Linear Systems (034032)
lecture no. 6

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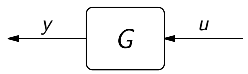
Reminder: systems

- Systems are constraints imposed on interdependent signals of interest.
- I/O systems map input signals (action) to output signals (reaction).
- The relation between signals in a system $G : u \mapsto y$ is denoted $y = Gu$.
- Models are relations between involved signals described by an abstract (mathematical) language under simplifying assumptions.
- Block-diagrams represent information flow in I/O systems, like



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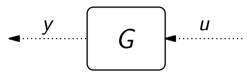
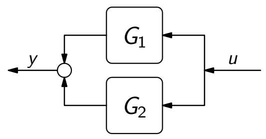


and are convenient to visualize interconnected systems



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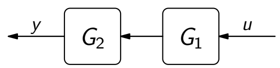

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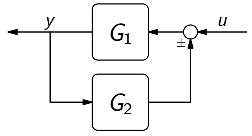


Outline

Gain

It is a system $u \mapsto y$ such that

$$y = ku$$

for a given gain

- either constant, $k \in \mathbb{R}$
- or varying,
  - $k : \mathbb{R} \to \mathbb{R}$, in the continuous-time case — so $y(t) = k(t)u(t)$
  - $k : \mathbb{Z} \to \mathbb{R}$, in the discrete-time case — so $y[t] = k[t]u[t]$

Think of an amplifier, gearbox, currency converter, ... as examples of such systems.

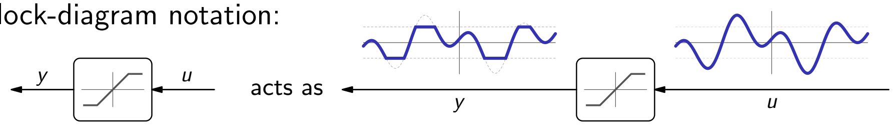
Saturation

It is a system $u \mapsto y$, which we denote $\mathrm{sat}_{[a,b]}$, such that

$$y(t) = \begin{cases} a & \text{if } u(t) < a \\ u(t) & \text{if } a \le u(t) \le b \\ b & \text{if } u(t) > b \end{cases} \quad \text{or} \quad y[t] = \begin{cases} a & \text{if } u[t] < a \\ u[t] & \text{if } a \le u[t] \le b \\ b & \text{if } u[t] > b \end{cases}$$

for given $a < b$. We use the short notation $\mathrm{sat}_a := \mathrm{sat}_{[-a,a]}$ for some $a > 0$. Think of a gas pedal in cars, water tap, integer overflow in computers, etc.

Block-diagram notation:



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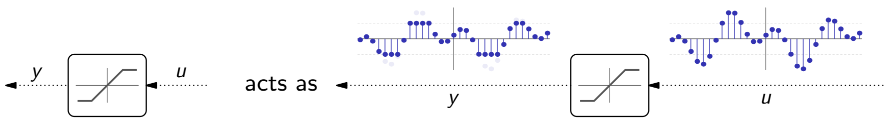


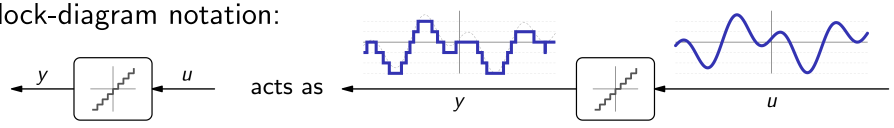
Quantizer

It is a system $u \mapsto y$, which we denote qnt_q, such that

$$y = q\left\lfloor \frac{u}{q} \right\rfloor \quad \text{or, maybe, } y = q\left\lfloor \frac{u}{q} + \frac{1}{2} \right\rfloor$$

for a given threshold $q > 0$. Think of an incremental encoder sensing, finite word-length effects in computers, etc.

Block-diagram notation:



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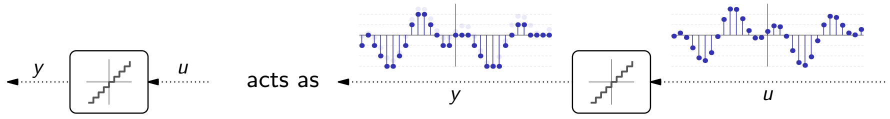


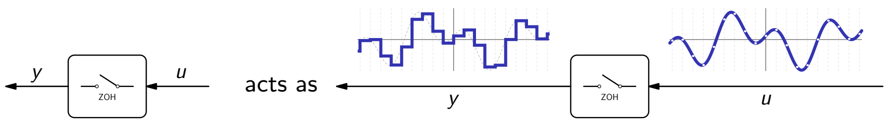
Sample-and-hold

It is a continuous-time system $G_{\mathrm{S\&H}} : u \mapsto y$ such that

$$y(t) = u\Bigl(\Bigl\lfloor \frac{t}{h} \Bigr\rfloor h\Bigr) \quad \Longleftrightarrow \quad y(ih + \tau) = u(ih), \quad \forall \tau \in (0, h)$$

for a given sampling interval $h > 0$. It is a basic block of any sampled-data control / signal processing system.

Block-diagram notation:



Integrator

In continuous time: the system $G_{\mathrm{int}} : u \mapsto y$ such that

$$y(t) = \int_{-\infty}^{t} u(s)\,\mathrm{d}s$$

Think of a water tank with no outflow, odometer, electricity / water meters, etc

In discrete time: the system $G_{\mathrm{int}} : u \mapsto y$ such that

$$y[t] = \sum_{s=-\infty}^{t} u[s]$$

Think of your (well-organized) mail account Inbox, Trash folder, etc.

Finite-memory integrator

In continuous time: the system $G_{\mathrm{fmi},\mu} : u \mapsto y$ such that

$$y(t) = \int_{t-\mu}^{t} u(s)\mathrm{d}s \quad \text{or} \quad y(t) = \int_{t-\mu(t)}^{t} u(s)\mathrm{d}s$$

for a given $\mu > 0$ or function $\mu : \mathbb{R} \to \mathbb{R}_+$. Think of a pipeline (if its content is of interest as the output), vehicle line at a traffic light (abroad), etc.

In discrete time: the system $G_{\mathrm{fmi},\mu} : u \mapsto y$ such that

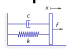
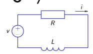
$$y[t] = \sum_{s=t-\mu}^{t} u[s] \quad \text{or} \quad y[t] = \sum_{s=t-\mu[t]}^{t} u[s]$$

for some $\mu \in \mathbb{N}$ or function $\mu : \mathbb{Z} \to \mathbb{N}$. Think of a circular buffer, a FIFO queue, etc.

First-order system

In continuous time: the system $G_{1\mathrm{o}} : u \mapsto y$ such that

$$\dot{y}(t) + a_0 y(t) = b_0 u(t)$$

for constant real parameters a_0 and $b_0 \neq 0$. Think of a thermometer , spring-damper , and RL-circuit , which we already saw.

In discrete time: the system $G_{1\mathrm{o}} : u \mapsto y$ such that

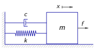
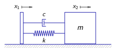
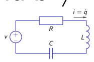
$$y[t+1] + a_0 y[t] = b_1 u[t+1] + b_0 u[t]$$

Think of a saving program, loan return, simple population grows model (if the birth and mortality rates are *assumed* to be in constant proportions to the total population and the input is migrations), etc.

Second-order system

In continuous time: the system $G_{2\mathrm{o}} : u \mapsto y$ such that

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_1 \dot{u}(t) + b_0 u(t)$$

for constant real parameters a_1, a_0, and $b \neq 0$. Think of mass-spring-damper  or  and RLC-circuit , which we already saw.

In discrete time: the system $G_{1\mathrm{o}} : u \mapsto y$ such that

$$y[t+2] + a_1 y[t+1] + a_0 y[t] = b_2 u[t+2] + b_1 u[t+1] + b_0 u[t]$$

Some believe (Keynesian economics) that this is how the GNP evolves as a function of governmental expenditure[1]; the Fibonacci sequence is generated by this system[2], ...

[1] With $a_1 = -\alpha(1+\beta)$, $a_0 = \alpha\beta$, $b_2 = 1$, and $b_1 = b_0 = 0$, where α is the part of the GNP going to consumption and β is the part of the GNP grows invested.

[2] With $a_1 = a_0 = -1$, $b_2 = b_0 = 0$, $b_1 = 1$, and $u = \delta$.

Delay element

In continuous time: the system $\bar{D}_\tau : u \mapsto y$ such that $y = \mathbb{S}_{-\tau} u$, i.e.

$$y(t) = u(t-\tau) \quad \text{or} \quad y(t) = u(t - \tau(t))$$

for a given $\tau > 0$ or function $\tau : \mathbb{R} \to \mathbb{R}_+$. Think of a conveyor belt, toaster, satellite transmission, optical fibers, etc.

In discrete time: the system $\bar{D}_\tau : u \mapsto y$ such that $y = \mathbb{S}_{-\tau} u$, i.e.

$$y[t] = u[t-\tau] \quad \text{or} \quad y[t] = u[t - \tau[t]]$$

for some $\tau \in \mathbb{N}$ or a function $\tau : \mathbb{Z} \to \mathbb{N}$. Think of a digital communication channel, exam grading, US presidential inauguration, etc.

Outline

Some standard systems

Principal properties

LTI systems in the time domain

Static & dynamic systems

A system $u \mapsto y$ is said to be static if

- y at every time instance t_c can depend only on u at $t = t_c$ and not at $t < t_c$ or $t > t_c$.

Otherwise, i.e. if at some time instances the output may depend on past or future inputs, the system is dubbed dynamic. In other words, static systems are memoryless, while dynamic systems have memory.

Examples

- static systems: gain, saturation, quantizer
- dynamic systems: integrator, finite-memory integrator, 1- and 2-order systems, delay

SISO & MIMO systems

If both u and y are scalar-valued, we say that a system $G : u \mapsto y$ is SISO (single-input / single-output). If either of those signals is vector-values, then we say that G is MIMO (multiple-input / multiple-output).

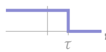
Causality

A system $u \mapsto y$ is said to be causal if

- y at every time instance t_c can depend only on u at $t \le t_c$ and not at $t > t_c$.

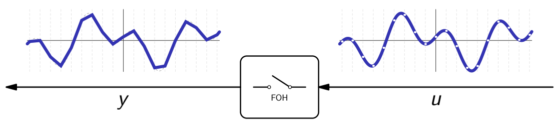
Every static system is obviously causal. A formal way to define causality is to require

$$\chi_\tau(Gu) = \chi_\tau\big(G(\chi_\tau u)\big), \quad \forall \tau \in \mathbb{R}$$

where the signal $\chi_\tau := \mathbb{S}_{-\tau}(\mathbb{P}_{-1}\mathbb{1}) =$ , meaning that inputs at $t > \tau$ do not affect outputs at $t \le \tau$.

Examples

- causal systems: everything studied today
- non-causal systems: sample-and-hold with the first-order hold



Linearity

A system $G : u \mapsto y$ is said to be linear if

$$G(a_1 u_1 + a_2 u_2) = a_1(Gu_1) + a_2(Gu_2)$$

for all admissible inputs u_1 and u_2 and all scalars a_1 and a_2. In other words, linearity requires superposition property to hold. An alternative definition is in terms of additivity and homogeneity, i.e.

$$G(u_1 + u_2) = Gu_1 + Gu_2 \quad \text{and} \quad G(au) = a(Gu)$$

Remark. By homogeneity, $u = 0 \implies Gu = 0$ whenever G is linear. In other words, the zero input necessarily produces the zero output.

Linearity is a very useful property, greatly simplifying the analysis. So it is

- often assumed, but seldom (if ever) holds in reality.

Still, linear models approximate the nonlinear reality well in many situations.

Linearity: examples

Integrator:

$$G_{\text{int}}(a_1u_1 + a_2u_2) = \int_{-\infty}^{t} (a_1u_1(s) + a_2u_2(s))\,\mathrm{d}s$$
$$= a_1 \int_{-\infty}^{t} u_1(s)\mathrm{d}s + a_2 \int_{-\infty}^{t} u_2(s)\mathrm{d}s$$
$$= a_1(G_{\text{int}}u_1) + a_2(G_{\text{int}}u_2)$$

i.e. it is linear (as well as finite-memory integrator, 1- and 2-order systems).

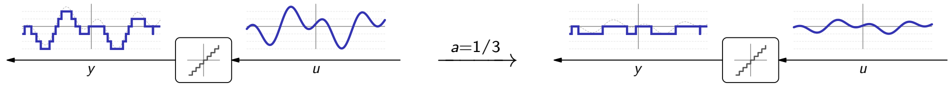
Delay:

$$\bar{D}_\tau(a_1u_1 + a_2u_2) = (a_1u_1 + a_2u_2)(t-\tau) = (a_1u_1)(t-\tau) + (a_2u_2)(t-\tau)$$
$$= a_1u_1(t-\tau) + a_2u_2(t-\tau)$$
$$= a_1(\bar{D}_\tau u_1) + a_2(\bar{D}_\tau u_2)$$

i.e. it is linear.

Linearity: examples (contd)

Quantizer: cf.

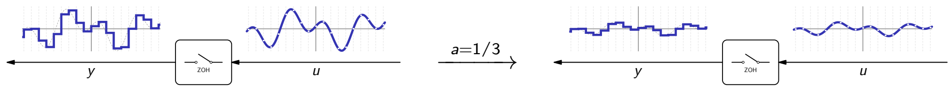


i.e. it is nonlinear because homogeneity does not hold (so is the saturation).

Sample-and-hold:

$$G_{\text{S\&H}}(a_1u_1 + a_2u_2) = (a_1u_1 + a_2u_2)\Big(\Big\lfloor \frac{t}{h} \Big\rfloor h\Big) = a_1u_1\Big(\Big\lfloor \frac{t}{h} \Big\rfloor h\Big) + a_2u_2\Big(\Big\lfloor \frac{t}{h} \Big\rfloor h\Big)$$
$$= a_1(G_{\text{S\&H}}u_1) + a_2(G_{\text{S\&H}}u_2)$$

i.e. it is linear, cf.



Time (shift) invariance

A system $G : u \mapsto y$ is said to be time invariant (or shift invariant) if

$$G(\mathbb{S}_\tau u) = \mathbb{S}_\tau(Gu)$$

for all admissible inputs u and all $\tau \in \mathbb{R}\,/\,\tau \in \mathbb{Z}$. In other words, every time shift of the input results in the same time shift of the output. Otherwise, a system is called time varying. If the property above holds only if $\tau = Tk$ for some $T > 0$ and every $k \in \mathbb{Z}$, then G is said to be T-periodic.

If G is both linear and time invariant, it is called LTI (linear time invariant).

Time invariance: examples

Gain:

$$(k(\mathbb{S}_\tau u))(t) = k(t)\big(u(t)|_{t\to t+\tau}\big) = k(t)u(t+\tau)$$
$$(\mathbb{S}_\tau(ku))(t) = \big(k(t)u(t)\big)|_{t\to t+\tau} = k(t+\tau)u(t+\tau)$$

so it is time invariant iff $k(t+\tau) = k(t)$ for all τ and t, i.e. iff $k = \text{const}$.

Finite-memory integrator: if the memory length μ is constant, then

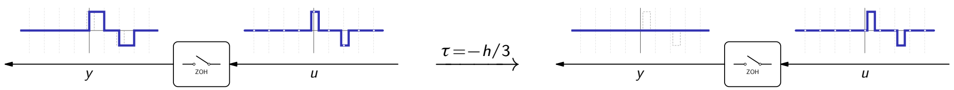
$$(G_{\text{fmi},\mu}(\mathbb{S}_\tau u))(t) = \int_{t-\mu(t)}^{t} \big(u(s)|_{s\to s+\tau}\big)\mathrm{d}s = \int_{t+\tau-\mu(t)}^{t+\tau} u(s)\mathrm{d}s$$
$$(\mathbb{S}_\tau(G_{\text{fmi},\mu}u))(t) = \Big(\int_{t-\mu(t)}^{t} u(s)\mathrm{d}s\Big)\Big|_{t\to t+\tau} = \int_{t+\tau-\mu(t+\tau)}^{t+\tau} u(s)\mathrm{d}s$$

so it is time invariant iff $\mu(t+\tau) = \mu(t)$ for all τ and t, i.e. iff $\mu = \text{const}$.

1st and 2nd order systems: are time invariant (try to show that).

Time invariance: examples (contd)

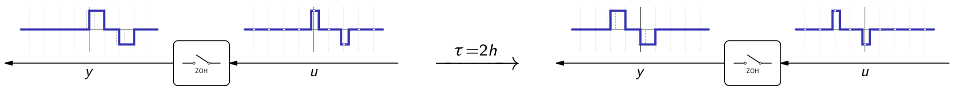
Sample-and-hold: cf.



i.e. it is time varying. However, a shift by kh for every $k \in \mathbb{Z}$ yields

$$\begin{aligned}(G_{\text{S\&H}}(\mathbb{S}_{kh}u))(t) &= u\Big(\Big\lfloor \frac{t+kh}{h}\Big\rfloor h\Big) = u\Big(\Big\lfloor \frac{t}{h}+k\Big\rfloor h\Big) = u\Big(\Big\lfloor \frac{t}{h}\Big\rfloor h + kh\Big) \\ &= (\mathbb{S}_{kh}(G_{\text{S\&H}}u))(t)\end{aligned}$$

i.e. it is h-periodic, cf.



I/O stability

A continuous-time system $G : u \mapsto y$ is said to be

– L_q-stable if there are $\gamma, \beta \in \mathbb{R}$ such that $\|y\|_q \le \gamma\|u\|_q + \beta$, $\forall u \in L_q$

for $q \in \{1, 2, \infty\}$. L_q-stability implies that $y \in L_q$ for all $u \in L_q$. L_∞-stable systems are also known as BIBO (bounded input / bounded output) stable.

If G is linear, then we may always take $\beta = 0$ in the stability definition. I.e. a linear system is L_q-stable if $\|y\|_q \le \gamma\|u\|_q$ for some γ (independent of u). If G is stable, then the smallest γ for which the inequality above holds true is called the L_q-induced norm of G,

$$\|G\|_{\text{ind},q} := \sup_{u\neq 0} \frac{\|Gu\|_q}{\|u\|_q} = \sup_{\|u\|_q=1} \|Gu\|_q.$$

In the discrete-time case, the ℓ_q-stability and ℓ_q-induced norms are defined similarly.

Stability: examples

Showing that a system is stable, w/o understanding general systems theory, might not be easy. But in some cases it is.

Delay. If $\tau = \text{const}$, then

$$\|y\|_\infty = \sup_{t\in\mathbb{R}}|y(t)| = \sup_{t\in\mathbb{R}}|u(t-\tau)| = \sup_{t\in\mathbb{R}}|u(t)| = \|u\|_\infty$$

and

$$\|y\|_2 = \int_{\mathbb{R}}|y(t)|^2\mathrm{d}t = \int_{\mathbb{R}}|u(t-\tau)|^2\mathrm{d}t\Big|_{s=t-\tau} = \int_{\mathbb{R}}|u(s)|^2\mathrm{d}s = \|u\|_2$$

Thus, $\bar{D}_\tau$ with a constant τ is both L_2- and L_∞-stable, with induced norms

$$\|\bar{D}_\tau\|_{\text{ind},\infty} = \|\bar{D}_\tau\|_{\text{ind},2} = 1.$$

Remark. Curiously, $\bar{D}_\tau$ might be L_2-unstable for varying τ. Try to find an example of $\tau(t)$ for which the delay is unstable and the corresponding destabilizing input.

Stability: examples (contd)

Showing that a system is unstable may be easier. One "just" needs to dream up a destabilizing input.

Integrator. If $u = \mathbb{1} \in L_\infty$, then

$$y(t) = \int_{-\infty}^{t} \mathbb{1}(s)\mathrm{d}s = \int_0^t \mathrm{d}s = t\mathbb{1}(t) = \mathrm{ramp}(t)$$

Because ramp $\notin L_\infty$, the integrator is not BIBO stable. If $u = \exp_\lambda \mathbb{1} \in L_2$ for some $\lambda < 0$, then

$$y(t) = \int_{-\infty}^{t} \mathrm{e}^{\lambda s}\mathbb{1}(s)\mathrm{d}s = \int_0^t \mathrm{e}^{\lambda s}\mathrm{d}s = \frac{\mathrm{e}^{\lambda s}}{\lambda}\Big|_0^t = \frac{\mathrm{e}^{\lambda t}-1}{\lambda}\mathbb{1}(t)$$

converges to $-\frac{1}{\lambda} \neq 0$ and thus does not belong to L_2. Thus, the integrator is not L_2-stable either.

Outline

Some standard systems

Principal properties

LTI systems in the time domain

Response of continuous-time LTI systems

Let $G : u \mapsto y$ be LTI. We know (Lect. 2, Slide 15) that $u = \delta * u$, i.e.

$$u(t) = \int_{\mathbb{R}} \delta(t-s)u(s)\mathrm{d}s \quad \Longrightarrow \quad u = \int_{\mathbb{R}} (\mathbb{S}_{-s}\delta)u(s)\mathrm{d}s$$

Passing over convergence-related issues (really delicate), we then have that

$$\begin{aligned} y(t) = (Gu)(t) &= \Bigl(G\Bigl(\int_{\mathbb{R}} (\mathbb{S}_{-s}\delta)u(s)\mathrm{d}s\Bigr)\Bigr)(t) \\ &= \int_{\mathbb{R}} \Bigl(G\bigl(\mathbb{S}_{-s}\delta u(s)\bigr)\Bigr)(t)\mathrm{d}s && \text{by linearity} \\ &= \int_{\mathbb{R}} \bigl(G(\mathbb{S}_{-s}\delta)\bigr)(t)u(s)\mathrm{d}s && \text{by linearity, } u(s) \text{ is constant on } t \text{ axis} \\ &= \int_{\mathbb{R}} \bigl(\mathbb{S}_{-s}(G\delta)\bigr)(t)u(s)\mathrm{d}s && \text{by time invariance} \\ &= \int_{\mathbb{R}} (G\delta)(t-s)u(s)\mathrm{d}s \end{aligned}$$

i.e. the relation $y = Gu$ is equivalent to $y = (G\delta) * u$ in the LTI case.

Response of discrete-time LTI systems

Let $G : u \mapsto y$ be LTI. We know (Lect. 2, Slide 28) that $u = \delta * u$, i.e.

$$u[t] = \sum_{s \in \mathbb{Z}} \delta[t-s]u[s] \quad \Longrightarrow \quad u = \sum_{s \in \mathbb{Z}} (\mathbb{S}_{-s}\delta)u[s]$$

Passing over convergence-related issues (also delicate), we then have that

$$\begin{aligned} y[t] = (Gu)[t] &= \Bigl(G\Bigl(\sum_{s \in \mathbb{Z}} (\mathbb{S}_{-s}\delta)u[s]\Bigr)\Bigr)[t] \\ &= \sum_{s \in \mathbb{Z}} \bigl(G(\mathbb{S}_{-s}\delta u[s])\bigr)[t] && \text{by linearity} \\ &= \sum_{s \in \mathbb{Z}} \bigl(G(\mathbb{S}_{-s}\delta)\bigr)[t]u[s] && \text{by linearity, } u[s] \text{ is constant on } t \text{ axis} \\ &= \sum_{s \in \mathbb{Z}} \bigl(\mathbb{S}_{-s}(G\delta)\bigr)[t]u[s] && \text{by time invariance} \\ &= \sum_{s \in \mathbb{Z}} (G\delta)[t-s]u[s] \end{aligned}$$

i.e. the relation $y = Gu$ is equivalent to $y = (G\delta) * u$ in the LTI case, again.

The impulse response

The signal

$$g := G\delta$$

which is the response of G to the Dirac delta / unit pulse applied at $t = 0$, is known as the impulse response of G. The relation

$$y = Gu \quad \Longleftrightarrow \quad y = g * u$$

implies that the processing of every input signal by an LTI system amounts to convolving that input with the impulse response of the system. Thus,

- properties of LTI systems are fully shaped by their impulse responses.

The impulse response: examples

Gain: if G acts as $y(t) = ku(t)$, then $g = k\delta$ (i.e. $g(t) = k\delta(t)$).

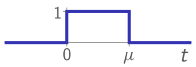
Delay: if G acts as $y(t) = u(t-\tau)$, then $g = \mathbb{S}_{-\tau}\delta$ (i.e. $g(t) = \delta(t-\tau)$).

Integrator: G_{int} acts as (cf. Lect. 2, Slide 16)

$$\left.\begin{aligned} y(t) &= \int_{-\infty}^{t} u(s)\mathrm{d}s \\ y[t] &= \sum_{s=-\infty}^{t} u[s] \end{aligned}\right\} \implies g_{\text{int}} = \mathbb{1} \quad \text{i.e.} \begin{cases} g_{\text{int}}(t) = \mathbb{1}(t) \\ g_{\text{int}}[t] = \mathbb{1}[t] \end{cases}$$

Finite-memory integrator: $G_{\text{fmi},\mu}$ acts as (cf. Lect. 2, Slide 12)

$$y(t) = \int_{t-\mu}^{t} u(s)\mathrm{d}s \implies g_{\text{fmi},\mu} = \mathbb{S}_{-\mu/2}\text{rect}_{\mu} =$$



The impulse response: examples (contd)

First-order system: if G acts as $\dot{y}(t) + a_0 y(t) = b_0 u(t)$ and $u = \delta$, then we may apply the Laplace transform and get

$$sY(s) + a_0 Y(s) = b_0 \implies Y(s) = \frac{b_0}{s + a_0} \implies y(t) = b_0 \mathrm{e}^{-a_0 t}\mathbb{1}(t)$$

i.e. $g_{1o} = b_0 \exp_{-a_0}\mathbb{1}$.

If G acts as $y[t+1] + a_0 y[t] = b_1 u[t+1] + b_0 u[t]$ and $u = \delta$, then we may apply the z transform and get $zY(z) + a_0 Y(z) = b_1 z + b_0$, from which

$$Y(z) = \frac{b_1 z + b_0}{z + a_0} = b_1 + (b_0 - b_1 a_0) z^{-1} \frac{z}{z + a_0}$$

and then, by the linearity and time shift properties

$$y[t] = b_1 + (b_0 - b_1 a_0)(-a_0)^{t-1}\mathbb{1}[t-1]$$

i.e. $g_{1o} = b_1\delta + (b_0 - b_1 a_0)\,\mathbb{S}_{-1}(\exp_{-a_0}\mathbb{1})$.

What if G is not LTI

If G is linear, but not time invariant, then the impulse response g is a signal $\mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{Z}^2 \to \mathbb{R}$, with $g(t,s)$ / $g[t,s]$ being the value of the response at the time instance t to an impulse / pulse applied at the time instance s. In this case

$$y(t) = \int_{\mathbb{R}} g(t,s)u(s)\mathrm{d}s \quad \text{or} \quad y[t] = \sum_{s\in\mathbb{Z}} g[t,s]u[s]$$

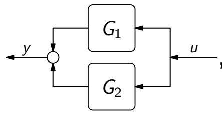
known as the kernel representation (not convolutions anymore). In terms of this impulse response the time invariance reads $g(t,s) = g(t+\tau, s+\tau)$ for all $\tau \in \mathbb{R}$, so that $g(t,s) = g(t-s, 0)$ (the discrete case is alike).

If G is nonlinear, then no universal representation is known.

> *Linear systems are all alike; every nonlinear system is nonlinear in its own way.* could have been coined by L. Tolstoy

System interconnections via impulse responses

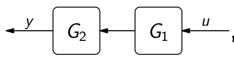
Parallel: If $G =$



, then $g = g_1 + g_2$

follows by $(x + y) * z = x * z + y * z$

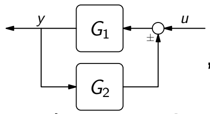
Cascade: If $G =$



, then $g = g_2 * g_1$

follows by $x * (y * z) = (x * y) * z$

Feedback: If $G =$



, then $(\delta \mp g_1 * g_2) * y = g_1 * u$

Resolving that equation in y, aka the *deconvolution* operation, is not trivial in terms of signals and results are not quite intuitive. For instance, is it obvious that $(\delta + \mathbb{1})^{-1} = \delta - \exp_{-1}\mathbb{1}$?

Causality via impulse responses

Theorem

An LTI system G with the impulse response g is causal iff $\operatorname{supp}(g) \subset \mathbb{R}_+$ *in the continuous-time case or* $\operatorname{supp}(g) \subset \mathbb{Z}_+$ *in the discrete-time case.*

Disclaimer. The continuous-time case may be delicate if g includes Dirac deltas. Hereafter we consider that $\operatorname{supp}(\delta) \subset \mathbb{R}_+$, although this statement is mathematically dubious.

Proof (discrete): The response $y = g * u$ reads at every t as

$$y[t] = \sum_{s\in\mathbb{Z}} g[t-s]u[s] = \overbrace{\sum_{s=-\infty}^{t} g[t-s]u[s]}^{\text{past and present}} + \overbrace{\sum_{s=t+1}^{\infty} g[t-s]u[s]}^{\text{future}}$$

Thus, $y[t]$ cannot depend on $u[s]$ for $s > t$ iff $g[t-s] = 0$ for all $s > t$, i.e. $g[t] = 0$ for all $t < 0$. □

If an LTI $G : u \mapsto y$ is causal, then

$$y(t) = \int_{-\infty}^{t} g(t-s)u(s)\mathrm{d}s \quad \text{and} \quad y[t] = \sum_{s=-\infty}^{t} g[t-s]u[s]$$

Stability via impulse responses

Theorem

An LTI system G with the impulse response g is BIBO stable iff $g \in L_1$ in the continuous-time case or $g \in \ell_1$ in the discrete-time case. If G is BIBO stable, then $\|G\|_{\text{ind},\infty} = \|g\|_1$.

Disclaimer. The continuous-time case may be delicate if g includes Dirac deltas here as well. Strictly speaking, δ is not a function. A more rigorous approach would be to limit the class of considered systems to those, whose impulse response could be decomposed in the form $g = \tilde{g} + \sum_i \gamma[i]\,\mathbb{S}_{t_i}\delta$ for some function $\tilde{g}$ and an increasing sequence of time instances $\{t_i\}$. Such systems are BIBO stable iff $\tilde{g} \in L_1$ and $\gamma \in \ell_1$. Nonetheless, we proceed with regarding the Dirac delta as absolutely integrable, merely to simplify the exposition.

Remark 1. The mere decaying of g might not be enough for the BIBO stability of G. For example, if $g(t) = 1/(1+t)\,\mathbb{1}(t)$ and $u = \mathbb{1} \in L_\infty$, then for all $t \geq 0$

$$y(t) = \int_{-\infty}^{t} \frac{1}{1+t-s}\mathbb{1}(t)\mathrm{d}s = \int_{0}^{t} \frac{1}{1+t-s}\mathrm{d}s = -\ln(1+t-s)\big|_0^t = \ln(1+t)$$

and it is not bounded. Hence, this system is not BIBO stable (and indeed, this $g \notin L_1$).

Remark 2. The L_2-stability is not easy to verify directly in terms of the impulse response.

Stability via impulse responses: proof (discrete)

Sufficiency: If $g \in \ell_1$, then for every $u \in \ell_\infty$

$$|y[t]| = \Big|\sum_{s\in\mathbb{Z}} g[t-s]u[s]\Big| \leq \sum_{s\in\mathbb{Z}} |g[t-s]u[s]| \leq \sum_{s\in\mathbb{Z}} |g[t-s]|\,|u[s]|$$
$$\leq \|u\|_\infty \sum_{s\in\mathbb{Z}} |g[t-s]|\Big|_{r=t-s} = \sum_{r\in\mathbb{Z}} |g[r]|\,\|u\|_\infty = \|g\|_1\|u\|_\infty$$

Hence, $\|y\|_\infty \leq \gamma\|u\|_\infty$ for $\gamma = \|g\|_1 < \infty$ and the system is BIBO stable.

Necessity: Choose $u = \operatorname{sign}(\mathbb{P}_{-1}g)$, i.e. $u[t] = \operatorname{sign}(g[-t])$. Its $\|u\|_\infty = 1$ so $u \in \ell_\infty$. With this choice,

$$y[0] = \sum_{s\in\mathbb{Z}} g[-s]u[s] = \sum_{s\in\mathbb{Z}} g[-s]\operatorname{sign}(g[-s]) = \sum_{s\in\mathbb{Z}} |g[-s]| = \sum_{s\in\mathbb{Z}} |g[s]|$$

and $g \in \ell_1$ is necessary for the boundedness of $y[0]$ and, therefore, that of $\|y\|_\infty \geq |y[0]|$. This also proves the $\|G\|_{\text{ind},\infty}$ part. □

Stability via impulse responses: examples

Integrator: $g_{\text{int}} = \mathbb{1} \notin L_1$, with agrees with what we already saw.

Finite-memory integrator: $g_{\text{fmi},\mu} = \mathbb{S}_{-\mu/2}\text{rect}_\mu \in L_1$, i.e. it is BIBO stable, with $\|G_{\text{fmi},\mu}\|_{\text{ind},\infty} = \|g_{\text{fmi},\mu}\|_1 = \mu$

First-order system (in continuous time): $\dot{y}(t) + a_0 y(t) = b_0 u(t)$

$$g_{1\text{o}} = b_0 \exp_{-a_0}\mathbb{1} \in L_1 \iff a_0 > 0$$

i.e. it's BIBO stable iff $a_0 > 0$, with $\|G_{1\text{o}}\|_{\text{ind},\infty} = \frac{1}{a_0}$ (cf. Lect. 2, Slide 19).

First-order system (in discrete time): $y[t+1] + a_0 y[t] = b_1 u[t+1] + b_0 u[t]$

$$g_{1\text{o}} = b_1\delta + (b_0 - b_1 a_0)\,\mathbb{S}_{-1}(\exp_{-a_0}\mathbb{1}) \in \ell_1 \iff |a_0| < 1$$

i.e. it's BIBO stable iff $|a_0| < 1$, with (cf. Lect. 2, Slide 30)

$$\|G_{1\text{o}}\|_{\text{ind},\infty} = |b_1| + \frac{|b_0 - b_1 a_0|}{1 - |a_0|}$$