Linear Systems (034032)
lecture no. 7

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Outline

LTI systems response: from time to Laplace / z domain

We know that if $G : u \mapsto y$ is LTI (linear time invariant), then

$$y = g * u$$

where $g = G\delta$ is the impulse response of G, i.e. its response to δ applied at $t = 0$. By the convolution property of the Laplace / z transform,

$$Y(s) = G(s)U(s) \quad \text{or} \quad Y(z) = G(z)U(z),$$

In other words, (dynamic) LTI systems in the Laplace / z domain act as the (static) multiplication of the transformed impulse response and input. The function

$$G(s) = (\mathfrak{L}\{g\})(s) \quad \text{or} \quad G(z) = (\mathfrak{Z}\{g\})(z)$$

is called the transfer function of G. Transfer function may also be viewed as

– the ratio of the Laplace / z transforms of the output and input signals,

$$G(s) = \frac{Y(s)}{U(s)} \quad \text{or} \quad G(z) = \frac{Y(z)}{U(z)}$$

Transfer functions: examples

Gain: its impulse response $g = k\delta$, so $G(s) = k$ and $G(z) = k$.

Delay: its impulse response $g = \mathbb{S}_{-\tau}\delta$, so $G(s) = e^{-\tau s}$ and $G(z) = z^{-\tau}$.

Integrator: its impulse response $g_{\text{int}} = \mathbb{1}$, so

$$G_{\text{int}}(s) = \frac{1}{s} \quad \text{and} \quad G_{\text{int}}(z) = \frac{z}{z-1}.$$

has a single pole at $s = 0$ and $z = 1$, respectively.

Finite-memory integrator: its impulse response $g_{\text{fmi},\mu} = \mathbb{S}_{-\mu/2}\text{rect}_\mu$, so

$$G_{\text{fmi},\mu}(s) = \frac{1 - e^{-\mu s}}{s} = \int_0^\mu e^{-st}\,dt$$

is an entire function.

Transfer function description

The fact that *dynamic* LTI systems can be described by *algebraic* relations,

$$Y(s) = G(s)U(s) \quad \text{or} \quad Y(z) = G(z)U(z),$$

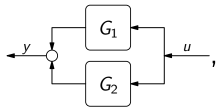
is a handy property, having a potential of greatly simplifying the analysis. It has no counterparts in the realm of general time-varying/nonlinear systems, making the class of LTI systems so special and sought after.

Still, we yet to know how to

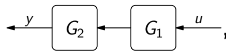
– express properties of LTI systems in terms of their transfer functions,

without which algebraic relations are not more than a curiosity.

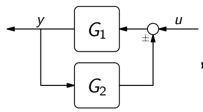
System interconnections via transfer functions

Parallel: If $G =$ 

, then $G(s) = G_1(s) + G_2(s)$ sum

Cascade: If $G =$ 

, then $G(s) = G_2(s)G_1(s)$ product

Feedback: If $G =$ 

, then $G(s) = \dfrac{G_1(s)}{1 \mp G_1(s)G_2(s)}$

by

$$Y = G_1(U \pm G_2 Y) \iff (1 \mp G_1 G_2)Y = G_1 U$$

Causality via transfer functions: discrete systems

Theorem

Let G be an LTI discrete-time system, whose transfer function $G(z)$ has its RoC $\subset \{z \in \mathbb{C} \mid |z| > \alpha_g\}$ *for some $\alpha_g \in \mathbb{R}_+$. Such G is causal iff*

$$\limsup_{|z|\to\infty} |G(z)| < \infty.$$

Proof (outline): G is causal iff $\operatorname{supp}(g) \subset \mathbb{Z}_+$. The transfer function

$$G(z) = (\mathfrak{Z}\{g\})(z) = \sum_{t\in\mathbb{Z}} g[t]z^{-t} = \sum_{t=-\infty}^{-1} g[t]z^{-t} + \sum_{t=0}^{\infty} g[t]z^{-t}$$

If $|z| \to \infty$, then $|z^{-t}| \to \infty$ for every $t \in \mathbb{Z} \setminus \mathbb{Z}_+$ as well and the first term above remains bounded only if $\operatorname{supp}(g) \subset \mathbb{Z}_+$. If $\operatorname{supp}(g) \subset \mathbb{Z}_+$, then the second term is in the RoC, so $|G(z)|$ is bounded, if $|z|$ is large enough. □

Remark: There is no similar result for continuous-time systems. For example, se^{-s} is not bounded if $\operatorname{Re} s \to +\infty$, but it corresponds to the causal system acting as $y(t) = \dot{u}(t-1)$.

L_2 / ℓ_2 stability via transfer functions

Theorem

A continuous-time LTI system is causal and L_2-stable iff its transfer function is holomorphic and bounded in $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$.

Theorem

A discrete-time LTI system is causal and ℓ_2-stable iff its transfer function is holomorphic and bounded in $\{z \in \mathbb{C} \mid |z| > 1\}$.

Example 1: we know that $G_{\text{fmi},\mu}(s) = \int_0^\mu e^{-st} dt$, which is entire. Now,

$$|G_{\text{fmi},\mu}(s)| = \left|\int_0^\mu e^{-st} dt\right| \le \int_0^\mu |e^{-st}| dt = \int_0^\mu e^{-\operatorname{Re} s\, t} dt < \int_0^\mu dt = \mu$$

for all $\operatorname{Re} s > 0$. Thus, $G_{\text{fmi},\mu}(s)$ is not only holomorphic, but also bounded in $\mathbb{C}_0$, meaning that $G_{\text{fmi},\mu}$ is L_2-stable.

Example 2: Consider $G : u \mapsto y$ acting as $y(t) = \dot{u}(t)$. Its transfer function $G(s) = s$ is entire, but not bounded in $\mathbb{C}_0$, meaning that G is not L_2-stable.

Outline

Transfer functions of LTI systems

LTI systems with rational transfer functions

Steady-state and transient responses

Step responses of stable 1-order systems

Step responses of stable 2-order systems

LTI systems described by ODE

Let $G : u \mapsto y$ be an analog LTI system described as

$$\begin{aligned} y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0 y(t) \\ = b_m u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + \cdots + b_1\dot{u}(t) + b_0 u(t) \end{aligned}$$

for some $a_i, b_i \in \mathbb{R}$. In the Laplace domain this relation reads

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}\, U(s).$$

Hence, the transfer function of this system is

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

In other words,

- transfer functions of LTI systems described by ODE are always rational

(actually, real-rational if the coefficients are real).

LTI systems described by difference equations

Let $G : u \mapsto y$ be a discrete LTI system described as

$$\begin{aligned} y[t+n] + a_{n-1}y[t+n-1] + \cdots + a_1 y[t+1] + a_0 y[t] \\ = b_m[t+m] + b_{m-1}u[t+m-1] + \cdots + b_1 u[t+1] + b_0 u[t] \end{aligned}$$

for some $a_i, b_i \in \mathbb{R}$. In the Laplace domain this relation reads

$$Y(z) = \frac{b_m z^m + b_{m-1}z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}\, U(z).$$

Hence, the transfer function of this system is

$$G(z) = \frac{b_m z^m + b_{m-1}z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}$$

again real-rational.

Interconnections

If

$$G_1(s) = \frac{N_1(s)}{D_1(s)} \quad \text{and} \quad G_2(s) = \frac{N_2(s)}{D_2(s)},$$

then

$$\begin{aligned} G_1(s) + G_2(s) &= \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)} = \frac{N_1(s)D_2(s) + N_2(s)D_1(s)}{D_1(s)D_2(s)} \\ G_2(s)G_1(s) &= \frac{N_2(s)}{D_2(s)}\frac{N_1(s)}{D_1(s)} = \frac{N_2(s)N_1(s)}{D_2(s)D_1(s)} \\ \frac{G_1(s)}{1 \mp G_1(s)G_2(s)} &= \frac{\frac{N_1(s)}{D_1(s)}}{1 \mp \frac{N_1(s)}{D_1(s)}\frac{N_2(s)}{D_2(s)}} = \frac{\frac{N_1(s)}{D_1(s)}}{\frac{D_1(s)D_2(s) \mp N_1(s)N_2(s)}{D_1(s)D_2(s)}} \\ &= \frac{N_1(s)D_2(s)}{D_1(s)D_2(s) \mp N_1(s)N_2(s)} \end{aligned}$$

are rational as well. In other words,

- parallel, cascade, & feedback interconnections of systems with rational transfer functions result in systems with rational transfer functions.

Impulse response of continuous-time systems

Let G be an LTI system with a rational proper transfer function of the form

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}, \quad n \ge m$$

Let it have k distinct poles $p_i \in \mathbb{C}$ of order $n_i \in \mathbb{N}$. In this case

$$G(s) = G(\infty) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{c_{ij}}{(s - p_i)^j}$$

The inverse Laplace transforms g_{ij} of $G_{ij}(s) := c_{ij}/(s - p_i)^j$ are

$$g_{i1}(t) = c_{i1} e^{p_i t} \mathbb{1}(t) \quad \text{and} \quad g_{ij}(t) = \frac{c_{ij}}{(j-1)!} t^{j-1} e^{p_i t} \mathbb{1}(t), \quad j > 1$$

(by the t-modulation property). Therefore, by linearity

$$g(t) = G(\infty)\,\delta(t) + \sum_{i=1}^{k} \Bigl(c_{i1} + \sum_{j=2}^{n_i} \frac{c_{ij}}{(j-1)!} t^{j-1} \Bigr) e^{p_i t} \mathbb{1}(t),$$

i.e. it is a superposition of (t-modulated) exponential functions.

Impulse response of continuous-time systems (contd)

If

- Re $p_i \ge 0$, then $\sum_{j=1}^{n_i} g_{ij} \notin L_1$ (terms do not decay)
- Re $p_i < 0$, then

$$\int_0^\infty \Bigl| \frac{c_{ij}}{(j-1)!} t^{j-1} e^{p_i t} \Bigr| dt = \frac{|c_{ij}|}{(-\operatorname{Re} p_i)^j} \quad \implies \quad \sum_{j=1}^{n_i} g_{ij} \in L_1$$

If Im $p_{i_1} \ne 0$, then $\exists\, i_2 \ne i_1$ such that $p_{i_2} = \overline{p_{i_1}}$ and also $c_{i_2 1} = \overline{c_{i_1 1}}$. Then

$$\frac{c_{i_1 1}}{s - p_{i_1}} + \frac{c_{i_2 1}}{s - p_{i_2}} = \frac{c_{i_1 1}}{s - p_{i_1}} + \frac{\overline{c_{i_1 1}}}{s - \overline{p_{i_1}}} = \frac{c_{i_1 1}(s - \overline{p_{i_1}}) + \overline{c_{i_1 1}}(s - p_{i_1})}{(s - p_{i_1})(s - \overline{p_{i_1}})}$$
$$= \frac{2 \operatorname{Re} c_{i_1 1}(s - \operatorname{Re} p_{i_1}) - 2 \operatorname{Im} c_{i_1 1} \operatorname{Im} p_{i_1}}{(s - \operatorname{Re} p_{i_1})^2 + (\operatorname{Im} p_{i_1})^2}$$

Thus, if all poles are simple, then we can always end up with an alternative partial fraction expansion with 1- and 2-order real-rational terms only. And we know that they correspond to exponentials and (modulated) sine waves.

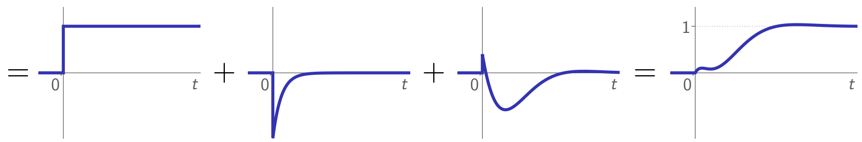
Impulse response of continuous-time systems: example

Consider

$$G(s) = \frac{s^2 - 2s + 6}{s^4 + 5s^3 + 8s^2 + 6s} = \frac{s^2 - 2s + 6}{s(s+3)(s^2 + 2s + 2)}$$
$$= \frac{\operatorname{Res}(G, 0)}{s} + \frac{\operatorname{Res}(G, -3)}{s+3} + \frac{\operatorname{Res}(G, -1 + j)}{s + 1 - j} + \frac{\operatorname{Res}(G, -1 - j)}{s + 1 + j}$$
$$= \frac{1}{s} - \frac{1.4}{s+3} + \frac{0.2 + 1.4j}{s + 1 - j} + \frac{0.2 - 1.4j}{s + 1 + j}$$
$$= \frac{1}{s} - \frac{1.4}{s+3} + \frac{0.4s - 2.8}{s^2 + 2s + 2} = \frac{1}{s} - \frac{1.4}{s+3} + 2\sqrt{2}\, \frac{\frac{\sqrt{2}}{10}(s+1) - \frac{7\sqrt{2}}{10}}{(s+1)^2 + 1}$$

Hence, taking into account that $\arccos(-0.7\sqrt{2}) = 2.999695599 \approx 3$,

$$g(t) = \mathbb{1}(t) - 1.4 e^{-3t} \mathbb{1}(t) + 2\sqrt{2}\, e^{-t} \sin(t + 3) \mathbb{1}(t)$$



Impulse response of discrete-time systems

Rational transfer functions in the z-domain are also decomposed to partial fractions. Time responses are then also superposition of exponential terms. However, discrete-time responses of corresponding terms are

- way messier than their continuous-time counterparts and less intuitive,

with some additional properties, like possibly oscillatory response of 1-order terms. For those reasons we won't discuss details.

I/O stability via transfer functions

Theorem

If the transfer function $G(s)$ of a continuous-time LTI system G is rational, then G is causal and I/O stable iff

- *$G(s)$ is proper and has no poles in* $\bar{\mathbb{C}}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$

Theorem

If the transfer function $G(z)$ of a discrete-time LTI system G is rational, then

- *G is causal iff $G(z)$ is proper and*
- *G is I/O stable iff $G(z)$ has no poles in* $\mathbb{C} \setminus \mathbb{D}_1 = \{z \in \mathbb{C} \mid |z| \geq 1\}$

Example 2 (contd): The differentiator $y(t) = \dot{u}(t)$ has the rational transfer function $G(s) = s$. But it is not proper, attesting to what we already saw.

Example 3: The discrete system $G : u \mapsto y$ acting as $y[t] = u[t+4] - u[t]$ has the (rational) transfer function $G(z) = z^4 - 1$. This $G(z)$ is non-proper and has no poles. Hence, G is not causal, but is stable.

Outline

Persistent test signals

Responses of systems obviously vary depending on inputs. It is not realistic to have a universal understanding of responses of dynamic systems to every possible input. Instead, we shall try to understand responses to (relatively) simple signals, which may nevertheless be *representative* in many situations (unlike the impulse). Such signals, termed test signals, normally

- have support in $\mathbb{R}_+$ (so the action starts at $t = 0$)
- are persistent (i.e. do not decay)
- have some regularity (like convergent or periodic)

Some commonly used test signals:

- step $u(t) = \mathbb{1}(t)$
  represents a quick change of an otherwise constant or slowly varying environment
- sine wave $u(t) = \sin(\omega_u t + \phi_u)\mathbb{1}(t)$
- T-periodic

with only pure imaginary poles as singularities of their Laplace transforms.

Steady-state and transients of the step responses

Consider a continuous-time LTI G with a rational transfer function. If G is stable, all poles of $G(s)$ are in $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$, so that it is holomorphic and bounded at $s = 0$. The Laplace transform of the step response is

$$Y(s) = \frac{G(s)}{s} = \frac{G(s) - G(0)}{s} + \frac{G(0)}{s} =: Y_{\text{tr}}(s) + Y_{\text{ss}}(s).$$

Transient response is the signal y_{tr}. Its Laplace transform $Y_{\text{tr}}(s)$ is rational, proper ($Y_{\text{tr}}(\infty) = Y(\infty) = 0$), and its singularity at $s = 0$ is removable, as

$$\lim_{s \to 0} Y_{\text{tr}}(s) = \lim_{s \to 0} \frac{G(s) - G(0)}{s} = G'(0).$$

Hence, y_{tr} is a superposition of *decaying* exponents and $\lim_{t \to \infty} y_{\text{tr}}(t) = 0$, meaning that

- the transient response vanishes asymptotically.

Steady-state and transients of the step responses (contd)

Consider a continuous-time LTI G with a rational transfer function. If G is stable, all poles of $G(s)$ are in $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$, so that it is holomorphic and bounded at $s = 0$. The Laplace transform of the step response is

$$Y(s) = \frac{G(s)}{s} = \frac{G(s) - G(0)}{s} + \frac{G(0)}{s} =: Y_{\text{tr}}(s) + Y_{\text{ss}}(s).$$

Stead-state response is the scaled step $y_{\text{ss}} = G(0)\mathbb{1}$. The constant

- $G(0)$ is called the static gain of G.

Thus, the step response of a stable LTI system G converges asymptotically to the step signal scaled by its static gain $G(0)$.

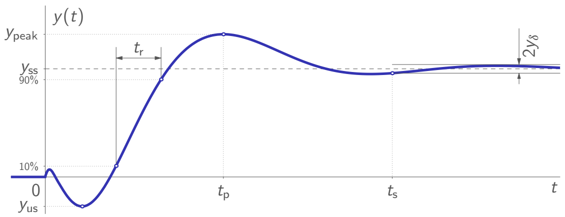
Remark: We refer to $G(0)$ as the static gain of G in the unstable case as well. If $G(0)$ is finite, then we may still think of $y_{\text{ss}} = G(0)\mathbb{1}$ as the steady-state response of an unstable system. However, the transients do not decay then (might even diverge).

Steady-state and transients of the step responses (contd)

Practically,

- steady-state response shows what the response will eventually be
  - what is the mercury temperature in a thermometer
  - what floor an elevator reaches
  - what position the pointer of a spring scale stops
  - ...
- transient response shows how the steady state is reached
  - how fast a thermometer catch the ambient temperature
  - how fast and smooth (comfortable) an elevator moves between floors
  - how fast the pointer of a spring scale stops
  - how smooth a vehicle responds to road bumps
  - how large an electromagnetic pulse is on switching electrical circuits
  - ...

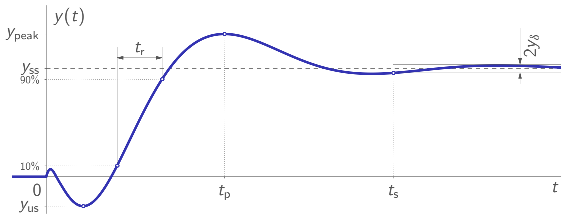
Transients: important characteristics



Smoothness of transients may be measured by the

- overshoot,
$$\text{OS} := \frac{y_{\text{peak}} - y_{\text{ss}}}{y_{\text{ss}}} > 0 \quad (\text{in } \%),$$
where y_{peak} is the highest peak in the direction of y_{ss},
- undershoot,
$$\text{US} := -\frac{y_{\text{us}}}{y_{\text{ss}}} > 0 \quad (\text{in } \%),$$
where y_{us} is the highest peak against the direction of y_{ss}.

Transients: important characteristics (contd)



Speed of transients may be measured by the

- rise time, t_r (time that takes y to rise from $0.1y_{\text{ss}}$ to $0.9y_{\text{ss}}$)
- peak time, t_p (time that takes y to reach its highest peak)

Duration of transients may be measured by the

- settling time, t_s (the smallest t_s such that $\left|\frac{y(t)}{y_{\text{ss}}} - 1\right| \le \frac{\delta}{100}$, $\forall t \ge t_s$)

for a given

- settling level $\delta := \left|\frac{y_\delta}{y_{\text{ss}}}\right|$ (in %, most common $\delta = 2\%$ and $\delta = 5\%$)

Outline

Fundamental 1-order system

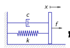
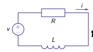
An LTI system $G : u \mapsto y$ with the transfer function

$$G(s) = \frac{k_{st}}{\tau s + 1},$$

where

- $k_{st} \neq 0$ is its static gain ($k_{st} = G(0)$)
- $\tau > 0$ is its time constant

Examples:

- spring-damper, , with $c\dot{x}(t) + kx(t) = f(t)$ has $\tau = \frac{c}{k}$ & $k_{st} = \frac{1}{k}$;
- mercury thermometer, , with $\tau\dot{\theta}(t) = \theta_{amb}(t) - \theta(t)$ has τ as its time constant and $k_{st} = 1$;
- RL-circuit, , with $L\dot{i}(t) + Ri(t) = v(t)$ has $\tau = \frac{L}{R}$ & $k_{st} = \frac{1}{R}$.

Step response

If $u = \mathbb{1}$, then

$$Y(s) = \frac{k_{st}}{\tau s + 1}\frac{1}{s} = k_{st}\Big(\frac{1}{s} - \frac{1}{s + 1/\tau}\Big)$$

and

$$y(t) = k_{st}(1 - e^{-t/\tau})\mathbb{1}(t),$$

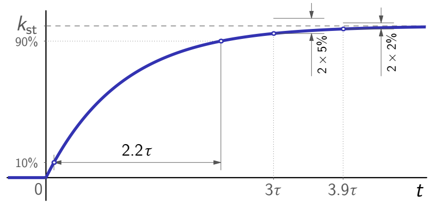
i.e.

- $y_{ss} = k_{st}\mathbb{1}$ is shaped by k_{st}
- $y_{tr} = -k_{st}(\mathbb{P}_{1/\tau}\exp_{-1})\mathbb{1}$ is shaped mainly by τ — k_{st} only scales it

By the initial value theorem,

$$\dot{y}(0) = \lim_{s\to\infty} s \cdot sY(s) = \lim_{s\to\infty} \frac{k_{st}s}{\tau s + 1} = \frac{k_{st}}{\tau} \neq 0$$

Step response: transients



This response has

- overshoot OS = 0% and undershot US = 0% — monotonic
- raise time $t_r = (\ln 9)\tau \approx 2.2\,\tau$
- settling time $t_s = (\ln \frac{100}{\delta})\tau$ — $y_{tr}(t_s) = -\frac{\delta}{100}k_{st}$

Qualitatively, the speed and duration of transients in first-order systems are proportional to the time constant τ, viz.

- the larger τ is, the slower transients are.

Outline

Fundamental 2-order system

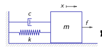
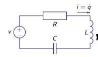
An LTI system $G : u \mapsto y$ with the transfer function

$$G(s) = \frac{k_{st}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where

- $k_{st} \neq 0$ is its static gain ($k_{st} = G(0)$)
- $\omega_n > 0$ is its natural frequency
- $\zeta \geq 0$ is its damping factor

Examples:

- mass-spring-damper, , with $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$ has $\omega_n = \sqrt{k/m}$, $\zeta = 0.5c/\sqrt{km}$, and $k_{st} = 1/k$;
- RLC-circuit, , with $L\ddot{q}(t) + R\dot{q}(t) + Cq(t) = v(t)$, where q is the charge, has $\omega_n = \sqrt{C/L}$, $\zeta = 0.5R/\sqrt{LC}$, and $k_{st} = 1/C$.

Step response

If $u = \mathbb{1}$, then

$$Y(s) = \frac{k_{st}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}$$

and the roots of its denominator are

$$s_1 = 0 \quad \text{and} \quad s_{2,3} = \bigl(-\zeta \pm \sqrt{\zeta^2 - 1}\bigr)\omega_n$$

Three cases shall be studied separately

1. if $\zeta > 1$, then $s_{2,3}$ are real and simple — overdamped
2. if $\zeta = 1$, then $s_{2,3}$ are real and equal — critically damped
3. if $\zeta < 1$, then $s_{2,3}$ are complex conjugate — underdamped

if $\zeta = 0$, then we say that the system is *undamped*; no need in a separate analysis

In all cases,

$$\dot{y}(0) = \lim_{s\to\infty} s \cdot sY(s) = \lim_{s\to\infty} \frac{k_{st}\omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = 0$$

Step response of overdamped systems

If $\zeta > 1$, then

$$Y(s) = k_{st}\Bigl(\frac{1}{s} - \frac{\beta}{s - \lambda_1} + \frac{\beta - 1}{s - \lambda_2}\Bigr)$$

where

$$\lambda_{1,2} = -(\zeta \mp \sqrt{\zeta^2 - 1})\omega_n < 0 \quad \text{and} \quad \beta = \frac{1}{2}\Bigl(\frac{\zeta}{\sqrt{\zeta^2 - 1}} + 1\Bigr) > 1$$

In the time domain

$$y(t) = k_{st}\bigl(1 - \beta e^{\lambda_1 t} + (\beta - 1)e^{\lambda_2 t}\bigr)\mathbb{1}(t)$$

with

$$\dot{y}(t) = k_{st}\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{2\sqrt{\zeta^2 - 1}}\mathbb{1}(t) > 0, \quad \forall t > 0$$

(because $\lambda_1 > \lambda_2$), meaning that y is monotonically increasing.

The transient response, $y_{tr}(t) = -k_{st}\beta e^{\lambda_1 t}\mathbb{1}(t) + k_{st}(\beta - 1)e^{\lambda_2 t}\mathbb{1}(t)$, is the superposition of transient responses of 1-order systems.

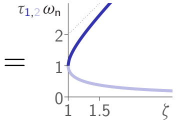
Step response of overdamped systems (contd)

Rewrite

$$y_{\mathrm{tr}}(t) = \overbrace{-k_{\mathrm{st}}\beta \mathrm{e}^{-t/\tau_1}\mathbb{1}(t)}^{y_{\mathrm{tr1}}(t)} + \overbrace{k_{\mathrm{st}}(\beta-1)\mathrm{e}^{-t/\tau_2}\mathbb{1}(t)}^{y_{\mathrm{tr2}}(t)},$$

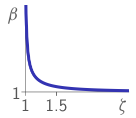
where the time constants and gains are

$$\tau_{1,2} = \frac{\zeta \pm \sqrt{\zeta^2-1}}{\omega_{\mathrm{n}}} =$$



and

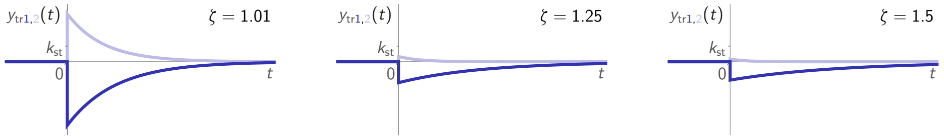
$$\beta = \frac{\zeta + \sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}} =$$



As ζ grows,

- τ_1 grows with respect to τ_2 $\implies$ y_{tr2} decays faster than y_{tr1}
- β grows with respect to $\beta - 1$ $\implies$ y_{tr2} becomes smaller than y_{tr1}

like in



meaning that y_{tr2} may be neglected for large ζ's, say for $\zeta \geq 1.25$.

Step response of critically damped systems

If $\zeta = 1$, then

$$Y(s) = k_{\mathrm{st}}\Big(\frac{1}{s} - \frac{1}{s+\omega_{\mathrm{n}}} - \frac{\omega_{\mathrm{n}}}{(s+\omega_{\mathrm{n}})^2}\Big)$$

In the time domain

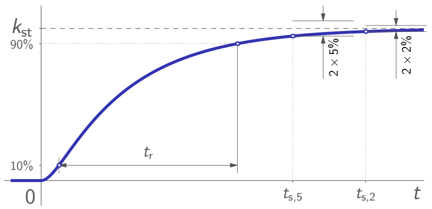
$$y(t) = k_{\mathrm{st}}\big(1 - (1+\omega_{\mathrm{n}}t)\mathrm{e}^{-\omega_{\mathrm{n}}t}\big)\mathbb{1}(t),$$

with

$$\dot{y}(t) = k_{\mathrm{st}}\omega_{\mathrm{n}}^2 t\mathrm{e}^{-\omega_{\mathrm{n}}t}\mathbb{1}(t) > 0, \quad \forall t > 0,$$

meaning that y is monotonically increasing in this case as well.

Step response of over- and critically damped systems



$$t_{\mathrm{r}} \approx \frac{1}{\omega_{\mathrm{n}}}\begin{cases} 4.9\zeta - 1.5 & \text{if } 1 \leq \zeta < 2 \\ 4.4\zeta - 0.5 & \text{if } \zeta \geq 2 \end{cases}$$

$$t_{\mathrm{s,5}} \approx \frac{1}{\omega_{\mathrm{n}}}\begin{cases} -0.6\zeta^2 + 8.5\zeta - 3.1 & \text{if } 1 \leq \zeta < 2 \\ 6\zeta - 0.5 & \text{if } \zeta \geq 2 \end{cases}$$

$$t_{\mathrm{s,2}} \approx \frac{1}{\omega_{\mathrm{n}}}\begin{cases} -1.1\zeta^2 + 12.3\zeta - 5.3 & \text{if } 1 \leq \zeta < 2 \\ 7.9\zeta - 0.9 & \text{if } \zeta \geq 2 \end{cases}$$

This response has

- overshoot OS $= 0\%$ and undershot US $= 0\%$ monotonic
- raise time t_{r}
  - monotonically increases with ζ
  - is inversely proportional to ω_{n}
- settling time t_{s}
  - monotonically increases with ζ
  - is inversely proportional to ω_{n}

Step response of underdamped systems

If $0 \leq \zeta < 1$, then

$$Y(s) = k_{\mathrm{st}}\Big(\frac{1}{s} - \frac{s+2\zeta\omega_{\mathrm{n}}}{s^2+2\zeta\omega_{\mathrm{n}}s+\omega_{\mathrm{n}}^2}\Big) = k_{\mathrm{st}}\Big(\frac{1}{s} - \frac{(s+\zeta\omega_{\mathrm{n}})+\zeta\omega_{\mathrm{n}}}{(s+\zeta\omega_{\mathrm{n}})^2+(1-\zeta^2)\omega_{\mathrm{n}}^2}\Big)$$

$$= k_{\mathrm{st}}\Big(\frac{1}{s} - \frac{1}{\sqrt{1-\zeta^2}}\frac{\sqrt{1-\zeta^2}(s+\zeta\omega_{\mathrm{n}})+\zeta\omega_{\mathrm{d}}}{(s+\zeta\omega_{\mathrm{n}})^2+\omega_{\mathrm{d}}^2}\Big)$$

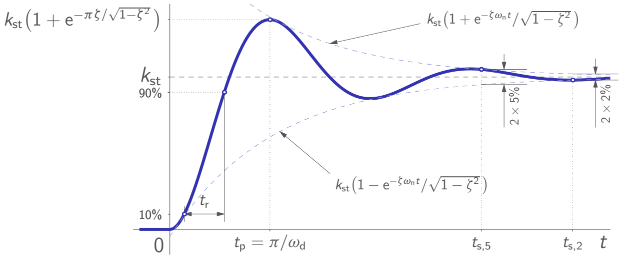
where $\omega_{\mathrm{d}} := \sqrt{1-\zeta^2}\,\omega_{\mathrm{n}}$ is the damped natural frequency. Hence,

$$y(t) = k_{\mathrm{st}}\bigg(1 - \frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_{\mathrm{n}}t}\sin(\omega_{\mathrm{d}}t + \arccos\zeta)\bigg)\mathbb{1}(t)$$

and it is always within the envelope

$$k_{\mathrm{st}}\bigg(1 - \frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_{\mathrm{n}}t}\bigg)\mathbb{1}(t) \leq y(t) \leq k_{\mathrm{st}}\bigg(1 + \frac{1}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_{\mathrm{n}}t}\bigg)\mathbb{1}(t).$$

Step response of underdamped systems: transients



$$t_r \approx \frac{1.6\zeta^3 - 0.17\zeta^2 + 0.92\zeta + 1.02}{\omega_n}$$

$$t_{s,5} \approx \frac{2.88}{(-\zeta^2 + 0.64\zeta + 0.96)\,\zeta\omega_n}$$

$$t_{s,2} \approx \frac{6.49}{(-\zeta^2 + 0.45\zeta + 1.65)\,\zeta\omega_n}$$

This response has

- overshoot $\mathrm{OS} = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \cdot 100\%$ and undershot $\mathrm{US} = 0\%$
- raise time t_r is inversely proportional to ω_n
- settling time t_s is inversely proportional to ω_n

Qualitatively,

- the smaller ζ is, the more oscillatory the response is

(larger overshoot, up to $\mathrm{OS} = 100\%$ for $\zeta = 0$, and longer t_s).

Step response of underdamped systems: effect of zeros

Let

$$G_\alpha(s) = \frac{k_{st}(\alpha\omega_n s + \omega_n^2)}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

for $\alpha \in \mathbb{R}$. This transfer function is said to have a zero at $s = -\omega_n/\alpha$ since $G_\alpha(-\omega_n/\alpha) = 0$. In this case

$$Y_\alpha(s) = \frac{k_{st}(\alpha\omega_n s + \omega_n^2)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = Y_0(s) + \frac{\alpha}{\omega_n} s Y_0(s)$$

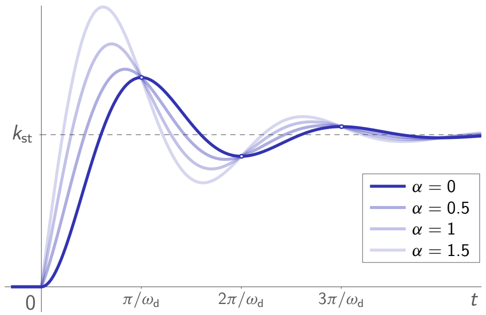
and

$$y_\alpha(t) = y_0(t) + \frac{\alpha}{\omega_n}\dot{y}_0(t),$$

where y_0 is the response with $\alpha = 0$ (no zeros) and

$$\frac{\alpha}{\omega_n}\dot{y}_0(t) = \frac{\alpha}{\sqrt{1-\zeta^2}}\sin(\omega_d t).$$

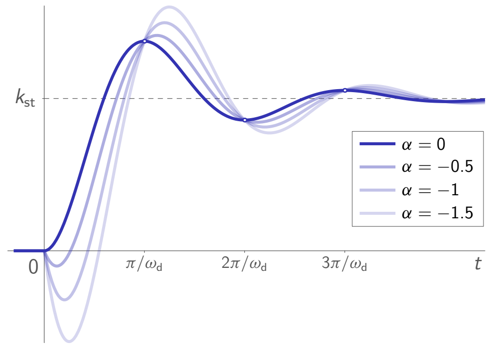
Step response of underdamped systems: $\alpha > 0$



As α grows,

- the overshoot OS increases
- the raise time t_r decreases
- the settling time t_s increases

Step response of underdamped systems: $\alpha < 0$



As α decreases,

- the overshoot OS increases
- the undershoot US increases
- the raise time t_r decreases
- the settling time t_s increases