Linear Systems (034032)
lecture no. 8

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Outline

Stability and characteristic polynomials

Hurwitz polynomials

Schur polynomials

Frequency-domain analysis of LTI systems

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Frequency-domain analysis of LTI systems

Reminder: I/O stability via transfer functions

Let G be a system, whose transfer function $G(s)$ is rational, i.e.

$$G(s) = \frac{N_G(s)}{D_G(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \quad b_m \neq 0.$$

It is said to be proper if $n \geq m$. If $N_G(s)$ and $D_G(s)$ have no common roots (i.e. are *coprime*), then the poles of $G(s)$ are the roots of $D_G(s)$.

Theorem

If the transfer function $G(s)$ of a continuous-time LTI system G is rational, then G is causal and I/O stable iff

- $G(s)$ *is proper and has no poles in* $\bar{\mathbb{C}}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$

Theorem

If the transfer function $G(z)$ of a discrete-time LTI system G is rational, then

- G *is causal iff* $G(z)$ *is proper and*
- G *is I/O stable iff* $G(z)$ *has no poles in* $\mathbb{C} \setminus \mathbb{D}_1 = \{z \in \mathbb{C} \mid |z| \geq 1\}$

I/O stability: key question

While properness is easy to check, finding roots of polynomials may not be, especially if coefficients depend on some parameters of interest. Try it for

- $D_2(s) = s^2 + ks + 1 - k^3$
- $D_3(s) = s^3 + 2s^2 + ks + k^2$
- $D_4(s) = s^4 - ks^3 + 2s^2 + s + 3k$
- $D_5(s) = s^5 + 2s^4 + (k+2)s^3 + 3s^2 + ks + 1$

or

- $\bar{D}_2(z) = z^2 + kz + 0.75/k$
- $\bar{D}_3(s) = z^3 + (1-k)z^2 + z + k$
- $\bar{D}_4(s) = z^4 - z^3 - kz^2 + 2kz - 1$

But we do not really need root locations. All we need to know is whether

- all roots are in a "good" area of the complex plane.

The latter task may be simpler, as we shall see soon . . .

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The latter task may be simpler, as we shall see soon . . .

Background on polynomials

Terminology: a polynomial is said to be

- Hurwitz if all its roots are in the open left-half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$
- Schur if all its roots are in the open unit disk $\mathbb{D}_1 := \{z \in \mathbb{C} \mid |z| < 1\}$

A polynomial is said to be monic if its leading coefficient is equal to 1.

Basics: by the fundamental theorem of algebra,

- a polynomial of degree n has exactly n roots, counting multiplicities.

Every degree-n monic polynomial $D(s)$ can be written as

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)$$

where λ_i are roots of $D(s)$. Moreover, if all coefficients of a polynomial are real and λ_i with $\operatorname{Im}\lambda_i \neq 0$ is its root, then so is $\overline{\lambda_i} = \operatorname{Re}\lambda_i - \mathrm{j}\operatorname{Im}\lambda_i$ and

$$(s - \lambda_i)(s - \overline{\lambda_i}) = s^2 - 2\operatorname{Re}\lambda_i\, s + |\lambda_i|^2,$$

so $D(s)$ can be factored into 1- and 2-order factors with real coefficients.

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Hurwitz polynomials: the necessary condition

If $D(s)$ is Hurwitz, then its real factors,

$$s + (-\lambda_i) \quad \text{and} \quad s^2 + (-2\,\mathrm{Re}\,\lambda_j)s + |\lambda_j|^2,$$

have only strictly positive coefficients. This implies that

- a monic $D(s)$ is Hurwitz only if all its coefficients are positive,

If the monic assumption is dropped, then the condition can be stated as

- $D(s)$ is Hurwitz only if all its coefficients are nonzero and of the same sign.

Example: $D_4(s) = s^4 - ks^3 + 2s^2 + s + 3k$ has all its coefficients positive iff $(-k > 0) \wedge (3k > 0) = \varnothing$, i.e. it is Hurwitz for no k.

This condition is not sufficient though. For example, $s^3 + s^2 + 4s + 30$ has roots in $\{-3, 1 \pm j3\}$, i.e. two of them are in the right-half plane.

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This condition is not sufficient though. For example, $s^3 + s^2 + 4s + 30$ has roots in $\{-3, 1 \pm \mathrm{j}3\}$, i.e. two of them are in the right-half plane.

Routh table

Associate with the polynomial $a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$ the table

$$\begin{array}{c|llll}
0 & r_{0,1} = a_n & r_{0,2} = a_{n-2} & r_{0,3} = a_{n-4} & \cdots \\
1 & r_{1,1} = a_{n-1} & r_{1,2} = a_{n-3} & r_{1,3} = a_{n-5} & \cdots \\
2 & r_{2,1} & r_{2,2} & r_{2,3} & \cdots \\
\vdots & \vdots & \vdots & & \\
n-2 & r_{n-2,1} & r_{n-2,2} & r_{n-2,2} & \\
n-1 & r_{n-1,1} & r_{n-1,2} & & \\
n & r_{n,1} & & &
\end{array}$$

where for each $i \in \mathbb{Z}_{2..n}$

$$\begin{bmatrix} r_{i,1} & r_{i,2} & \cdots \end{bmatrix} = \begin{bmatrix} r_{i-2,2} & r_{i-2,3} & \cdots \end{bmatrix} - \frac{r_{i-2,1}}{r_{i-1,1}} \begin{bmatrix} r_{i-1,2} & r_{i-1,3} & \cdots \end{bmatrix}$$

and if the last required column of an involved row is empty, 0 is taken. It is called *regular* if all $r_{i,1} \neq 0$. Otherwise, it is *singular* (cannot be completed).

Routh–Hurwitz criterion

Theorem

Let $D(s)$ be a polynomial of degree n.

1. *$D(s)$ is Hurwitz iff the associated Routh table is regular and all $n+1$ elements of its first column have the same sign.*
2. *If the associated Routh table is regular, then $D(s)$ has no roots in $\mathrm{j}\mathbb{R}$ and the number of its roots in $\mathbb{C}_0$ equals the number of sign changes in the first column of the table.*

Routh–Hurwitz criterion: degree-2 polynomials

If $n = 2$, i.e. $D(s) = a_2 s^2 + a_1 s + a_0$ for $a_2 \neq 0$, the Routh table is

$$\begin{array}{c|cc} 0 & a_2 & a_0 \\ 1 & a_1 & 0 \\ 2 & a_0 & \end{array}$$

and we conclude that a degree-2 polynomial is

- Hurwitz iff all its coefficients are nonzero and have the same sign

(do memorize this condition). In other words, the necessary condition is also sufficient in this case.

Routh–Hurwitz criterion: degree-3 polynomials

If $n = 3$, i.e. $D(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$ for $a_3 \neq 0$, the Routh table is

$$\begin{array}{c|cc} 0 & a_3 & a_1 \\ 1 & a_2 & a_0 \\ 2 & a_1 - a_0 a_3/a_2 & 0 \\ 3 & a_0 & \end{array}$$

All elements of its first column have the same sign iff

$$\frac{a_2}{a_3} > 0, \quad \frac{a_1}{a_3} - \frac{a_0}{a_2} > 0, \quad \text{and} \quad \frac{a_0}{a_3} > 0.$$

The second inequality is equivalent to $a_1/a_3 > (a_0/a_3)/(a_2/a_3) > 0$. Thus, we conclude that a degree-3 polynomial is

– Hurwitz iff all its coefficients are nonzero, have the same sign, and, in addition, $a_1 a_2 > a_0 a_3$.

Memorize this condition as well.

Routh–Hurwitz criterion: degree-4 polynomials

If $n = 4$, i.e. $D(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$ for $a_4 \neq 0$, the table is

0	a_4	a_2	a_0
1	a_3	a_1	0
2	$a_2 - a_1 a_4 / a_3$	a_0	
3	$a_1 - a_0 a_3^2 / (a_2 a_3 - a_1 a_4)$	0	
4	a_0		

All elements of its first column have the same sign iff

$$\frac{a_3}{a_4} > 0, \quad \frac{a_2}{a_4} - \frac{a_1}{a_3} > 0, \quad \frac{a_1}{a_4} - \frac{a_0 a_3^2}{a_4(a_2 a_3 - a_1 a_4)} > 0, \quad \text{and} \quad \frac{a_0}{a_4} > 0$$

and it can be shown that a degree-4 polynomial is

- Hurwitz iff all its coefficients are nonzero, have the same sign, and, in addition, $a_1 a_2 > a_0 a_3 + a_1^2 a_4 / a_3$

(try to prove that, don't need to memorize).

Routh–Hurwitz criterion: examples

$D_2(s) = s^2 + ks + 1 - k^3$ is Hurwitz iff

$$(k > 0) \wedge (1 - k^3 > 0) \iff 0 < k < 1$$

$D_3(s) = s^3 + 2s^2 + ks + k^2$ is Hurwitz iff

$$(k > 0) \wedge (2k > k^2) \iff (k > 0) \wedge (0 < k < 2) \iff 0 < k < 2$$

If $D_5(s) = s^5 + 2s^4 + (k+2)s^3 + 3s^2 + ks + 1$, then the Routh table is

0	1	$2+k$	k
1	2	3	1
2	$k + 1/2$	$k - 1/2$	0
3	$(2k+5)/(2k+1)$	1	
4	$(2k-3)/(2k+5)$	0	
5	1		

and the polynomial is Hurwitz iff $k > \dfrac{3}{2}$ (show that).

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 & j_{0,n+1} = a_0 & j_{0,n} = a_1 & \cdots & j_{0,2} = a_{n-1} & j_{0,1} = a_n \\
1 & j_{1,1} & j_{1,2} & \cdots & j_{1,n} & \\
 & j_{1,n} & j_{1,n-1} & \cdots & j_{1,1} & \\
\vdots & \vdots & \vdots & & & \\
n-1 & j_{n-1,1} & j_{n-1,2} & & & \\
 & j_{n-1,2} & j_{n-1,1} & & & \\
n & j_{n,1} & & & &
\end{array}
$$

where for each $i \in \mathbb{Z}_{1..n}$

$$
\left[\, j_{i,1} \cdots j_{i,n+1-i} \,\right] = \left[\, j_{i-1,1} \cdots j_{i-1,n+1-i} \,\right] - \frac{j_{i-1,n+2-i}}{j_{i-1,1}} \left[\, j_{i-1,n+2-i} \cdots j_{i-1,2} \,\right]
$$

(the ith row has $n+1-i$ elements). The Jury table is said to be *regular* if all $j_{i,1} \neq 0$. Otherwise, it is *singular*.

Jury criterion

Theorem

Let $D(z)$ be a polynomial of degree n with $a_n > 0$.

1. *$D(z)$ is Schur iff the associated Jury table is regular and $j_{i,1} > 0$ for all $i \in \mathbb{Z}_{1..n}$.*
2. *If the Jury table is regular, then $D(z)$ has no roots in $\{z \in \mathbb{C} \mid |z| = 1\}$ and the number of its roots in $\{z \in \mathbb{C} \mid |z| > 1\}$ equals the number of negative elements among $j_{i,1}$.*

It can be shown that if $j_{i,1} > 0$ for all $i \in \mathbb{Z}_{0..n-1}$, then $j_{n,1} > 0$ iff

- $D(1) > 0$ and $(-1)^n D(-1) > 0$

which is an easy-to-check necessary condition for $D(z)$ to be Schur (again, assuming that $a_n > 0$).

Example If $\bar{D}_4(z) = z^4 - z^3 - kz^2 + 2kz - 1$, then $\bar{D}_4(1) = k - 1 > 0$ iff $k > 1$ and $\bar{D}_4(-1) = 1 - 3k > 0$ iff $k < 1/3$. Thus, it is Schur for no k.

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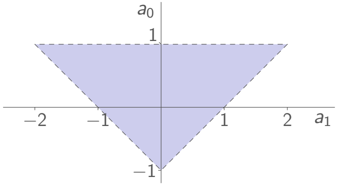
Jury criterion: degree-2 polynomials

If $n = 2$ and $D(z)$ is monic, i.e. $D(z) = z^2 + a_1 z + a_0$, the Jury table is

0	1	a_1	a_0
	a_0	a_1	1
1	$1 - a_0^2$	$a_1(1 - a_0)$	
	$a_1(1 - a_0)$	$(1 + a_0)(1 - a_0)$	
2	$1 - a_0^2 - a_1^2(1 - a_0)/(1 + a_0)$		

and we conclude that a degree-2 polynomial is Schur iff

$$\begin{cases} |a_0| < 1 \\ |a_1| < 1 + a_0 \end{cases} \iff$$



(more complex than in the continuous-time case).

Jury criterion: examples

$\bar{D}_2(z) = z^2 + kz + 0.75/k$ is Schur iff

$$(|0.75/k| < 1) \wedge (|k| < 1 + 0.75/k) \iff \frac{3}{4} < k < \frac{3}{2}$$

If $\bar{D}_3(z) = z^3 + (1-k)z^2 + z + k$, then the Jury table is

0	1	$1-k$	1	k
	k	1	$1-k$	1
1	$1-k^2$	$1-2k$	$1-k+k^2$	
	$1-k+k^2$	$1-2k$	$1-k^2$	
2	$\frac{(2-k)}{1-k^2}k(1-2k)$	$\frac{k(1-2k)^2}{1-k^2}$		
	$\frac{k(1-2k)^2}{1-k^2}$	$\frac{(2-k)k(1-2k)}{1-k^2}$		
3	$\frac{k(1-2k)}{(1-k^2)(2-k^2)}(7-3k-k^2)$			

from which the polynomial is Schur iff $0 < k < \frac{1}{2}$ (show that).

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 & k & 1 & 1-k & 1 \\
1 & 1-k^2 & 1-2k & 1-k+k^2 & \\
 & 1-k+k^2 & 1-2k & 1-k^2 & \\
2 & \frac{(2-k)}{1-k^2}k(1-2k) & \frac{k(1-2k)^2}{1-k^2} & & \\
 & \frac{k(1-2k)^2}{1-k^2} & \frac{(2-k)k(1-2k)}{1-k^2} & & \\
3 & \frac{k(1-2k)}{(1-k^2)(2-k^2)}(7-3k-k^2) & & &
\end{array}$$

from which the polynomial is Schur iff $0 < k < \frac{1}{2}$ (show that).

Bilinear transformation

Consider the mapping

$$z \to \frac{1+s}{1-s} = \frac{1+\operatorname{Re} s + \mathrm{j}\operatorname{Im} s}{1-\operatorname{Re} s - \mathrm{j}\operatorname{Im} s} \qquad \Longleftrightarrow \qquad s \to \frac{z-1}{z+1}$$

between the z and s complex planes. In this case

$$|z|^2 = \frac{(1+\operatorname{Re} s)^2 + (\operatorname{Im} s)^2}{(1-\operatorname{Re} s)^2 + (\operatorname{Im} s)^2} = 1 + \frac{4\operatorname{Re} s}{(1-\operatorname{Re} s)^2 + (\operatorname{Im} s)^2}$$

so that

$$|z| < 1 \iff \operatorname{Re} s < 0, \quad |z| > 1 \iff \operatorname{Re} s > 0, \quad |z| = 1 \iff \operatorname{Re} s = 0$$

i.e.

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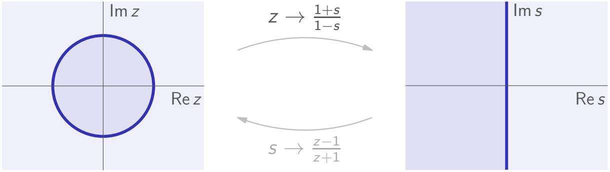
i.e.



Bilinear transformation and Schur polynomials

If $D(z) = a_n z^n + \cdots + a_1 z + a_0$ for $a_n \neq 0$, then

$$D(z)\big|_{z=(1+s)/(1-s)} = a_n \frac{(1+s)^n}{(1-s)^n} + a_{n-1} \frac{(1+s)^{n-1}}{(1-s)^{n-1}} + \cdots + a_1 \frac{1+s}{1-s} + a_0$$
$$= \frac{\tilde{D}(s)}{(1-s)^n},$$

where

$$\tilde{D}(s) = a_n(1+s)^n + a_{n-1}(1+s)^{n-1}(1-s) + \cdots + a_1(1+s)(1-s)^{n-1} + a_0(1-s)^n$$

is a degree-n polynomial such that $\tilde{D}(1) = 2^n a_n \neq 0$. Thus,

- s_0 is a root of $\tilde{D}(s)$ iff $z_0 = (1+s_0)/(1-s_0)$ is a root of $D(z)$

and

- $D(z)$ is Schur iff $\tilde{D}(s)$ is Hurwitz

(because $|z| < 1 \iff \mathrm{Re}\,\frac{z-1}{z+1} < 0$). This yields an algorithm of checking if a polynomial is Schur via checking if another polynomial is Hurwitz.

Bilinear transformation and Schur polynomials

If $D(z) = a_n z^n + \cdots + a_1 z + a_0$ for $a_n \neq 0$, then

$$
\begin{aligned}
D(z)\big|_{z=(1+s)/(1-s)} &= a_n \frac{(1+s)^n}{(1-s)^n} + a_{n-1}\frac{(1+s)^{n-1}}{(1-s)^{n-1}} + \cdots + a_1 \frac{1+s}{1-s} + a_0 \\
&= \frac{\tilde{D}(s)}{(1-s)^n},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{D}(s) = a_n(1+s)^n + a_{n-1}(1+s)^{n-1}(1-s) + \cdots \\
+ a_1(1+s)(1-s)^{n-1} + a_0(1-s)^n
\end{aligned}
$$

is a degree-n polynomial such that $\tilde{D}(1) = 2^n a_n \neq 0$. Thus,

- s_0 is a root of $\tilde{D}(s)$ iff $z_0 = (1+s_0)/(1-s_0)$ is a root of $D(z)$

and

- $D(z)$ is Schur iff $\tilde{D}(s)$ is Hurwitz

(because $|z| < 1 \iff \mathrm{Re}\,\frac{z-1}{z+1} < 0$). This yields an algorithm of checking if a polynomial is Schur via checking if another polynomial is Hurwitz.

Outline

Stability and characteristic polynomials

Hurwitz polynomials

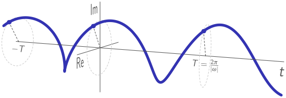
Schur polynomials

Frequency-domain analysis of LTI systems

Reminder: harmonic signal

Signal $a\exp_{j\omega} : \mathbb{R} \to \mathbb{C}$,

$$a e^{j\omega t} = |a| e^{j(\omega t + \arg(a))} =$$



for $a \in \mathbb{C}$ and $\omega \in \mathbb{R}$ is called harmonic signal with frequency ω, amplitude $|a|$, and initial phase $\arg(a)$. Via Euler's formula we can derive that

$$\sin(\omega t + \phi) = a e^{j\omega t} + \bar{a} e^{-j\omega t}, \quad a = 0.5 e^{j(\phi - \pi/2)}$$

(harmonics with ω and $-\omega$ come together in real-valued signals). Its power

$$P_{a\exp_{j\omega}} := \lim_{M\to\infty} \frac{1}{M} \int_{-M/2}^{M/2} |a e^{j\omega t}|^2 \mathrm{d}t = \frac{1}{T} \int_0^T |a e^{j\omega t}|^2 \mathrm{d}t = |a|^2.$$

Reminder: Fourier transform

The Fourier transform $\mathfrak{F}\{x\}$ of a signal x is the signal $X : \mathbb{R} \to \mathbb{C}$ such that

$$X(\mathrm{j}\omega) = (\mathfrak{F}\{x\})(\mathrm{j}\omega) := \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$$

It is well defined if $x \in L_1$ (plus some mild technical assumptions) and then

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega =: (\mathfrak{F}^{-1}\{X\})(t)$$

at every t (the inverse Fourier transform). The later means that

- every transformable signal is a superposition of elementary harmonics

and also that

- harmonics with largest $|X(\mathrm{j}\omega)|$ dominate x.

Response of LTI systems to harmonic signals

Let $G : u \mapsto y$ be LTI and BIBO stable. If $u = a\exp_\omega$, then the response

$$y(t) = \int_{\mathbb{R}} g(s) a e^{j\omega(t-s)} ds = \int_{\mathbb{R}} g(s) e^{-j\omega s} ds\, a e^{j\omega t} = G(j\omega) a e^{j\omega t},$$

where $G(j\omega)$ is the Fourier transform of the impulse response g of G, which is well defined because $g \in L_1$. Thus, an LTI system merely

– scales the amplitude of a harmonic input by $|G(j\omega)|$ and
– shifts the phase of a harmonic input by $\arg(G(j\omega))$

but does not change its frequency.

If u is harmonic, then

$$|G(j\omega)|^2 = \frac{P_y}{P_u}.$$

We say that a harmonic input

– $u = a\exp_\omega$ passes G if $P_y \geq \frac{P_u}{2}$, which amounts to $|G(j\omega)| \geq \frac{1}{\sqrt{2}}$.

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Frequency response

Given a stable LTI system G, the function $G = \mathfrak{F}\{g\} : \mathbb{R} \to \mathbb{C}$ is known as its frequency response. For stable systems, $G(\mathrm{j}\omega) = G(s)|_{s=\mathrm{j}\omega}$, i.e.

- the frequency response at ω equals the value of the transfer function at the pure imaginary point $s = \mathrm{j}\omega$

(always in the RoC of transfer functions of BIBO stable systems).

Remark: For stable discrete systems, the frequency response $G(\mathrm{e}^{\mathrm{j}\theta}) = G(z)|_{z=\mathrm{e}^{\mathrm{j}\theta}}$.

The convention of defining the frequency response as the transfer function at the imaginary axis applies to unstable systems as well. For example, the

- frequency response of an integrator, whose $g_{\text{int}} = \mathbb{1}$, is $G_{\text{int}}(\mathrm{j}\omega) = \dfrac{1}{\mathrm{j}\omega}$

rather than $(\mathfrak{F}\{\mathbb{1}\})(\mathrm{j}\omega) = 1/(\mathrm{j}\omega) + \pi\delta(\omega)$.

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[1] Convenient, because unstable systems can be stabilized via feedback interconnections.

Response of LTI systems to periodic signals

Let $G : u \mapsto y$ be LTI and BIBO stable. If u is T-periodic, then

$$u(t) = \sum_{k \in \mathbb{Z}} U[k] \mathrm{e}^{\mathrm{j}\omega_0 k t}, \quad \omega_0 = \frac{2\pi}{T}$$

where

$$U[k] = \frac{1}{T} \int_{-T/2}^{T/2} u(t) \mathrm{e}^{-\mathrm{j}\omega_0 k t} \mathrm{d}t$$

are the Fourier coefficients. By linearity,

$$y(t) = \sum_{k \in \mathbb{Z}} G(\mathrm{j}\omega_0 k) U[k] \mathrm{e}^{\mathrm{j}\omega_0 k t},$$

is also T-periodic, with the Fourier coefficients

$$Y[k] = G(\mathrm{j}\omega_0 k) U[k].$$

The frequency response of G shapes the changes in Fourier coefficients

Response of LTI systems to sine wave test signals

Let $G : u \mapsto y$ be LTI, have a real-rational transfer function $G(s)$, and be stable. If $u(t) = \sin(\omega t + \phi)\mathbb{1}(t)$, then

$$Y(s) = G(s)\frac{s\sin\phi + \omega\cos\phi}{s^2 + \omega^2} = G_{\text{tr}}(s) + \frac{\operatorname{Res}(Y, \mathrm{j}\omega)}{s - \mathrm{j}\omega} + \frac{\operatorname{Res}(Y, -\mathrm{j}\omega)}{s + \mathrm{j}\omega}$$

for a proper $G_{\text{tr}}(s)$ having the same poles as $G(s)$ (all in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$). Now,

$$\operatorname{Res}(Y, \mathrm{j}\omega) = G(\mathrm{j}\omega)\frac{\mathrm{j}\omega\sin\phi + \omega\cos\phi}{\mathrm{j}2\omega} = |G(\mathrm{j}\omega)|\frac{\mathrm{e}^{\mathrm{j}\phi_y}}{2\mathrm{j}}$$

and

$$\operatorname{Res}(Y, -\mathrm{j}\omega) = G(-\mathrm{j}\omega)\frac{-\mathrm{j}\omega\sin\phi + \omega\cos\phi}{-\mathrm{j}2\omega} = -|G(\mathrm{j}\omega)|\frac{\mathrm{e}^{-\mathrm{j}\phi_y}}{2\mathrm{j}}$$

where $\phi_y := \phi + \arg(G(\mathrm{j}\omega))$ and the facts that

$$|G(-\mathrm{j}\omega)| = |G(\mathrm{j}\omega)| \quad \text{and} \quad \arg(G(-\mathrm{j}\omega)) = -\arg(G(\mathrm{j}\omega))$$

(remember, $g : \mathbb{R} \to \mathbb{R} \implies G(-\mathrm{j}\omega) = \overline{G(\mathrm{j}\omega)}$, Lect. 3, Slide 31) are used.

Response of LTI systems to sine wave test signals (contd)

Thus,

$$Y(s) = G_{\mathrm{tr}}(s) + |G(\mathrm{j}\omega)| \frac{s \sin\phi_y + \omega \cos\phi_y}{s^2 + \omega^2}$$

or, in the time domain,

$$y(t) = y_{\mathrm{tr}}(t) + |G(\mathrm{j}\omega)| \sin(\omega t + \phi_y)$$

where y_{tr} decays. The signal

$$y_{\mathrm{ss}}(t) = |G(\mathrm{j}\omega)| \sin\big(\omega t + \phi + \arg(G(\mathrm{j}\omega))\big)$$

is the steady-state response of G to a sine wave test input and the system

- scales the amplitude of the input by $|G(\mathrm{j}\omega)|$ and
- shifts the phase of the input by $\arg(G(\mathrm{j}\omega))$

in steady state, but does not alter its frequency, again.

Example 1

A mercury thermometer, , $G_{\text{therm}} : \theta_{\text{amb}} \mapsto \theta$ has the transfer function

$$G_{\text{therm}}(s) = \frac{1}{\tau s + 1}$$

for some time constant $\tau > 0$. Its frequency response

$$G_{\text{therm}}(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\tau\omega + 1} = \frac{1 - \mathrm{j}\tau\omega}{1 + \tau^2\omega^2} = \frac{1}{\sqrt{1 + \tau^2\omega^2}} \mathrm{e}^{-\mathrm{j}\arctan(\tau\omega)}$$

Its response to a sine wave test signal is (here $\phi_\omega := \phi - \arctan(\tau\omega)$)

$$\begin{aligned} \Theta(s) &= \frac{1}{\tau s + 1} \frac{s \sin\phi + \omega \cos\phi}{s^2 + \omega^2} \\ &= -\frac{\sin\phi_\omega}{\sqrt{1 + \tau^2\omega^2}} \frac{1}{s + 1/\tau} + \frac{1}{\sqrt{1 + \tau^2\omega^2}} \frac{s \sin\phi_\omega + \omega \cos\phi_\omega}{s^2 + \omega^2} \end{aligned}$$

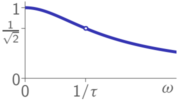
and

$$\theta(t) = \overbrace{-\frac{\sin\phi_\omega}{\sqrt{1 + \tau^2\omega^2}} \mathrm{e}^{-t/\tau} \mathbb{1}(t)}^{\text{(decaying) transients}} + \overbrace{\frac{1}{\sqrt{1 + \tau^2\omega^2}} \sin(\omega t + \phi_\omega) \mathbb{1}(t)}^{\text{steady state}}$$

Example 1 (contd)

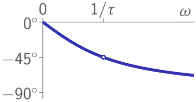
Scaling amplitude and shifting phase,

$$\frac{1}{\sqrt{1+\tau^2\omega^2}} =$$



and

$$-\arctan(\tau\omega) =$$



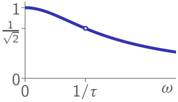
,

have a negligible effect on the steady-state response only if $\tau\omega \ll 1$. But if $\tau\omega > 1$, then the steady-state response is quite different from the input. In other words,

- thermometer could reliably measure a harmonic environment only if its time constant is small enough relatively to the signal frequency.

Example 1 (contd)

Scaling amplitude and shifting phase,

$$\frac{1}{\sqrt{1+\tau^2\omega^2}} =$$ 

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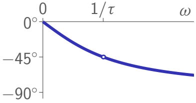
$$-\arctan(\tau\omega) =$$ 

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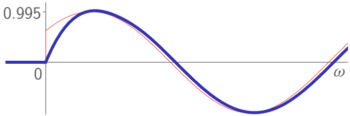
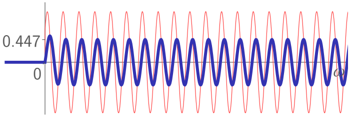
For example,

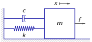


and



for $\tau\omega = 0.1$ and $\tau\omega = 2$, respectively (the red thin lineis θ_{amb}).

Example 2

Consider a mass-spring-damper system, 

, $G_{\text{msd}} : f \mapsto x$ with

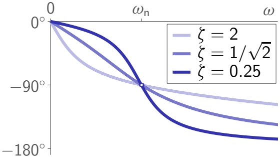
$$G_{\text{msd}}(s) = \frac{1}{ms^2 + cs + k} = \frac{k_{\text{st}}\omega_{\text{n}}^2}{s^2 + 2\zeta\omega_{\text{n}} s + \omega_{\text{n}}^2},$$

where $\omega_{\text{n}} = \sqrt{\frac{k}{m}}$, $\zeta = \frac{c}{2\sqrt{km}}$, & $k_{\text{st}} = \frac{1}{k} > 0$. Its frequency response

$$G_{\text{msd}}(\mathrm{j}\omega) = \frac{k_{\text{st}}\omega_{\text{n}}^2}{\omega_{\text{n}}^2 - \omega^2 + \mathrm{j}2\zeta\omega_{\text{n}}\omega} = \frac{k_{\text{st}}}{\sqrt{(1-\omega^2/\omega_{\text{n}}^2)^2 + 4\zeta^2\omega^2/\omega_{\text{n}}^2}}\,\mathrm{e}^{-\mathrm{j}\phi_2(\omega)},$$

where $\phi_2(\omega)$ is the unique value in the range $[-\pi, 0]$ satisfying

$$\tan\phi_2(\omega) = -\frac{2\zeta\omega/\omega_{\text{n}}}{1-\omega^2/\omega_{\text{n}}^2} \quad\Longrightarrow\quad \phi_2(\omega) =$$

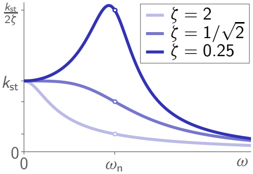


which is how the phase of an input sine wave signal is shifted in this case.

Example 2 (contd)

Magnitude of the input is then scaled as

$$|G_{\mathrm{msd}}(\mathrm{j}\omega)| = \frac{k_{\mathrm{st}}}{\sqrt{(1-\omega^2/\omega_{\mathrm{n}}^2)^2 + 4\zeta^2\omega^2/\omega_{\mathrm{n}}^2}} =$$



and G_{msd} can amplify inputs at certain frequencies if ζ is small enough. For example,

for $\zeta = \frac{1}{\sqrt{2}}$ and $\zeta = 0.25$, respectively, and $\omega = \omega_{\mathrm{n}}$ (the red thin line is f).

Example 2 (contd)

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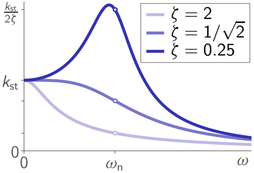

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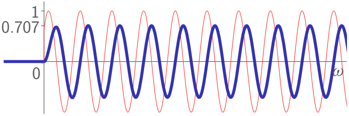
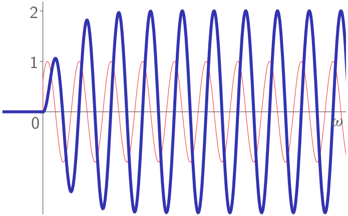


and



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