\text{ibp},[\omega_1,\omega_2]}(j\omega) = \operatorname{rect}_{2\omega_2}(\omega) - \operatorname{rect}_{2\omega_1}(\omega)$$

$$\Updownarrow$$

$$f_{\text{ibp},[\omega_1,\omega_2]}(t) = \frac{\omega_2}{\pi}\operatorname{sinc}(\omega_2 t) - \frac{\omega_1}{\pi}\operatorname{sinc}(\omega_1 t) =$$



- ideal band-stop $F_{\text{ibs},[\omega_1,\omega_2]} = 1 - F_{\text{ibp},[\omega_1,\omega_2]}$

[1] But, strangely enough, it is L_2-stable, try to prove it with the material of Lects. 3 & 6.

Outline

Systems as filters I

Frequency response plots

Systems as filters II

Decibels

Decibel (dB) is a unit of measurement expressing the ratio of two values of a root-power quantity on a logarithmic scale. Applying to $|G(j\omega)|$, it is

$$|G(j\omega)|_{(\mathrm{dB})} := 20\log_{10}|G(j\omega)|$$

Useful properties:

- $|G_1(j\omega)G_2(j\omega)|_{(\mathrm{dB})} = |G_1(j\omega)|_{(\mathrm{dB})} + |G_2(j\omega)|_{(\mathrm{dB})}$
- $|[G(j\omega)]^n|_{(\mathrm{dB})} = n|G(j\omega)|_{(\mathrm{dB})}$ for all $n \in \mathbb{R}$
- $\left|\dfrac{1}{G(j\omega)}\right|_{(\mathrm{dB})} = -|G(j\omega)|_{(\mathrm{dB})}$
- $\left|\dfrac{G_1(j\omega)}{G_2(j\omega)}\right|_{(\mathrm{dB})} = |G_1(j\omega)|_{(\mathrm{dB})} - |G_2(j\omega)|_{(\mathrm{dB})}$

Some common values[2] (memorize those in blue):

gain	1	$\sqrt{2}$	2	4	5	10	25	50	100	1000
dB	0	≈ 3	≈ 6	≈ 12	≈ 14	20	≈ 28	≈ 34	40	60

[2] The `mag2db` and `db2mag` commands of Matlab are handy.

Bode plot

Consists of

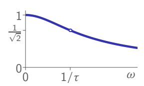
- Bode magnitude plot of $|G(j\omega)|$ (in dB) vs. ω (in logarithmic scale)
- Bode phase plot of $\arg(G(j\omega))$ (in deg) vs. ω (in logarithmic scale)

In the logarithmic scale the distance between ω_0 and $N\omega_0$ does *not* depend on ω_0 for a given $N \in \mathbb{R}_+$ ($N = 2$ is an octave, $N = 10$ is a decade).

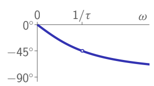
Example: For $G_{\mathrm{therm}}(s) = 1/(\tau s + 1)$,

$$G_{\mathrm{therm}}(j\omega) = \frac{1}{\sqrt{1+\tau^2\omega^2}}e^{-j\arctan(\tau\omega)}$$

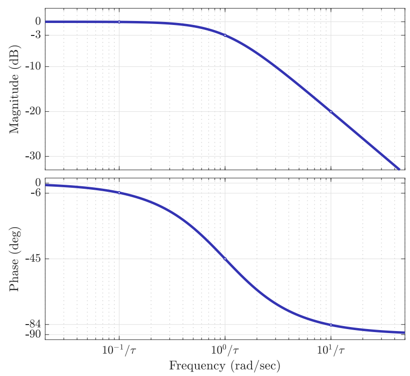
with the magnitude and phase as



and



respectively. The same on the Bode plot:



Bode plot: advantages

- factors add up on both magnitude and phase plots, meaning the Bode plots of systems with real-rational transfer functions, like

$$G(s) = \frac{b_m \prod_i (s - z_i) \prod_j (s^2 + 2\zeta_{z,i}\omega_{z,i}s + \omega_{z,i}^2)}{\prod_i (s - p_i) \prod_i (s^2 + 2\zeta_{p,i}\omega_{p,i}s + \omega_{p,i}^2)}$$

  can be built by superposing frequency responses of 3 basic blocks[3]
  0. static gain, like k for $k \in \mathbb{R} \setminus \{0\}$
  1. first-order factor, like $s + a$, for $a \in \mathbb{R}$
  2. second-order factor, like $s^2 + 2\zeta\omega_n s + \omega_n^2$, for $\zeta \in (-1, 1)$ & $\omega_n > 0$
- very large magnitudes are not that large in dB
- logarithmic frequency scale facilitates viewing wider frequency ranges

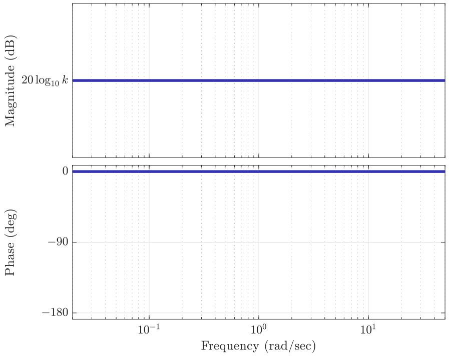
[3] If they are in denominators, then both their magnitude (dB) and phase change sign.

Basic blocks: static gain

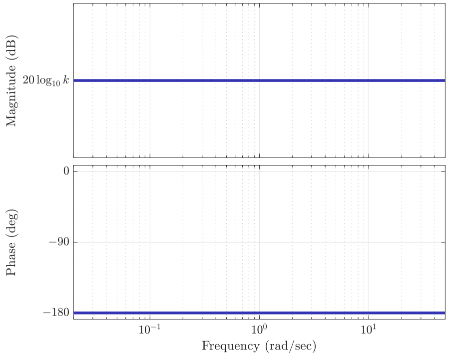
If $G(s) = k$, then

$$G(\mathrm{j}\omega) = k = |k| \begin{cases} \mathrm{e}^{\mathrm{j}0} & \text{if } k > 0 \\ \mathrm{e}^{-\mathrm{j}\pi} & \text{if } k < 0 \end{cases}$$

and both magnitude and phase plots are horizontal lines:



if $k > 0$



if $k < 0$

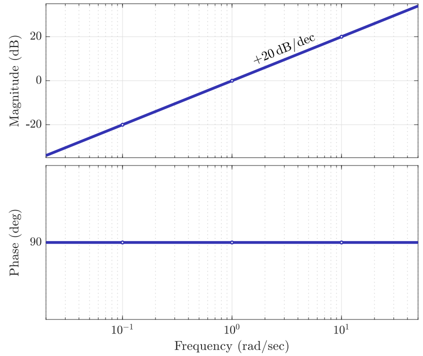
Basic blocks: 1-order factor, $a = 0$

If $G(s) = s$, then

$$G(\mathrm{j}\omega) = \mathrm{j}\omega = \omega \mathrm{e}^{\mathrm{j}\pi/2}$$

and

- the Bode magnitude plot is a straight line with the $+20\,$dB/dec slope, passing through the $0\,$dB level at $\omega = 1\,$rad/sec
- the Bode phase plot is a horizontal line



Basic blocks: 1-order factor, $a \neq 0$

If $a \neq 0$, then it is convenient to normalize the static gain of $s + a$. Hence, the basic block is $G(s) = \tau s + 1$ for $\tau = 1/a \neq 0$, for which

$$G(\mathrm{j}\omega) = 1 + \mathrm{j}\tau\omega = \sqrt{1 + \tau^2\omega^2}\, \mathrm{e}^{\mathrm{j}\arctan(\tau\omega)}$$

The magnitude can be approximated as

$$|G(\mathrm{j}\omega)| = \sqrt{1 + \tau^2\omega^2} \approx \begin{cases} 1 & \text{if } |\tau|\omega < 1 \\ |\tau|\omega & \text{if } |\tau|\omega > 1 \end{cases}$$

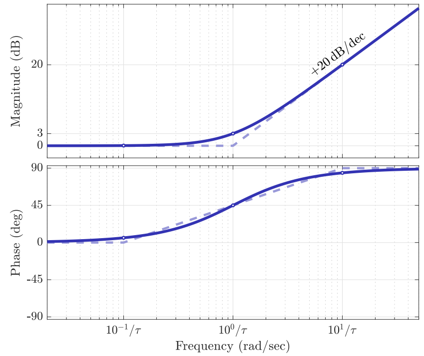
which corresponds to straight lines (the frequency $\omega = 1/\tau$ is known as the corner frequency). The phase can be approximated as

$$\arg(G(\mathrm{j}\omega)) = \arctan(\tau\omega) \approx \operatorname{sign}\tau \begin{cases} 0^\circ & \text{if } |\tau|\omega < \frac{1}{10} \\ 45^\circ(1 + \log_{10}\omega) & \text{if } \frac{1}{10} \leq |\tau|\omega \leq 10 \\ 90^\circ & \text{if } |\tau|\omega > 10 \end{cases}$$

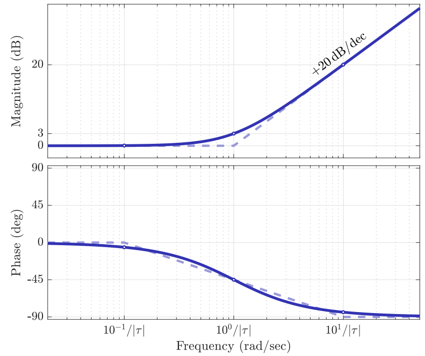
(error within $\pm 5.711^\circ$), which corresponds to straight lines too for $\log \omega$.

Basic blocks: 1-order factor, $a \neq 0$ (contd)

Thus, precise (solid) and approximate (dashed) Bode plots are



if $\tau > 0$



if $\tau < 0$

- drawing precise Bode is easy nowadays (e.g. bode in Matlab)
- approximate Bode is still useful for a quick mental grasping

Basic blocks: 2-order factor, $\zeta \neq 0$

If $G(s) = (s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1$ (with the normalized static gain), then

$$G(j\omega) = 1 - \frac{\omega^2}{\omega_n^2} + j\,2\zeta\frac{\omega}{\omega_n} = \sqrt{\Bigl(1 - \frac{\omega^2}{\omega_n^2}\Bigr)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}\, e^{j\arg(G(j\omega))},$$

where, assuming $\arctan\phi \in [-\pi/2, \pi/2]$,

$$\arg(G(j\omega)) = \arctan\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} + \begin{cases} 0 & \text{if } \omega \le \omega_n \\ \pi & \text{if } \omega \ge \omega_n \wedge \zeta > 0 \\ -\pi & \text{if } \omega \ge \omega_n \wedge \zeta < 0 \end{cases}$$

which is a continuous and monotonic function of ω (increasing if $\zeta > 0$ and decreasing if $\zeta < 0$). Both the magnitude and phase may be approximated by piecewise linear functions, but this is accurate only around $|\zeta| = 1/\sqrt{2}$.

Basic blocks: 2-order factor, $\zeta \neq 0$ (contd)

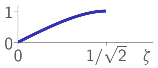
The derivative of the magnitude

$$\frac{|G(j\omega)|}{d\omega} = \underbrace{\frac{2}{|G(j\omega)|}\frac{\omega}{\omega_n}}_{>0,\ \forall\omega>0}\Bigl(\frac{\omega^2}{\omega_n^2} + 2\zeta^2 - 1\Bigr).$$

Hence,

- if $|\zeta| \ge \frac{1}{\sqrt{2}}$, then $|G(j\omega)|$ is monotonically increasing
- if $|\zeta| < \frac{1}{\sqrt{2}}$, then $|G(j\omega)|$
  - monotonically decreases for $\omega < \sqrt{1 - 2\zeta^2}\,\omega_n$
  - monotonically increases for $\omega > \sqrt{1 - 2\zeta^2}\,\omega_n$
  - has

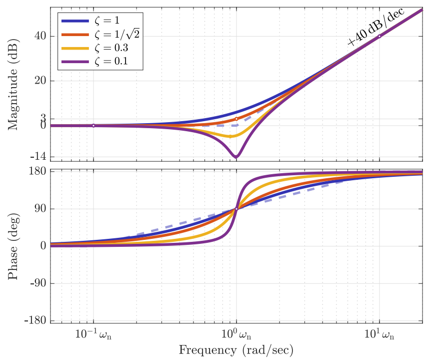
$$\min_\omega |G(j\omega)| = 2\zeta\sqrt{1 - \zeta^2} =$$



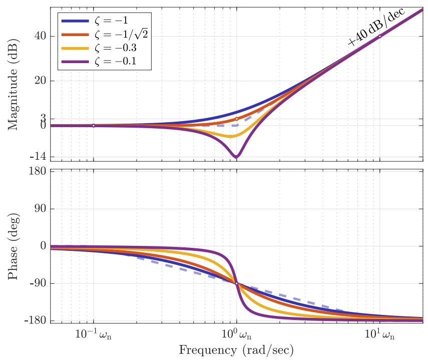
attainable at $\omega = \sqrt{1 - 2\zeta^2}\,\omega_n < \omega_n$

Basic blocks: 2-order factor, $\zeta \neq 0$ (contd)

Thus, precise (solid) and approximate (dashed) Bode plots are



if $\zeta > 0$



if $\zeta < 0$

If ζ is "far" from $1/\sqrt{2}$, the

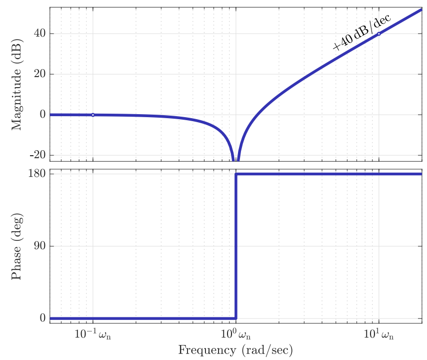
- approximate Bode plots are not quite accurate around $\omega = \omega_n$

Basic blocks: 2-order factor, $\zeta = 0$

If $G(s) = (s/\omega_n)^2 + 1$ (the static gain is again normalized), then

$$G(j\omega) = 1 - \frac{\omega^2}{\omega_n^2} = \Bigl|1 - \frac{\omega^2}{\omega_n^2}\Bigr| \begin{cases} e^{j0} & \text{if } \omega < \omega_n \\ e^{j\pi} & \text{if } \omega > \omega_n \end{cases}$$

resulting in

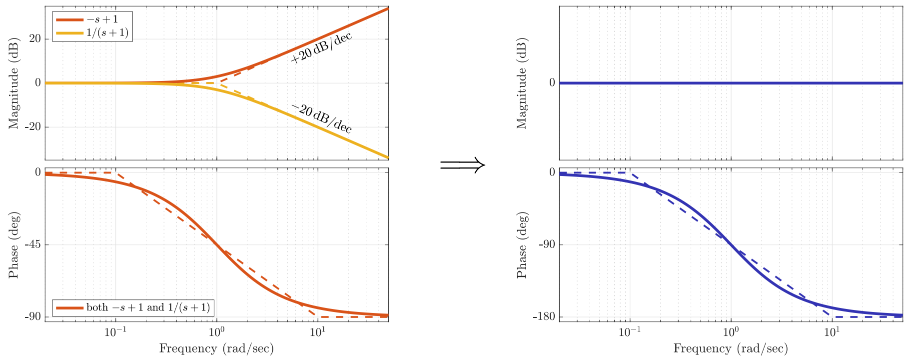


Example 1

If

$$G_1(s) = \frac{-s+1}{s+1}$$

then



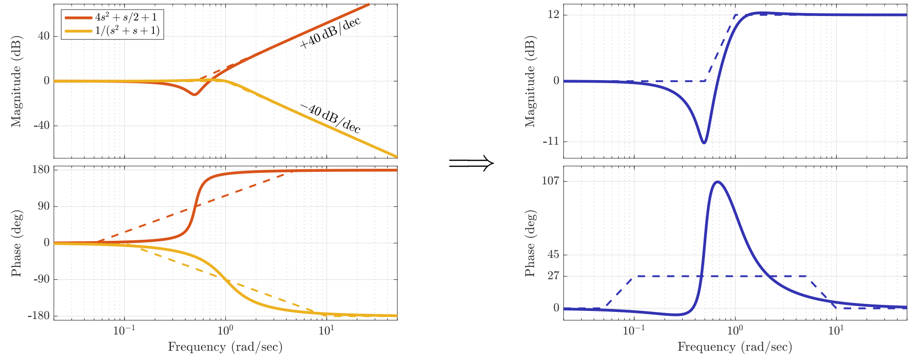
and the high-frequency gain is $G_1(\infty) = 1 = 0\,\text{dB}$.

Example 2

If

$$G_2(s) = \frac{4s^2 + s/2 + 1}{s^2 + s + 1}$$

then



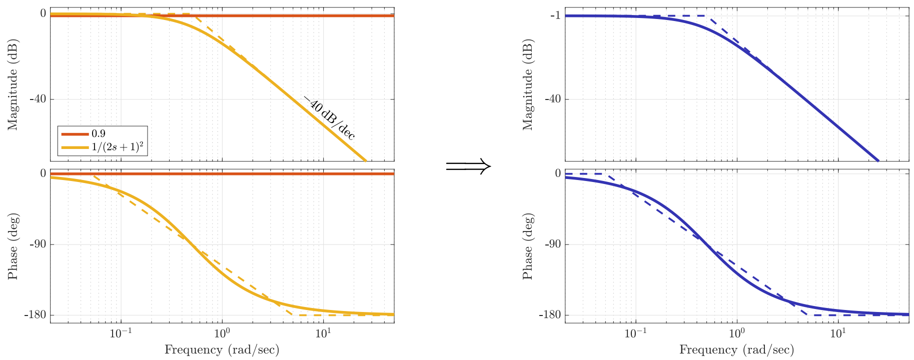
and the high-frequency gain is $G_2(\infty) = 4 \approx 12\,\text{dB}$.

Example 3

If

$$G_3(s) = \frac{0.9}{(2s+1)^2}$$

then



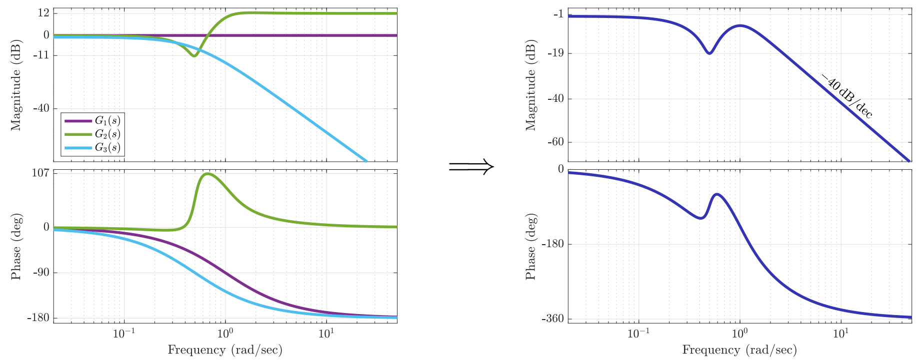
and the magnitude decays at high frequencies with a slope of $-40\,\text{dB/dec}$.

Example 4

If

$$G(s) = G_1(s)G_2(s)G_3(s) = \frac{0.9(-s+1)(4s^2+s/2+1)}{(s+1)(2s+1)^2(s^2+s+1)}$$

then



Asymptotic properties of Bode plots: magnitude

At low frequencies:

- every zero at the origin contributes a slope of $+20\,\mathrm{dB/dec}$
- every integrator (pole at the origin) contributes a slope of $-20\,\mathrm{dB/dec}$
- if no poles/zeros at the origin, starts as a horizontal line at $|G(0)|_{(\mathrm{dB})}$

At high frequencies:

- every zero adds a slope of $+20\,\mathrm{dB/dec}$
- every pole adds a slope of $-20\,\mathrm{dB/dec}$
- if $G(s)$ is bi-proper, ends as a horizontal line at $|G(\infty)|_{(\mathrm{dB})}$

Asymptotic properties of Bode plots: phase

At low frequencies:

- every zero at the origin contributes a phase lead of $+90^\circ$
- every integrator (pole at the origin) contributes a phase lag of -90°

At high frequencies:

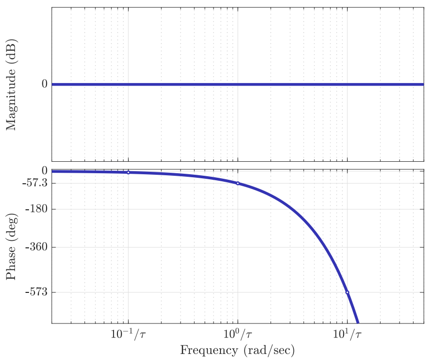
- every zero in $\mathbb{C} \setminus \mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ adds a phase lead of 90°
- every pole in $\mathbb{C} \setminus \mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ adds a phase lag of -90°
- every zero in $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ adds a phase lag of -90°
- every pole in $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ adds a phase lead of 90°

Bode for non-rational transfer functions

If

$$G(s) = \bar{D}_\tau(s) = \mathrm{e}^{-\tau s}, \quad \tau > 0$$

then $G(\mathrm{j}\omega) = \mathrm{e}^{-\mathrm{j}\tau\omega}$, so that $|G(\mathrm{j}\omega)| = 1$ and $\arg(G(\mathrm{j}\omega)) = -\tau\omega$ [rad]. We therefore have



Bode for non-rational transfer functions (contd)

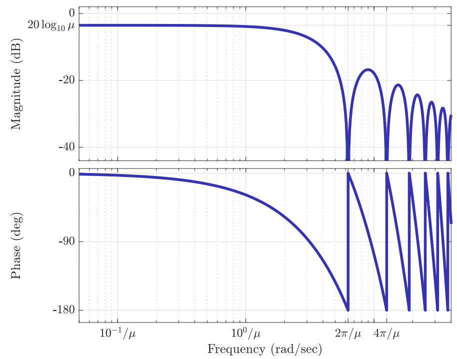
If

$$G(s) = G_{\mathrm{fmi},\mu}(s) = \frac{1 - \mathrm{e}^{-\mu s}}{s}, \quad \mu > 0$$

then

$$G(\mathrm{j}\omega) = \frac{1 - \mathrm{e}^{-\mathrm{j}\mu\omega}}{\mathrm{j}\omega} = \mu \frac{\mathrm{e}^{\mathrm{j}\mu\omega/2} - \mathrm{e}^{-\mathrm{j}\mu\omega/2}}{\mathrm{j}2\,\mu\omega/2}\mathrm{e}^{-\mathrm{j}\mu\omega/2} = \mu \operatorname{sinc}\Big(\frac{\mu\omega}{2}\Big)\mathrm{e}^{-\mathrm{j}\mu\omega/2}$$

and we have

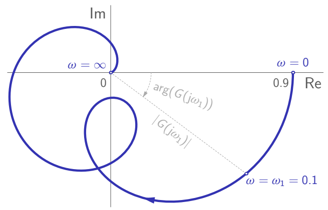


Polar plot

Shows Im $G(j\omega)$ vs. Re $G(j\omega)$ as the frequency ω grows from 0 to ∞, with an arrow indicating the growth direction of ω.

Example 4:

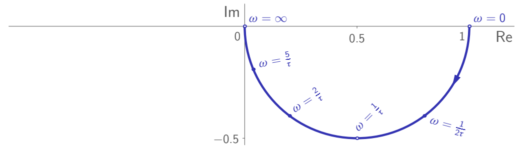
$$G(s) = \frac{0.9(-s+1)(4s^2+s/2+1)}{(s+1)(2s+1)^2(s^2+s+1)} \quad \Longrightarrow$$



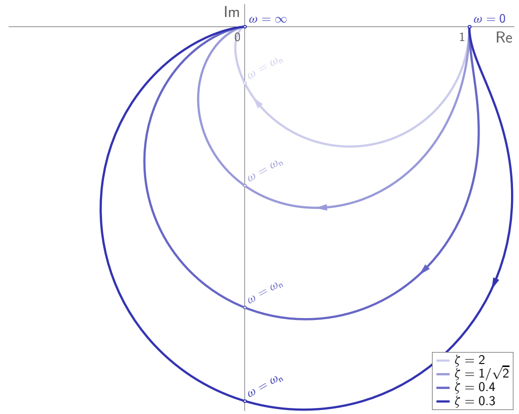
Polar plots are

- less informative than Bode — the frequency is hidden
- harder to draw manually than Bode — no superposition rules hold
- produced by the `nyquist` command of Matlab
  Draws the plot for $-\infty < \omega < \infty$ (aka the Nyquist diagram). To produce the plain polar plot, use `setoptions(nyquistplot(G),'ShowFullContour','off')`
- very important in feedback control applications (the Nyquist criterion)

Some polar plots

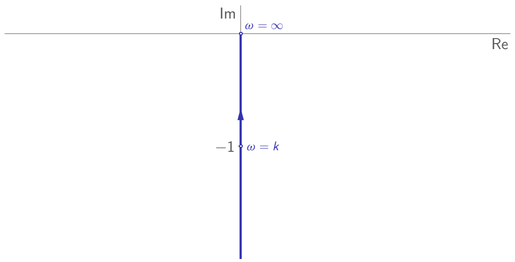


$$G(s) = \frac{1}{\tau s + 1}$$

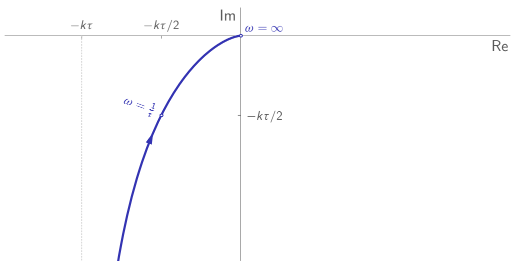


$$G(s) = \frac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1}$$

Some polar plots (contd)

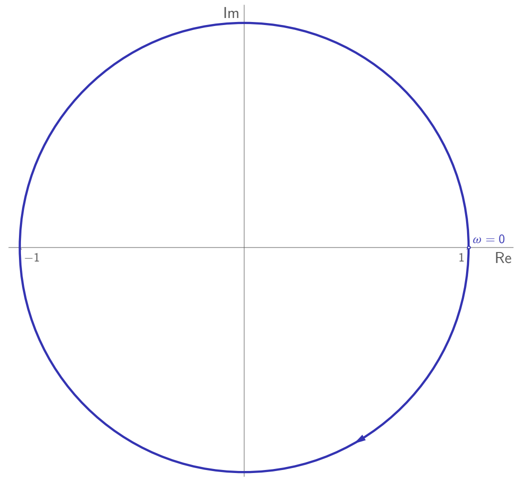


$$G(s) = \frac{k}{s},\ k > 0$$



$$G(s) = \frac{k}{s(\tau s + 1)},\ k > 0$$

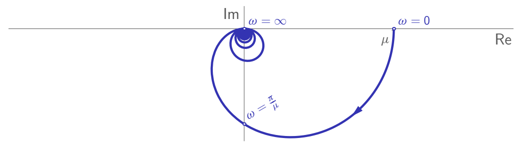
Some polar plots (contd)



$$G(s) = e^{-\tau s}$$

or

$$G(s) = \left(\frac{-\tau s + 1}{\tau s + 1}\right)^2$$



$$G(s) = \frac{1 - e^{-\mu s}}{s},\ \mu > 0$$

Outline

Systems as filters I

Frequency response plots

Systems as filters II

Butterworth polynomials

The Butterworth polynomial of degree n, $B_n(s)$, is the Hurwitz polynomials such that

$$|B_n(\mathrm{j}\omega)|^2 = 1 + \omega^{2n}$$

Its general form is (depending on whether n is even or odd)

$$B_n(s) = \prod_{i=1}^{n/2}(s^2 + 2\zeta_i s + 1) \quad \text{or} \quad B_n(s) = (s+1)\prod_{i=1}^{(n-1)/2}(s^2 + 2\zeta_i s + 1)$$

where

$$\zeta_i := \sin\Big(\frac{2i-1}{2n}\pi\Big) \in (0,1), \quad i \in \mathbb{Z}_{1..\lfloor n/2 \rfloor}$$

Roots of $B_n(s)$ are at equally-spaced points in $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0 \wedge |s| = 1\}$.
Particular cases:

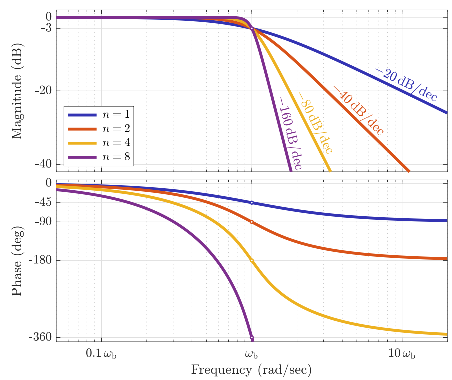
$$B_1(s) = s + 1, \quad B_2(s) = s^2 + \sqrt{2}s + 1, \quad B_3(s) = (s+1)(s^2 + s + 1).$$

Low-pass Butterworth filter

The n-order low-pass filter

$$F(s) = \frac{1}{B_n(s/\omega_\mathrm{b})} \quad \Longrightarrow \quad |F(\mathrm{j}\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_\mathrm{b})^{2n}}}$$

is monotonically decreasing with $|F(\mathrm{j}\omega_\mathrm{b})| = \frac{1}{\sqrt{2}}$ (i.e. ω_b is its bandwidth):

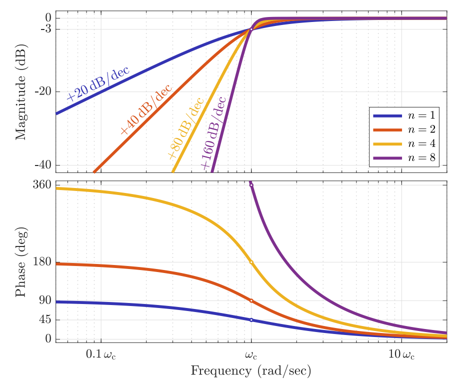


High-pass Butterworth filter

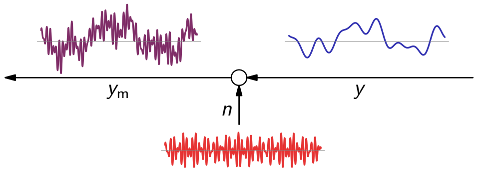
The n-order high-pass filter

$$F(s) = \frac{(s/\omega_\mathrm{c})^n}{B_n(s/\omega_\mathrm{c})} \quad \Longrightarrow \quad |F(\mathrm{j}\omega)| = \frac{(\omega/\omega_\mathrm{c})^n}{\sqrt{1 + (\omega/\omega_\mathrm{c})^{2n}}}$$

is monotonically increasing with $|F(\mathrm{j}\omega_\mathrm{c})| = \frac{1}{\sqrt{2}}$ (i.e. ω_c is its cut-off freq.):



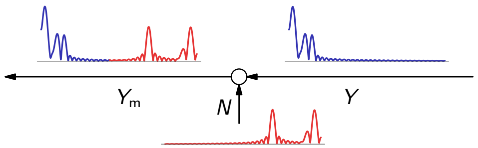
Example from Lect. 3



The question was

– how signal (y) can be recovered from its corrupt measurements (y_m)?

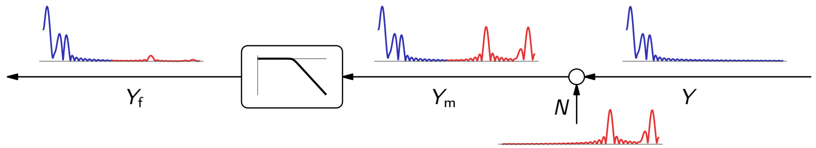
This might appear a tough task, for there is no way we can separate y from n in the time domain . . . But in the frequency domain these signals are well separated,



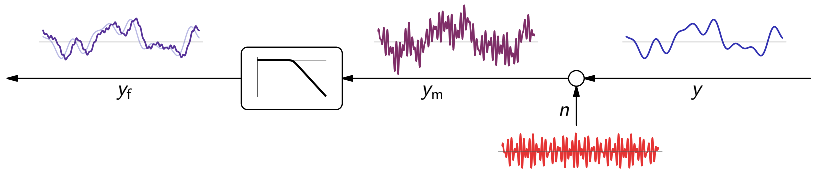
suggesting that the frequency-domain viewpoint is valuable.

Example from Lect. 3 (contd)

If we process the measurement by a low-pass filter (4-order Butterworth in this case), then the result,



separates the slow signal y from fast noise n. In the time domain,



although with certain phase lag . . .

Notch filter

Is a narrow stopband band-stop filter of the form

$$F(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \quad \Longrightarrow \quad |F(j\omega)| = \sqrt{\frac{(\omega^2 - \omega_0^2)^2}{(\omega^2 - \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$

is

– monotonically decreasing in $\omega < \omega_0$

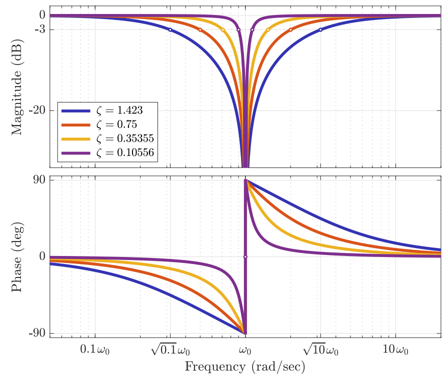
– monotonically increasing in $\omega > \omega_0$

with

$$|F(j\omega)| = \frac{1}{\sqrt{2}} \iff \omega = \begin{cases} \omega_1 := (\sqrt{1+\zeta^2} - \zeta)\omega_0 < \omega_0 \\ \omega_2 := (\sqrt{1+\zeta^2} + \zeta)\omega_0 > \omega_0 = 1/\omega_1 \end{cases}$$

i.e. (ω_1, ω_2) is its stopband (a decade if $\zeta = 0.45\sqrt{10} \approx 1.423$).

Notch filter



with stopbands $(\omega_0/\sqrt{10}, \sqrt{10}\omega_0)$ (decade), $(0.5\omega_0, 2\omega_0)$, $(\omega_0/\sqrt{2}, \sqrt{2}\omega_0)$ (octave), and $(0.9\omega_0, \omega_0/0.9)$.