Linear Systems (034032)
lecture no. 9

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Previously on Linear Systems ...

Let G be a stable continuous-time LTI system with the impulse response g. Its frequency response $G(\mathrm{j}\omega) = (\mathfrak{F}\{g\})(\mathrm{j}\omega)$ or

$$G(\mathrm{j}\omega) = G(s)|_{s=\mathrm{j}\omega} = (\mathfrak{L}\{g\})(\mathrm{j}\omega).$$

In the discrete-time case, the frequency response $G(\mathrm{e}^{\mathrm{j}\theta}) = (\mathfrak{F}\{g\})(\mathrm{e}^{\mathrm{j}\theta})$ or

$$G(\mathrm{e}^{\mathrm{j}\theta}) = G(z)|_{z=\mathrm{e}^{\mathrm{j}\theta}} = (\mathfrak{Z}\{g\})(\mathrm{e}^{\mathrm{j}\theta}).$$

The frequency response shapes

- the response to harmonic inputs
- the response to periodic inputs
- the steady-state response to test sine wave inpute

Outline

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Frequency-domain response of LTI systems

By the convolution property of the Fourier transform,

$$y(t) = (Gu)(t) \quad \Longleftrightarrow \quad Y(\mathrm{j}\omega) = G(\mathrm{j}\omega)U(\mathrm{j}\omega)$$

whenever the corresponding Fourier transforms exist. Hence, the frequency response of G just scales every harmonic component of the input, so that

- harmonics with frequencies ω at which $|G(\mathrm{j}\omega)| > 1$ are amplified
- harmonics with frequencies ω at which $|G(\mathrm{j}\omega)| < 1$ are attenuated
- harmonics with frequencies ω at which $|G(\mathrm{j}\omega)| < 1/\sqrt{2}$ do not pass
  this is a convention, facilitating categorical conclusions; take it with a grain of salt

This multiplication property

- facilitates the use of LTI system as filters,

whose task is to shape the spectrum of signals of interest (to pass "desired" components and to block "unwanted" ones).

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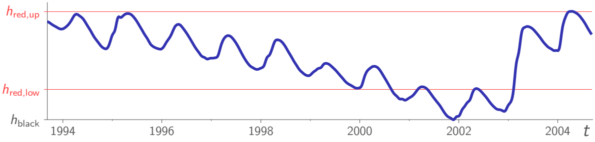
Example

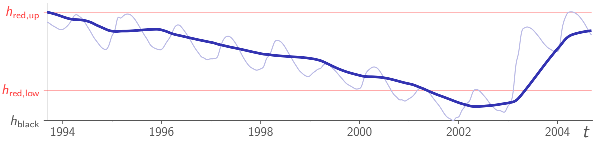
Kinneret water level h from Sep 1993 to Sep 2004



If processed by the finite-memory integrator, so that $h_a = G_{\text{fmi},1}h$ or

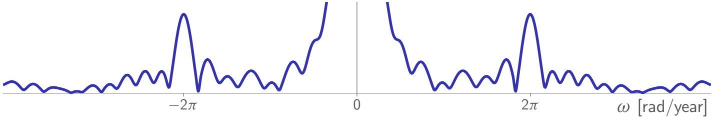
$$h_a(t) = \int_{t-1}^{t} h(s)\,ds \iff H_a(s) = \frac{1 - e^{-s}}{s} H(s)$$

(average over the last year), we end up with



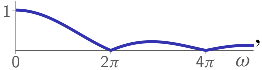
Example: frequency-domain insight

The amplitude spectrum of h (with $h_{\text{red,up}}$ taken as zero) is

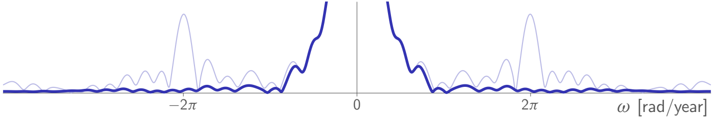


The frequency response magnitude of $G_{\text{fmi},1}$,

$$|G_{\text{fmi},1}(\mathrm{j}\omega)| = \left|\frac{1 - \mathrm{e}^{-\mathrm{j}\omega}}{\mathrm{j}\omega}\right| = \left|\operatorname{sinc}\left(\frac{\omega}{2}\right)\right| =$$



is zero at $\omega = 2\pi$, so the peaks at $\omega = \pm 2\pi$ [rad/year] are filtered out,



eliminating the effect of annual cycles.

Classification

Depending on their purpose, filters may be categorized as

- low-pass filters allow only harmonics with $\omega \leq \omega_\mathrm{b}$ to pass
  i.e. $|G(\mathrm{j}\omega)| \geq \frac{1}{\sqrt{2}} \iff \omega \leq \omega_\mathrm{b}$, which is known as the bandwidth of G
- high-pass filters allow only harmonics with $\omega \geq \omega_\mathrm{c}$ to pass
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- band-pass filters allow only harmonics with $\omega_1 \leq \omega \leq \omega_2$ to pass
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- band-stop filters allow only harmonics with $\omega \leq \omega_1$ and $\omega \geq \omega_2$ to pass
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The question is

- what are systems, whose frequency responses are such filters ?

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Ideal filters

– ideal low-pass has

$$F_{\mathrm{ilp},\omega_\mathrm{b}}(\mathrm{j}\omega) = \mathrm{rect}_{2\omega_\mathrm{b}}(\omega) \iff f_{\mathrm{ilp},\omega_\mathrm{b}}(t) = \frac{\omega_\mathrm{b}}{\pi}\,\mathrm{sinc}(\omega_\mathrm{b} t)$$

(see Lect. 3, Slide 37) and is non-causal and even BIBO *un*stable[1], for $f_{\mathrm{ilp},\omega_\mathrm{b}} \notin L_1$. Hence, not quite practical.

– ideal high-pass $F_{\mathrm{ihp},\omega_\mathrm{c}} = 1 - F_{\mathrm{ilp},\omega_\mathrm{c}}$ and $f_{\mathrm{ihp},\omega_\mathrm{c}}(t) = \delta(t) - \frac{\omega_\mathrm{c}}{\pi}\,\mathrm{sinc}(\omega_\mathrm{c} t)$

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$$F_{\mathrm{ibp},[\omega_1,\omega_2]}(\mathrm{j}\omega) = \mathrm{rect}_{2\omega_2}(\omega) - \mathrm{rect}_{2\omega_1}(\omega)$$

$$\Updownarrow$$

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– ideal band-stop $F_{\mathrm{ibs},[\omega_1,\omega_2]} = 1 - F_{\mathrm{ibp},[\omega_1,\omega_2]}$

[1] But, strangely enough, it is L_2-stable, try to prove it with the material of Lects. 3 & 6.

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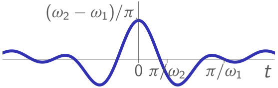

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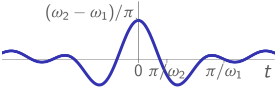

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Decibels

Decibel (dB) is a unit of measurement expressing the ratio of two values of a root-power quantity on a logarithmic scale. Applying to $|G(j\omega)|$, it is

$$|G(j\omega)|_{(dB)} := 20 \log_{10} |G(j\omega)|$$

Useful properties:

- $|G_1(j\omega) G_2(j\omega)|_{(dB)} = |G_1(j\omega)|_{(dB)} + |G_2(j\omega)|_{(dB)}$
- $|[G(j\omega)]^n|_{(dB)} = n|G(j\omega)|_{(dB)}$ for all $n \in \mathbb{R}$
- $\left|\dfrac{1}{G(j\omega)}\right|_{(dB)} = -|G(j\omega)|_{(dB)}$
- $\left|\dfrac{G_1(j\omega)}{G_2(j\omega)}\right|_{(dB)} = |G_1(j\omega)|_{(dB)} - |G_2(j\omega)|_{(dB)}$

Some common values[2] (memorize those in blue):

gain	1	$\sqrt{2}$	2	4	5	10	25	50	100	1000
dB	0	≈ 3	≈ 6	≈ 12	≈ 14	20	≈ 28	≈ 34	40	60

[2] The `mag2db` and `db2mag` commands of Matlab are handy.

Bode plot

Consists of

- Bode magnitude plot of $|G(\mathrm{j}\omega)|$ (in dB) vs. ω (in logarithmic scale)
- Bode phase plot of $\arg(G(\mathrm{j}\omega))$ (in deg) vs. ω (in logarithmic scale)

In the logarithmic scale the distance between ω_0 and $N\omega_0$ does *not* depend on ω_0 for a given $N \in \mathbb{R}_+$ ($N = 2$ is an octave, $N = 10$ is a decade).

Example. For $G_{\text{therm}}(s) = 1/(\tau s + 1)$,

$$G_{\text{therm}}(\mathrm{j}\omega) = \frac{1}{\sqrt{1 + \tau^2\omega^2}} \mathrm{e}^{-\mathrm{j}\arctan(\tau\omega)}$$

with the magnitude and phase as

respectively. The same on the Bode plot:

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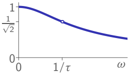
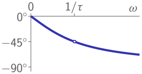


and



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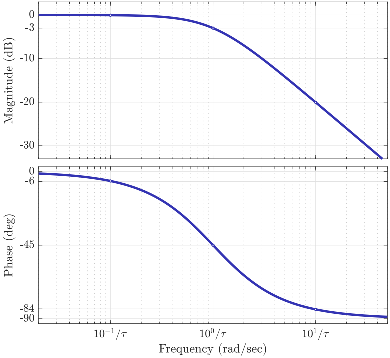


Bode plot: advantages

– factors add up on both magnitude and phase plots, meaning the Bode plots of systems with real-rational transfer functions, like

$$G(s) = \frac{b_m \prod_i (s - z_i) \prod_j (s^2 + 2\zeta_{z,i}\omega_{z,i} s + \omega_{z,i}^2)}{\prod_i (s - p_i) \prod_i (s^2 + 2\zeta_{p,i}\omega_{p,i} s + \omega_{p,i}^2)}$$

can be built by superposing frequency responses of 3 basic blocks[3]

0. static gain, like k for $k \in \mathbb{R} \setminus \{0\}$
1. first-order factor, like $s + a$, for $a \in \mathbb{R}$
2. second-order factor, like $s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2$, for $\zeta \in (-1, 1)$ & $\omega_\mathrm{n} > 0$

– very large magnitudes are not that large in dB

– logarithmic frequency scale facilitates viewing wider frequency ranges

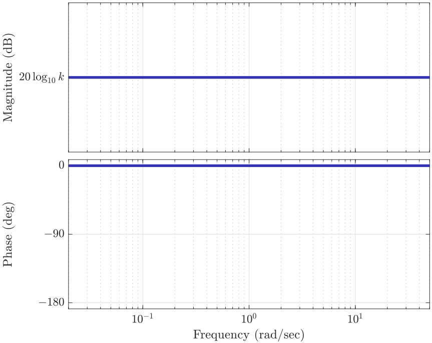
[3] If they are in denominators, then both their magnitude (dB) and phase change sign.

Basic blocks: static gain

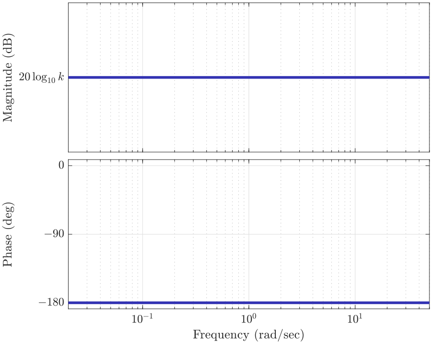
If $G(s) = k$, then

$$G(j\omega) = k = |k| \begin{cases} e^{j0} & \text{if } k > 0 \\ e^{-j\pi} & \text{if } k < 0 \end{cases}$$

and both magnitude and phase plots are horizontal lines:



if $k > 0$



if $k < 0$

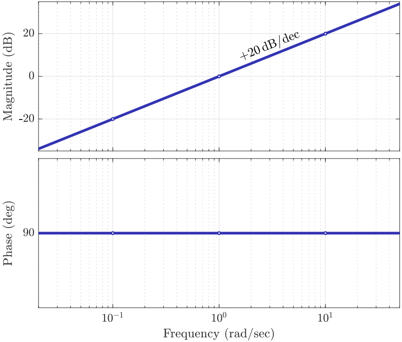
Basic blocks: 1-order factor, $a = 0$

If $G(s) = s$, then

$$G(j\omega) = j\omega = \omega e^{j\pi/2}$$

and

- the Bode magnitude plot is a straight line with the $+20$ dB/dec slope, passing through the 0 dB level at $\omega = 1$ rad/sec
- the Bode phase plot is a horizontal line



Basic blocks: 1-order factor, $a \neq 0$

If $a \neq 0$, then it is convenient to normalize the static gain of $s + a$. Hence, the basic block is $G(s) = \tau s + 1$ for $\tau = 1/a \neq 0$, for which

$$G(\mathrm{j}\omega) = 1 + \mathrm{j}\tau\omega = \sqrt{1 + \tau^2\omega^2}\,\mathrm{e}^{\mathrm{j}\arctan(\tau\omega)}$$

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The magnitude can be approximated as

$$|G(\mathrm{j}\omega)| = \sqrt{1 + \tau^2\omega^2} \approx \begin{cases} 1 & \text{if } |\tau|\omega < 1 \\ |\tau|\omega & \text{if } |\tau|\omega > 1 \end{cases}$$

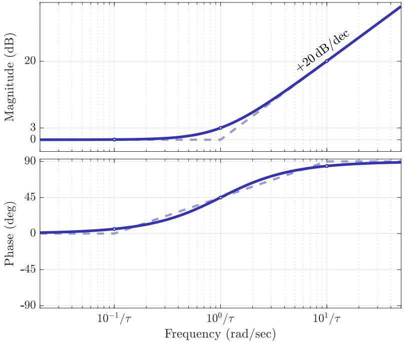
which corresponds to straight lines (the frequency $\omega = 1/\tau$ is known as the corner frequency). The phase can be approximated as

$$\arg(G(\mathrm{j}\omega)) = \arctan(\tau\omega) \approx \operatorname{sign}\tau \begin{cases} 0^\circ & \text{if } |\tau|\omega < \frac{1}{10} \\ 45^\circ(1 + \log_{10}\omega) & \text{if } \frac{1}{10} \leq |\tau|\omega \leq 10 \\ 90^\circ & \text{if } |\tau|\omega > 10 \end{cases}$$

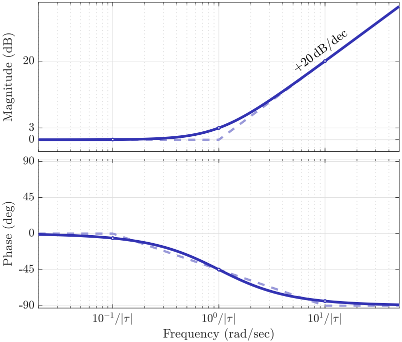
(error within $\pm 5.711^\circ$), which corresponds to straight lines too for $\log \omega$.

Basic blocks: 1-order factor, $a \neq 0$ (contd)

Thus, precise (solid) and approximate (dashed) Bode plots are



if $\tau > 0$



if $\tau < 0$

- drawing precise Bode is easy nowadays (e.g. `bode` in Matlab)
- approximate Bode is still useful for a quick mental grasping

Basic blocks: 2-order factor, $\zeta \neq 0$

If $G(s) = (s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1$ (with the normalized static gain), then

$$G(j\omega) = 1 - \frac{\omega^2}{\omega_n^2} + j\,2\zeta\frac{\omega}{\omega_n} = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}\, e^{j\arg(G(j\omega))},$$

where, assuming $\arctan\phi \in [-\pi/2, \pi/2]$,

$$\arg(G(j\omega)) = \arctan\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} + \begin{cases} 0 & \text{if } \omega \le \omega_n \\ \pi & \text{if } \omega \ge \omega_n \wedge \zeta > 0 \\ -\pi & \text{if } \omega \ge \omega_n \wedge \zeta < 0 \end{cases}$$

which is a continuous and monotonic function of ω (increasing if $\zeta > 0$ and decreasing if $\zeta < 0$). Both the magnitude and phase may be approximated by piecewise linear functions, but this is accurate only around $|\zeta| = 1/\sqrt{2}$.

Basic blocks: 2-order factor, $\zeta \neq 0$ (contd)

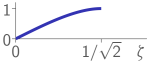
The derivative of the magnitude

$$\frac{|G(j\omega)|}{d\omega} = \underbrace{\frac{2}{|G(j\omega)|}\frac{\omega}{\omega_n}}_{>0,\ \forall\omega>0}\left(\frac{\omega^2}{\omega_n^2} + 2\zeta^2 - 1\right).$$

Hence,

- if $|\zeta| \geq \frac{1}{\sqrt{2}}$, then $|G(j\omega)|$ is monotonically increasing
- if $|\zeta| < \frac{1}{\sqrt{2}}$, then $|G(j\omega)|$
  - monotonically decreases for $\omega < \sqrt{1-2\zeta^2}\omega_n$
  - monotonically increases for $\omega > \sqrt{1-2\zeta^2}\omega_n$
  - has

$$\min_{\omega}|G(j\omega)| = 2\zeta\sqrt{1-\zeta^2} =$$



attainable at $\omega = \sqrt{1-2\zeta^2}\omega_n < \omega_n$

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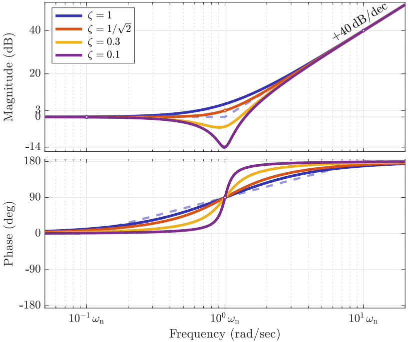

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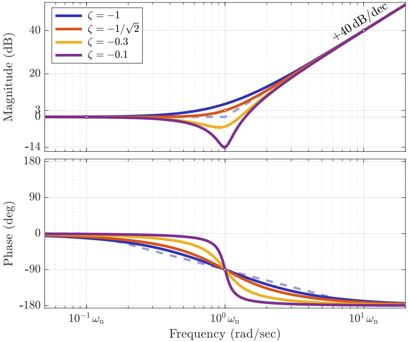


if $\zeta < 0$

If ζ is "far" from $1/\sqrt{2}$, the

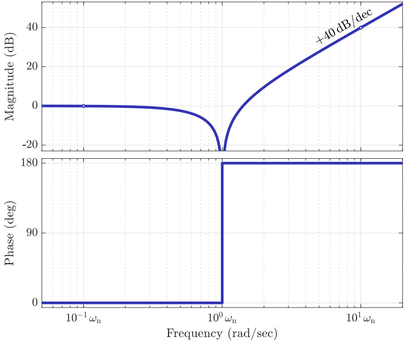
– approximate Bode plots are not quite accurate around $\omega = \omega_n$

Basic blocks: 2-order factor, $\zeta = 0$

If $G(s) = (s/\omega_\mathrm{n})^2 + 1$ (the static gain is again normalized), then

$$G(\mathrm{j}\omega) = 1 - \frac{\omega^2}{\omega_\mathrm{n}^2} = \left|1 - \frac{\omega^2}{\omega_\mathrm{n}^2}\right| \begin{cases} \mathrm{e}^{\mathrm{j}0} & \text{if } \omega < \omega_\mathrm{n} \\ \mathrm{e}^{\mathrm{j}\pi} & \text{if } \omega > \omega_\mathrm{n} \end{cases}$$

resulting in

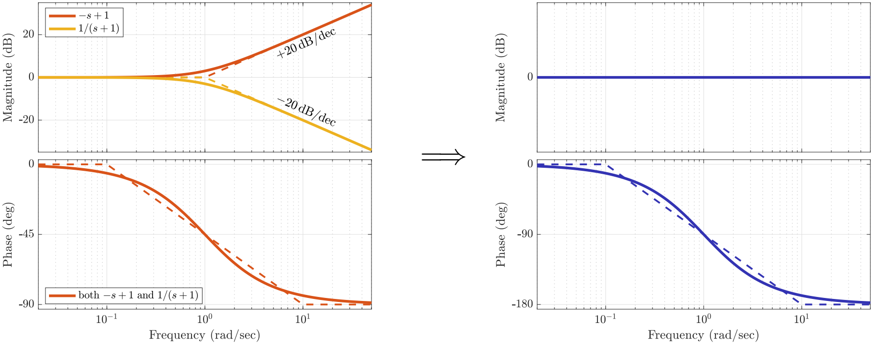


Example 1

If

$$G_1(s) = \frac{-s+1}{s+1}$$

then



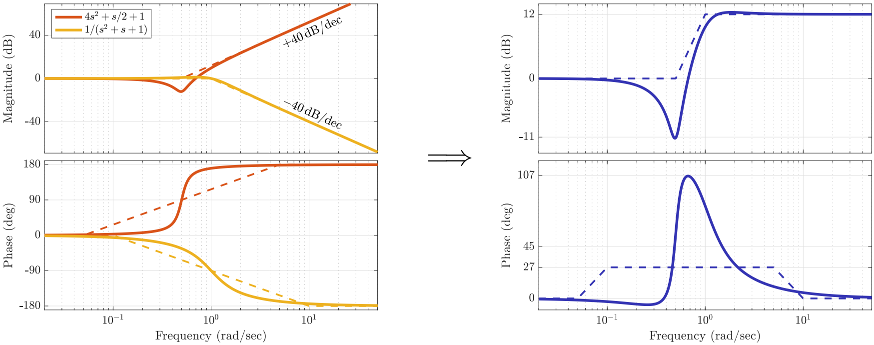
and the high-frequency gain is $G_1(\infty) = 1 = 0\,\text{dB}$.

Example 2

If

$$G_2(s) = \frac{4s^2 + s/2 + 1}{s^2 + s + 1}$$

then



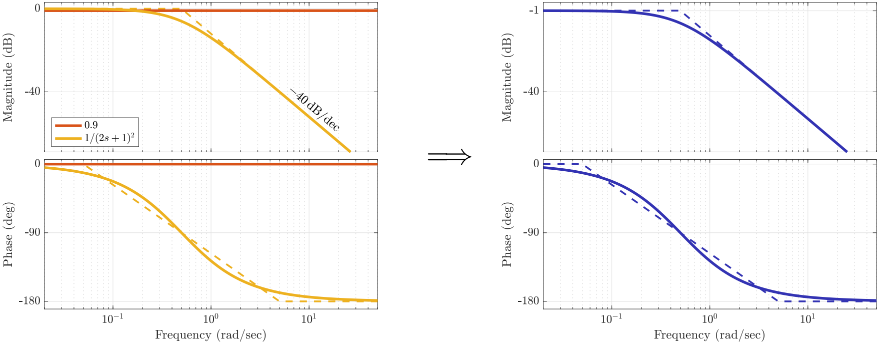
and the high-frequency gain is $G_2(\infty) = 4 \approx 12\,\text{dB}$.

Example 3

If

$$G_3(s) = \frac{0.9}{(2s+1)^2}$$

then



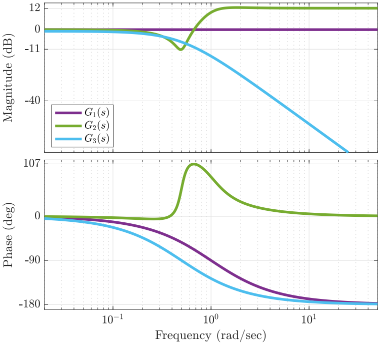
and the magnitude decays at high frequencies with a slope of $-40\,\text{dB/dec}$.

Example 4

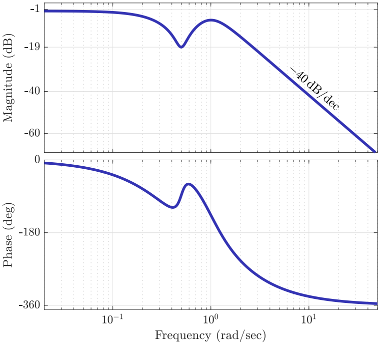
If

$$G(s) = G_1(s)G_2(s)G_3(s) = \frac{0.9(-s+1)(4s^2+s/2+1)}{(s+1)(2s+1)^2(s^2+s+1)}$$

then



$\Longrightarrow$



Asymptotic properties of Bode plots: magnitude

At low frequencies:

- every zero at the origin contributes a slope of $+20\,\text{dB/dec}$
- every integrator (pole at the origin) contributes a slope of $-20\,\text{dB/dec}$
- if no poles/zeros at the origin, starts as a horizontal line at $|G(0)|_{(\text{dB})}$

At high frequencies:

- every zero adds a slope of $+20\,\text{dB/dec}$
- every pole adds a slope of $-20\,\text{dB/dec}$
- if $G(s)$ is bi-proper, ends as a horizontal line at $|G(\infty)|_{(\text{dB})}$

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At low frequencies:

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- every integrator (pole at the origin) contributes a phase lag of -90°

At high frequencies:

- every zero in $\mathbb{C} \setminus \mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ adds a phase lead of 90°
- every pole in $\mathbb{C} \setminus \mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \le 0\}$ adds a phase lag of -90°
- every zero in $\mathbb{C}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ adds a phase lag of -90°
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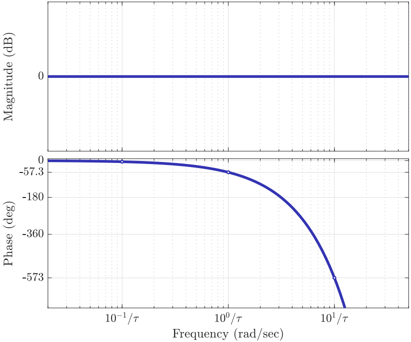
Bode for non-rational transfer functions

If

$$G(s) = \bar{D}_\tau(s) = \mathrm{e}^{-\tau s}, \quad \tau > 0$$

then $G(\mathrm{j}\omega) = \mathrm{e}^{-\mathrm{j}\tau\omega}$, so that $|G(\mathrm{j}\omega)| = 1$ and $\arg(G(\mathrm{j}\omega)) = -\tau\omega\,[\mathrm{rad}]$. We therefore have



Bode for non-rational transfer functions (contd)

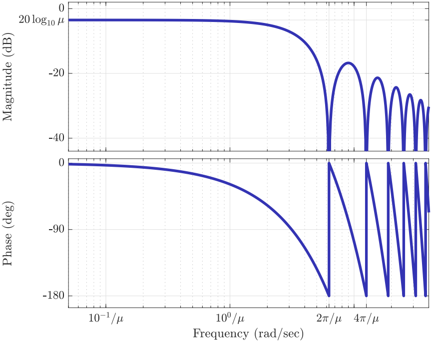
If

$$G(s) = G_{\mathsf{fmi},\mu}(s) = \frac{1 - \mathrm{e}^{-\mu s}}{s}, \quad \mu > 0$$

then

$$G(\mathrm{j}\omega) = \frac{1 - \mathrm{e}^{-\mathrm{j}\mu\omega}}{\mathrm{j}\omega} = \mu \frac{\mathrm{e}^{\mathrm{j}\mu\omega/2} - \mathrm{e}^{-\mathrm{j}\mu\omega/2}}{\mathrm{j}2\,\mu\omega/2} \mathrm{e}^{-\mathrm{j}\mu\omega/2} = \mu\,\mathrm{sinc}\Big(\frac{\mu\omega}{2}\Big)\mathrm{e}^{-\mathrm{j}\mu\omega/2}$$

and we have



Polar plot

Shows Im $G(\mathrm{j}\omega)$ vs. Re $G(\mathrm{j}\omega)$ as the frequency ω grows from 0 to ∞, with an arrow indicating the growth direction of ω.

Example 4:

$$G(s) = \frac{0.9(-s+1)(4s^2+s/2+1)}{(s+1)(2s+1)^2(s^2+s+1)} \implies$$

Polar plots are

- less informative than Bode — the frequency is hidden
- harder to draw manually than Bode — no superposition rules hold
- produced by the `nyquist` command of Matlab
  Draws the plot for $-\infty < \omega < \infty$ (aka the Nyquist diagram). To produce the plain polar plot, use `setoptions(nyquistplot(G),'ShowFullContour','off')`
- very important in feedback control applications (the Nyquist criterion)

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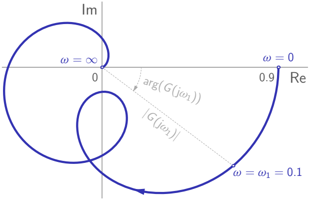

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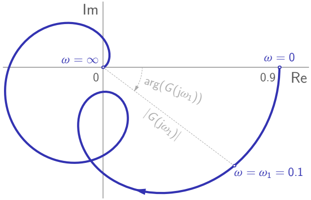

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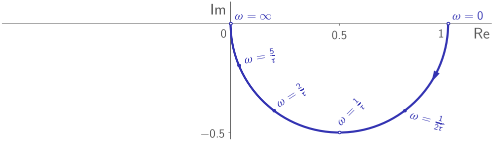
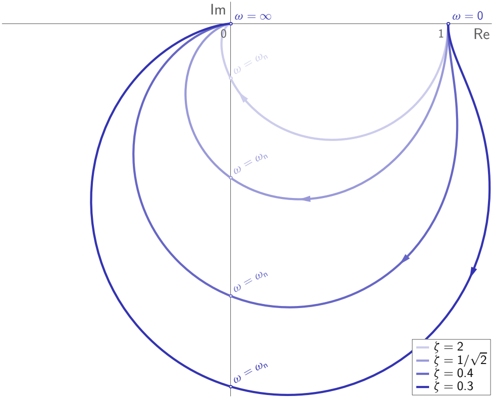
Some polar plots

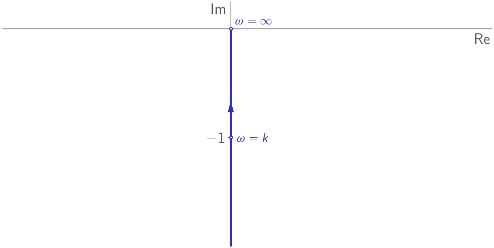


$$G(s) = \frac{1}{\tau s + 1}$$

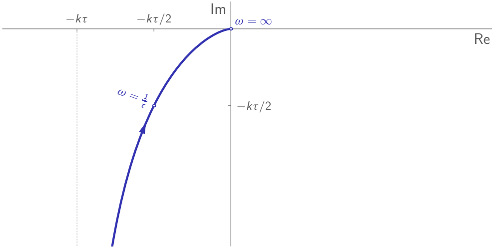


$$G(s) = \frac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1}$$

Some polar plots (contd)

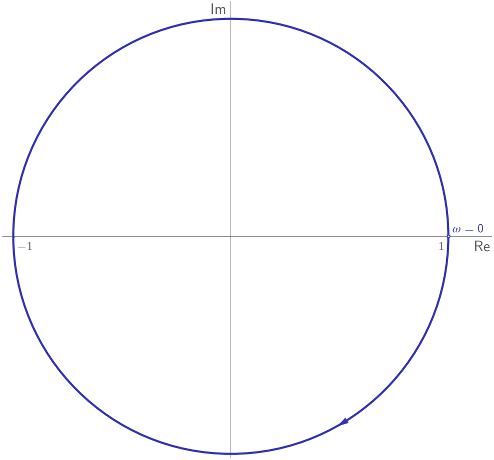


$$G(s) = \frac{k}{s},\ k > 0$$



$$G(s) = \frac{k}{s(\tau s + 1)},\ k > 0$$

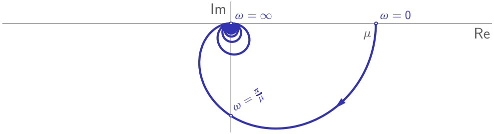
Some polar plots (contd)



$$G(s) = \mathrm{e}^{-\tau s}$$

or

$$G(s) = \left(\frac{-\tau s + 1}{\tau s + 1}\right)^2$$



$$G(s) = \frac{1 - \mathrm{e}^{-\mu s}}{s}, \ \mu > 0$$

Outline

Systems as filters I

Frequency response plots

Systems as filters II

Butterworth polynomials

The Butterworth polynomial of degree n, $B_n(s)$, is the Hurwitz polynomials such that

$$|B_n(\mathrm{j}\omega)|^2 = 1 + \omega^{2n}$$

Its general form is (depending on whether n is even or odd)

$$B_n(s) = \prod_{i=1}^{n/2}(s^2 + 2\zeta_i s + 1) \quad \text{or} \quad B_n(s) = (s+1)\prod_{i=1}^{(n-1)/2}(s^2 + 2\zeta_i s + 1)$$

where

$$\zeta_i := \sin\left(\frac{2i-1}{2n}\pi\right) \in (0,1), \quad i \in \mathbb{Z}_{1..\lfloor n/2 \rfloor}$$

Roots of $B_n(s)$ are at equally-spaced points in $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0 \wedge |s| = 1\}$.
Particular cases:

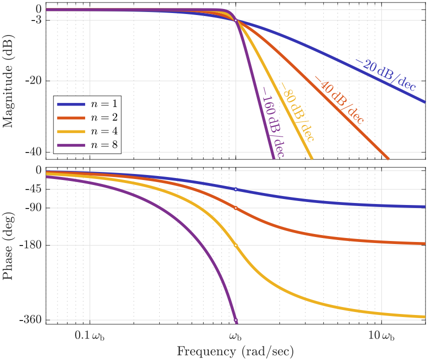
$$B_1(s) = s + 1, \quad B_2(s) = s^2 + \sqrt{2}s + 1, \quad B_3(s) = (s+1)(s^2 + s + 1).$$

Low-pass Butterworth filter

The *n*-order low-pass filter

$$F(s) = \frac{1}{B_n(s/\omega_\mathrm{b})} \quad \Longrightarrow \quad |F(\mathrm{j}\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_\mathrm{b})^{2n}}}$$

is monotonically decreasing with $|F(\mathrm{j}\omega_\mathrm{b})| = \frac{1}{\sqrt{2}}$ (i.e. ω_b is its bandwidth):

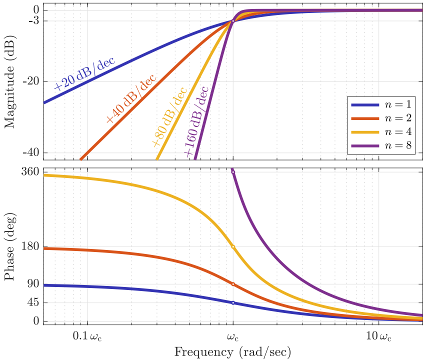


High-pass Butterworth filter

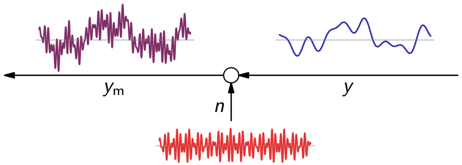
The n-order high-pass filter

$$F(s) = \frac{(s/\omega_c)^n}{B_n(s/\omega_c)} \quad \Longrightarrow \quad |F(j\omega)| = \frac{(\omega/\omega_c)^n}{\sqrt{1+(\omega/\omega_c)^{2n}}}$$

is monotonically increasing with $|F(j\omega_c)| = \frac{1}{\sqrt{2}}$ (i.e. ω_c is its cut-off freq.):



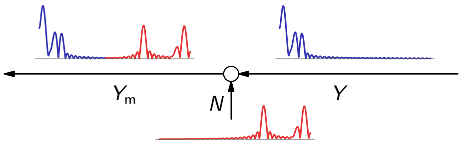
Example from Lect. 3



The question was

– how signal (y) can be recovered from its corrupt measurements (y_m)?

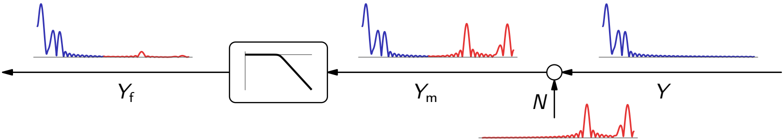
This might appear a tough task, for there is no way we can separate y from n in the time domain ... But in the frequency domain these signals are well separated,



suggesting that the frequency-domain viewpoint is valuable.

Example from Lect. 3 (contd)

If we process the measurement by a low-pass filter (4-order Butterworth in this case), then the result,



separates the slow signal y from fast noise n. In the time domain,

although with certain phase lag ...

Example from Lect. 3 (contd)

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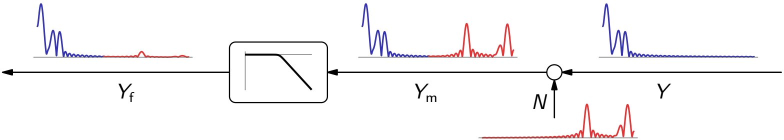

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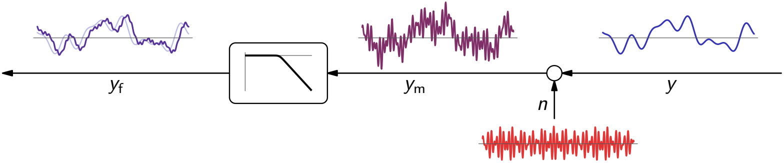


although with certain phase lag . . .

Notch filter

Is a narrow stopband band-stop filter of the form

$$F(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \quad \Longrightarrow \quad |F(\mathrm{j}\omega)| = \sqrt{\frac{(\omega^2 - \omega_0^2)^2}{(\omega^2 - \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$

is

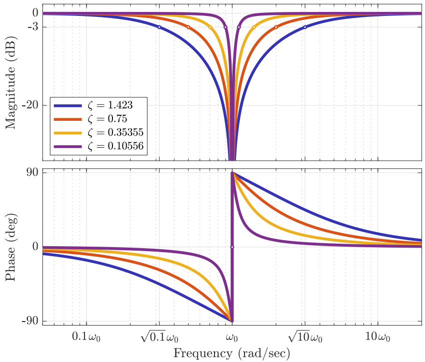
- monotonically decreasing in $\omega < \omega_0$
- monotonically increasing in $\omega > \omega_0$

with

$$|F(\mathrm{j}\omega)| = \frac{1}{\sqrt{2}} \iff \omega = \begin{cases} \omega_1 := (\sqrt{1+\zeta^2} - \zeta)\omega_0 < \omega_0 \\ \omega_2 := (\sqrt{1+\zeta^2} + \zeta)\omega_0 > \omega_0 = 1/\omega_1 \end{cases}$$

i.e. (ω_1, ω_2) is its stopband (a decade if $\zeta = 0.45\sqrt{10} \approx 1.423$).

Notch filter



with stopbands $(\omega_0/\sqrt{10}, \sqrt{10}\omega_0)$ (decade), $(0.5\omega_0, 2\omega_0)$, $(\omega_0/\sqrt{2}, \sqrt{2}\omega_0)$ (octave), and $(0.9\omega_0, \omega_0/0.9)$.