Linear Systems (034032)
lecture no. 5

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Outline

Background: (rudimentary) complex functions

(Bilateral) Laplace transform

(Bilateral) z transform

Solving differential equations with the Laplace transform

Functions of a complex variable

A complex function F is a mapping $F : \mathbb{C} \to \mathbb{C}$. The complex derivative of F at $s_0 \in \mathbb{C}$ is

$$F'(s_0) = \lim_{s \to s_0} \frac{F(s) - F(s_0)}{s - s_0},$$

where the limit is supposed to be independent of the (complex) path along which s approaches s_0. If the limit above exists, then the function F is said to be complex differentiable at s_0. If $\mathbb{O} \subset \mathbb{C}$ is an *open* set, we say that

– F is holomorphic (analytic) in $\mathbb{O}$ if $F'(s_0)$ exists for every $s_0 \in \mathbb{O}$.

Entire functions are those holomorphic in the whole $\mathbb{C}$. A function is said to be holomorphic at $s_0 \in \mathbb{C}$ if it is holomorphic in *some* neighbourhood of s_0. Every holomorphic function is locally the sum of a convergent power series.

Examples: Functions s^n for $n \in \mathbb{N}$ and e^{as} for $a \in \mathbb{R}$ are entire:

$$(s^n)' = ns^{n-1} \quad \text{and} \quad (e^{as})' = ae^{as}.$$

Function $1/s^n$ for $n \in \mathbb{N}$ is holomorphic in $\mathbb{C} \setminus \{0\}$, as $(1/s^n)' = -n/s^{n+1}$.

Isolated singularities

We say that F has an isolated singularity at a point $s_0 \in \mathbb{O}$, where $\mathbb{O}$ is an open set in $\mathbb{C}$, if F is holomorphic in $\mathbb{O} \setminus \{s_0\}$. Three cases are possible:

1. if F can be defined at s_0 so that the extended function is holomorphic in $\mathbb{O}$ (roughly, if $F(s_0)$ is bounded), then the singularity is removable;
   e.g. $F(s) = (1 - e^{-s})/s$ has a removable singularity at $s = 0$, because $F(0) = 1$
2. if there is $n \in \mathbb{N}$ such that $(s - s_0)^n F(s)$ has a removable singularity at s_0 and $\lim_{s \to s_0} (s - s_0)^n F(s) \neq 0$, then the singularity is a pole and n is its order[1] (or multiplicity);
   e.g. $F(s) = 1/s^2$ has a second-order pole at $s = 0$, because $sF(s) = s/s^2 = 1/s$ has a non-removable singularity at $s = 0$ and $s^2F(s) = s^2/s^2 = 1$ is entire and nonzero at $s = 0$
3. if no finite n as in item 2. exists, then the singularity is called essential.
   e.g. $F(s) = e^{1/s}$ has an essential singularity at $s = 0$

[1] If the order equals 1, then the pole is said to be *simple*.

Rational functions and partial fraction expansion

A function F is said to be rational if it is the quotient of two polynomials,

$$F(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} =: \frac{N(s)}{D(s)}, \quad b_m \neq 0$$

It's called proper / strictly proper / bi-proper if $n \geq m$ / $n > m$ / $n = m$ and non-proper if $n < m$. When all coefficients are real, F is called *real-rational*. Every root of $D(s) = 0$ that is not a root of $N(s) = 0$ is a pole of F. If F is proper, then

$$F(s) = F(\infty) + \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{c_{ij}}{(s - p_i)^j}$$

for distinct poles $p_i \in \mathbb{C}$ of F of order n_i and $c_{ij} \in \mathbb{C}$. This form is known as the partial fraction expansion of F. If p_i is a simple pole, i.e. $n_i = 1$, then

$$c_{i1} = \operatorname{Res}(F, p_i) := \lim_{s \to p_i} (s - p_i)F(s) \neq 0$$

is the residue of F at $s = p_i$.

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Pros and cons of the Fourier transform

The Fourier transform,

$$X(\mathrm{j}\omega) = (\mathfrak{F}\{x\})(\mathrm{j}\omega) := \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$$

decomposes x into elementary harmonic signals via

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega.$$

This

- $\ddot\smile$ offers a valuable insight into properties of x.

However, it

- $\ddot\frown$ applies to only a limited class of signals,

some widely used signals (like $\mathbb{1}$) require Dirac distributions and some (like $\exp_\lambda \mathbb{1}$ for $\lambda > 0$) are not transformable at all.

Definition

The bilateral (two-sided) Laplace transform $\mathfrak{L}\{x\}$ of a signal $x : \mathbb{R} \to \mathbb{F}$ is

$$X(s) = (\mathfrak{L}\{x\})(s) := \int_{\mathbb{R}} x(t)\mathrm{e}^{-st}\mathrm{d}t$$

defined over those $s \in \mathbb{C}$ for which the integral converges. The latter set is known as the region of convergence (RoC) of the transform. If $s = \sigma + \mathrm{j}\omega$, then

$$\int_{\mathbb{R}} x(t)\mathrm{e}^{-st}\mathrm{d}t = \int_{\mathbb{R}} \bigl(x(t)\mathrm{e}^{-\sigma t}\bigr)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$$

is the Fourier transform of $x \exp_{-\sigma}$, so that the condition $x \exp_{-\operatorname{Re} s} \in L_1$ is sufficient for this s to be in the RoC.

If the whole vertical line $\{s \in \mathbb{C} \mid \operatorname{Re} s = \sigma\} \subset \text{RoC}$, then

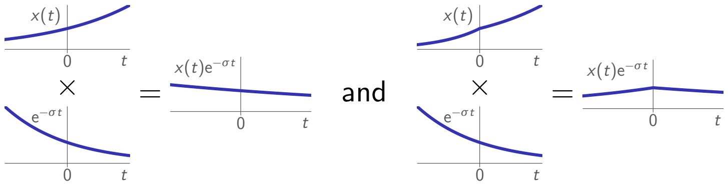
$$x(t) = \frac{1}{\mathrm{j}2\pi}\int_{\sigma + \mathrm{j}\mathbb{R}} X(s)\mathrm{e}^{st}\mathrm{d}s$$

(the inverse Laplace transform).

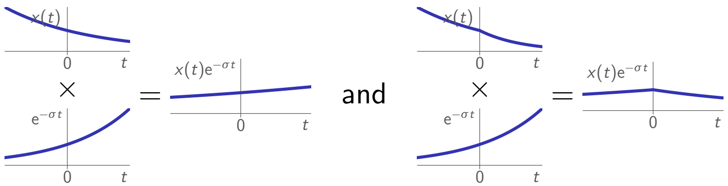
Remarks on RoC

The condition $x \exp_{-\sigma} \in L_1$ is draconian if $\operatorname{supp}(x) = \mathbb{R}$.

If $\sigma > 0$, then $x(t)$ should decay faster than $\mathrm{e}^{-\sigma t}$ grows in $t < 0$ and grow slower than $\mathrm{e}^{-\sigma t}$ decays in $t > 0$:

cf. 

If $\sigma < 0$, then $x(t)$ should decay faster than $\mathrm{e}^{-\sigma t}$ grows in $t > 0$ and grow slower than $\mathrm{e}^{-\sigma t}$ decays in $t < 0$:

cf. 

Remarks on RoC (contd)

Many hurdles are avoided if $\operatorname{supp}(x)$ is limited to finite interval or semi-axis. For instance, if $\operatorname{supp}(x) = \mathbb{R}_+$, then $\exists \alpha_x \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$\underbrace{\{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha_x\}}_{\mathbb{C}_{\alpha_x}} \subset \text{RoC} \quad \text{and} \quad \underbrace{\{s \in \mathbb{C} \mid \operatorname{Re} s < \alpha_x\}}_{\mathbb{C} \setminus \bar{\mathbb{C}}_{\alpha_x}} \cap \text{RoC} = \varnothing$$

(here $\bar{\mathbb{C}}_{\alpha_x}$ stands for the closure of $\mathbb{C}_{\alpha_x}$). Particular cases:

- $\alpha_x = -\infty \implies \text{RoC} = \mathbb{C}$ — e.g. $x(t) = \mathrm{e}^{-t^2} \mathbb{1}(t)$
- $\alpha_x = +\infty \implies \text{RoC} = \varnothing$ — e.g. $x(t) = \mathrm{e}^{t^2} \mathbb{1}(t)$

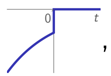
If x is bounded and $\operatorname{supp}(x) = [a, b]$, then $\text{RoC} = \mathbb{C}$.

If the RoC of x is nonempty, then X can be extended beyond its RoC to the whole $\mathbb{C}$ by the *analytic continuation* technique (don't ask what's that) and we treat $X = \mathfrak{L}\{x\}$ as a signal $X : \mathbb{C} \to \mathbb{C}$, which may contain singularities.

Mind the RoC

Example 1: if $x_1 = \exp_{-1} \mathbb{1}$, i.e. $x_1(t) = \mathrm{e}^{-t}\mathbb{1}(t) =$ , then

$$X_1(s) = \int_{\mathbb{R}} x_1(t)\mathrm{e}^{-st}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(s+1)t}\mathrm{d}t = -\frac{\mathrm{e}^{-(s+1)t}}{s+1}\bigg|_0^\infty = \frac{1}{s+1}$$

Example 2: if $x_2 = -\mathbb{P}_{-1}(\exp_1 \mathbb{1})$, i.e. $x_2(t) = -\mathrm{e}^{-t}\mathbb{1}(-t) =$ , then

$$X_2(s) = -\int_{\mathbb{R}} x_2(t)\mathrm{e}^{-st}\mathrm{d}t = -\int_{-\infty}^0 \mathrm{e}^{-(s+1)t}\mathrm{d}t = \frac{\mathrm{e}^{-(s+1)t}}{s+1}\bigg|_{-\infty}^0 = \frac{1}{s+1}$$

The only way to distinguish them is via their RoC's:

- $\text{RoC}_1 = \mathbb{C}_{-1}$ — $\lim_{t\to\infty} \mathrm{e}^{-(s+1)t} = 0 \iff \operatorname{Re}(s+1) > 0$
- $\text{RoC}_2 = \mathbb{C} \setminus \bar{\mathbb{C}}_{-1}$ — $\lim_{t\to-\infty} \mathrm{e}^{-(s+1)t} = 0 \iff \operatorname{Re}(s+1) < 0$

(note that $\text{RoC}_1 \cap \text{RoC}_2 = \varnothing$).

Basic properties

Assuming all involved signals have their support in $\mathbb{R}_+$,

property	time domain	s-domain	RoC
linearity	$x = a_1x_1 + a_2x_2$	$X(s) = a_1X_1(s) + a_2X_2(s)$	$\mathbb{C}_{\alpha_1} \cap \mathbb{C}_{\alpha_2}$
time shift	$y = \mathbb{S}_\tau x$	$Y(s) = \mathrm{e}^{\tau s}X(s)$	$\mathbb{C}_{\alpha_x}$
time scaling	$y = \mathbb{P}_\varsigma x,\ \varsigma > 0$	$Y(s) = \frac{1}{\varsigma}X(\frac{s}{\varsigma})$	$\mathbb{C}_{\varsigma\alpha_x}$
modulation	$y = x\exp_{s_0}$	$Y(s) = X(s - s_0)$	$\mathbb{C}_{\alpha_x + \operatorname{Re} s_0}$
t-modulation	$y = x\,\mathrm{ramp}$	$Y(s) = -\frac{\mathrm{d}}{\mathrm{d}s}X(s)$	$\mathbb{C}_{\alpha_x}$
differentiation[2]	$y = \dot{x}$	$Y(s) = sX(s)$	$\mathbb{C}_{\alpha_x}$
convolution	$z = x * y$	$Z(s) = X(s)Y(s)$	$\mathbb{C}_{\alpha_x} \cap \mathbb{C}_{\alpha_y}$

[2] If $\lim_{t\to\infty} x(t)\mathrm{e}^{-st} = 0$ for all $s \in \text{RoC}$, which is normally the case for $x\exp_{-\operatorname{Re} s} \in L_1$.

Laplace transform of the rectangular pulse

If $x = \text{rect}$, then

$$X(s) = \int_{\mathbb{R}} x(t)\mathrm{e}^{-st}\mathrm{d}t = \int_{-1/2}^{1/2} \mathrm{e}^{-st}\mathrm{d}t = \frac{\mathrm{e}^{-st}}{-s}\bigg|_{-1/2}^{1/2} = \frac{\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}}{s}$$

and the integral converges for all s. Hence, $\text{RoC} = \mathbb{C}$ for this signal.

Remark: Note that the function $(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2})/s$ is entire, which follows by either

$$\lim_{s\to 0} \frac{\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}}{s} = \lim_{s\to 0} \frac{\mathrm{e}^{s/2}/2 + \mathrm{e}^{-s/2}/2}{1} = 1 \qquad \text{or} \qquad \frac{\mathrm{e}^{s/2} - \mathrm{e}^{-s/2}}{s} = \int_{-1/2}^{1/2} \mathrm{e}^{-st}\mathrm{d}t,$$

the first by L'Hôpital's rule, so the singularity of $(\mathrm{e}^{s/2} - \mathrm{e}^{-s/2})/s$ at $s = 0$ is removable.

Consequence:

– by time scaling, if $y = \text{rect}_a$ for some $a > 0$, then

$$Y(s) = (\mathfrak{L}\{\mathbb{P}_{1/a}\text{rect}\})(s) = \frac{\mathrm{e}^{sa/2} - \mathrm{e}^{-sa/2}}{s} = \frac{\sinh(as/2)}{s/2}$$

Laplace transform of the step

If $x = \mathbb{1}$, then

$$X(s) = \int_{\mathbb{R}} x(t)\mathrm{e}^{-st}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}\mathrm{d}t = \lim_{T\to\infty}\int_0^T \mathrm{e}^{-st}\mathrm{d}t = \lim_{T\to\infty}\frac{1-\mathrm{e}^{-sT}}{s}$$

Cases:

– if $\operatorname{Re} s < 0$, then $\lim_{T\to\infty}|\mathrm{e}^{-sT}| = \lim_{T\to\infty}\mathrm{e}^{-(\operatorname{Re} s)T} = \infty$
– if $s = 0$, then $\lim_{s\to 0}\frac{1-\mathrm{e}^{-sT}}{s} = T$ diverge as $T \to \infty$
– if $s = \mathrm{j}\omega$ for $\omega \neq 0$, then $\frac{1-\mathrm{e}^{-\mathrm{j}\omega T}}{\mathrm{j}\omega}$ doesn't converge as $T \to \infty$
– if $\operatorname{Re} s > 0$, then $\lim_{T\to\infty}|\mathrm{e}^{-sT}| = \lim_{T\to\infty}\mathrm{e}^{-(\operatorname{Re} s)T} = 0$

Thus, the integral converges iff $\operatorname{Re} s > 0$ and then

$$X(s) = \frac{1}{s}$$

and $\text{RoC} = \mathbb{C}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. If treated as a signal $\mathbb{C} \to \mathbb{C}$, this X has a singularity (pole) at $s = 0$.

Laplace transform of the step (contd)

Consequences:

– if $y = \mathbb{1} * x$, i.e. $y(t) = \int_{-\infty}^{t} x(t)\mathrm{d}t$, then by the convolution property

$$Y(s) = \frac{X(s)}{s}$$

and its RoC is the intersection of $\mathbb{C}_0$ and the RoC of x.

– if $y = \frac{\text{ramp}^n}{n!}$ for $n \in \mathbb{N}$, i.e. $y(t) = \frac{t^n \mathbb{1}(t)}{n!}$, then by the t-modulation

$$Y(s) = \frac{1}{s^{n+1}}$$

and its RoC is still $\mathbb{C}_0$. For example,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}s}\left(\tfrac{1}{s}\right) = -\tfrac{1}{s^2} \quad &\Longrightarrow \quad \mathfrak{L}\{\mathbb{1}\cdot\text{ramp}\} = \mathfrak{L}\{\text{ramp}\} = \tfrac{1}{s^2} \\ \tfrac{\mathrm{d}}{\mathrm{d}s}\left(\tfrac{1}{s^2}\right) = -\tfrac{2}{s^3} \quad &\Longrightarrow \quad \mathfrak{L}\{\text{ramp}\cdot\text{ramp}\} = \mathfrak{L}\{\text{ramp}^2\} = \tfrac{2!}{s^3} \\ &\vdots \end{aligned}$$

Laplace transform of exponential

If $x = \exp_\lambda \mathbb{1}$ for $\lambda \in \mathbb{C}$, then

$$\begin{aligned} X(s) &= \int_{\mathbb{R}} x(t)\mathrm{e}^{-st}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(s-\lambda)t}\mathrm{d}t = \lim_{T\to\infty}\int_0^T \mathrm{e}^{-(s-\lambda)t}\mathrm{d}t \\ &= \lim_{T\to\infty}\frac{1-\mathrm{e}^{-(s-\lambda)T}}{s-\lambda} \end{aligned}$$

and by already familiar arguments

$$X(s) = \frac{1}{s-\lambda}$$

and $\text{RoC} = \mathbb{C}_{\operatorname{Re}\lambda} := \{s \in \mathbb{C} \mid \operatorname{Re} s > \operatorname{Re}\lambda\}$, where $\operatorname{Re}(s-\lambda) > 0$.

If treated as a signal $\mathbb{C} \to \mathbb{C}$, this X has a singularity (pole) at $s = \lambda$.

Laplace transform of sine wave

If $x(t) = \sin(\omega_x t + \phi)\mathbb{1}(t)$ for $\omega_x, \phi \in \mathbb{R}$, then (see Lect. 3, Slide 4)

$$x(t) = \left(\frac{e^{j(\phi-\pi/2)}}{2}e^{j\omega_x t} + \frac{e^{-j(\phi-\pi/2)}}{2}e^{-j\omega_x t}\right)\mathbb{1}(t)$$

Hence, by linearity and the transform of the exponential,

$$\begin{aligned} X(s) &= \frac{e^{j(\phi-\pi/2)}}{2(s-j\omega_x)} + \frac{e^{-j(\phi-\pi/2)}}{2(s+j\omega_x)} = \frac{-je^{j\phi}}{2(s-j\omega_x)} + \frac{je^{-j\phi}}{2(s+j\omega_x)} \\ &= \frac{-sj(e^{j\phi}-e^{-j\phi}) + \omega_x(e^{j\phi}+e^{-j\phi})}{2(s^2+\omega_x^2)} \\ &= \frac{s\sin\phi + \omega_x\cos\phi}{s^2+\omega_x^2} \end{aligned}$$

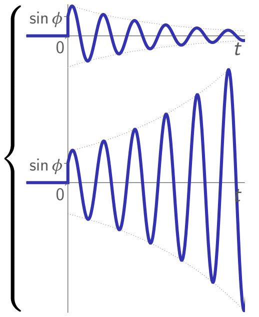
and $\text{RoC} = \mathbb{C}_0$.

If treated as a signal $\mathbb{C} \to \mathbb{C}$, this X has singularities (poles) at $s = \pm j\omega_x$.

Laplace transform of modulated sine wave

If

$$x(t) = e^{\lambda t}\sin(\omega_x t + \phi)\mathbb{1}(t) =$$



for $\omega_x, \phi, \lambda \in \mathbb{R}$, then by modulation

$$X(s) = \frac{(s-\lambda)\sin\phi + \omega_x\cos\phi}{(s-\lambda)^2+\omega_x^2} = \frac{s\sin\phi + \omega_x\cos\phi - \lambda\sin\phi}{s^2+2\lambda s+\omega_x^2+\lambda^2}$$

and $\text{RoC} = \mathbb{C}_\lambda$.

If treated as a signal $\mathbb{C} \to \mathbb{C}$, this X has singularities (poles) at $s = \lambda \pm j\omega_x$.

Laplace transform of the Dirac delta

If $x = \delta$, then

$$X(s) = \int_{\mathbb{R}} \delta(t)e^{-st}dt = e^{-st}|_{t=0} = 1$$

and $\text{RoC} = \mathbb{C}$.

Consequence:

- if $y = \mathbb{S}_\tau\delta$ for $\tau \in \mathbb{R}$, i.e. $y(t) = \delta(t+\tau)$, then

$$Y(s) = e^{\tau s}$$

  by the time shift property.

Unilateral (one-sided) transform

The transform of $x : \mathbb{R} \to \mathbb{F}$ of the form

$$\int_{0^-}^{\infty} x(t)e^{-st}dt$$

is known as its unilateral (or one-sided) Laplace transform. It is, in fact, the bilateral Laplace transform of $x\mathbb{1}$. The bi- and unilateral Laplace transforms coincide if $\text{supp}(x) \subset \mathbb{R}_+$, in which case $x = x\mathbb{1}$.

The properties of the one-sided Laplace transform are similar to those of its two-sided counterpart, with some deviations, like

- if $y(t) = \dot{x}(t)$, then its unilateral transform $Y(s) = sX(s) - x(0^-)$
- time shift is not well defined if only a part of the support is taken into account, as is done in the unilateral transform, requiring some tricks

Fourier vs. Laplace

The relations

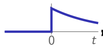
$$(\mathfrak{F}\{x\})(j\omega) = \int_{\mathbb{R}} x(t)e^{-j\omega t}dt \quad \text{and} \quad (\mathfrak{L}\{x\})(s) = \int_{\mathbb{R}} x(t)e^{-st}dt$$

suggest that $\mathfrak{F}\{x\} = \mathfrak{L}\{x\}|_{s=j\omega}$. This is why we use the notation $X(j\omega)$ for the Fourier. However, certain care shall be taken with this relation, it is

- true only if $j\mathbb{R} \subset \mathrm{RoC}$ of the Laplace transform of x,

i.e. only if x is Fourier transformable.

Example (1)

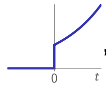
If $x(t) = e^{-t}\mathbb{1}(t) =$ , then $X(s) = 1/(s+1)$. If $Y = X|_{s=j\omega}$, then

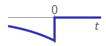
$$Y(j\omega) = \frac{1}{j\omega + 1} \quad \Longrightarrow \quad (\mathfrak{F}^{-1}\{Y\})(t) = e^{-t}\mathbb{1}(t) = x(t)$$

(see Lect. 3, Slide 40), because $j\mathbb{R} \subset \mathrm{RoC} = \mathbb{C}_{-1}$.

Fourier vs. Laplace (contd)

Example (2)

If $x(t) = e^{t}\mathbb{1}(t) =$ , then $X(s) = 1/(s-1)$. But $Y = X|_{s=j\omega}$ has

$$Y(j\omega) = \frac{1}{j\omega - 1} \quad \Longrightarrow \quad (\mathfrak{F}^{-1}\{Y\})(t) = -e^{t}\mathbb{1}(-t) =$$  $$\neq x(t)$$

because $j\mathbb{R} \not\subset \mathrm{RoC} = \mathbb{C}_1$.

Example (3)

If $x(t) = \mathbb{1}(t)$, then $X(s) = 1/s$. If $Y = X|_{s=j\omega}$, then

$$Y(j\omega) = \frac{1}{j\omega} \neq \frac{1}{j\omega} + \pi\delta(\omega) = X(j\omega)$$

(see Lect. 3, Slide 41), because $j\mathbb{R} \not\subset \mathrm{RoC} = \mathbb{C}_0$.

In Mathematica: `InverseFourierTransform` with `FourierParameters->{1,-1}`.

The final value theorem

Theorem

If $x : \mathbb{R} \to \mathbb{F}$ with $\mathrm{supp}(x) \subset \mathbb{R}_+$ is converging, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \mathrm{Res}(X, 0).$$

Example

Let $x(t) = e^{\lambda t}\sin(\omega_x t + \phi)\mathbb{1}(t)$, for which

$$X(s) = \frac{s\sin\phi + \omega_x\cos\phi - \lambda\sin\phi}{s^2 + 2\lambda s + \omega_x^2 + \lambda^2}$$

In this case

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s(s\sin\phi + \omega_x\cos\phi - \lambda\sin\phi)}{s^2 + 2\lambda s + \omega_x^2 + \lambda^2} = 0$$

equals $\lim_{t\to\infty} x(t)$ only if $\lambda < 0$, otherwise this 0 makes no sense.

The initial value theorem

Theorem

If $x : \mathbb{R} \to \mathbb{F}$ with $\mathrm{supp}(x) \subset \mathbb{R}_+$ is such that $x(0^+)$ exists, then

$$\lim_{t\to 0} x(t) = \lim_{s\in\mathbb{R}, s\to\infty} sX(s).$$

Example

Let $x(t) = e^{\lambda t}\sin(\omega_x t + \phi)\mathbb{1}(t)$, for which

$$X(s) = \frac{s\sin\phi + \omega_x\cos\phi - \lambda\sin\phi}{s^2 + 2\lambda s + \omega_x^2 + \lambda^2}$$

In this case

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{s^2\sin\phi + s(\omega_x\cos\phi - \lambda\sin\phi)}{s^2 + 2\lambda s + \omega_x^2 + \lambda^2} = \sin\phi = x(0),$$

indeed.

Outline

Background: (rudimentary) complex functions

(Bilateral) Laplace transform

(Bilateral) z transform

Solving differential equations with the Laplace transform

Definition

The bilateral (two-sided) z-transform $\mathfrak{Z}\{x\}$ of a signal $x : \mathbb{Z} \to \mathbb{F}$ is

$$X(z) = (\mathfrak{Z}\{x\})(z) := \sum_{t\in\mathbb{Z}} x[t]z^{-t}$$

defined over those $z \in \mathbb{C}$ for which the sum converges (again, the RoC). If $z = \gamma e^{j\theta}$, then

$$\sum_{t\in\mathbb{Z}} x[t]z^{-t} = \sum_{t\in\mathbb{Z}} \bigl(x[t]\gamma^{-t}\bigr)e^{-j\theta t}$$

is the DTFT of $x\exp_{1/\gamma}$ and the condition $x\exp_{1/|z|} \in \ell_1$ ensures that this z is in the RoC. Like in the Laplace transform case, we mostly z-transform signals x with $\operatorname{supp}(x) \subset \mathbb{Z}_+$. For such signals $\exists \alpha_x \in \mathbb{R}_+ \cup \{\infty\}$ such that $\{z \in \mathbb{C} \mid |z| > \alpha_x\} \subset \text{RoC}$ and $\{z \in \mathbb{C} \mid |z| < \alpha_x\} \cap \text{RoC} = \emptyset$.

If the RoC of x is nonempty, then X can be extended beyond its RoC to the whole $\mathbb{C}$ by the *analytic continuation* technique and we treat $X = \mathfrak{Z}\{x\}$ as a signal $X : \mathbb{C} \to \mathbb{C}$, which may contain singularities.

Basic properties

Assuming all involved signals have their support in $\mathbb{Z}_+$,

property	time domain	z-domain	RoC
linearity	$x = a_1x_1 + a_2x_2$	$X(z) = a_1X_1(z) + a_2X_2(z)$	$\cap$
time shift	$y = \mathbb{S}_\tau x$	$Y(z) = z^\tau X(z)$	RoC_x
modulation	$y = x\exp_\lambda$	$Y(z) = X(z/\lambda)$	$\lvert\lambda\rvert\text{RoC}_x$
t-modulation	$y = x\,\mathrm{ramp}$	$Y(z) = -z\frac{d}{dz}X(z)$	RoC_x
convolution	$w = x * y$	$W(z) = X(z)Y(z)$	$\cap$

z-transform of the pulse

If $x = \delta$, then

$$X(z) = \sum_{t\in\mathbb{Z}} \delta[t]z^{-t} = z^0 = 1$$

and $\text{RoC} = \mathbb{C}$.

Consequence:

– if $y = \mathbb{S}_\tau\delta$ for $\tau \in \mathbb{Z}$, i.e. $y[t] = \delta[t+\tau]$, then

$$Y(z) = z^\tau$$

by the time shift property.

z-transform of the step

If $x = \mathbb{1}$, then

$$X(z) = \sum_{t\in\mathbb{Z}} x[t]z^{-t} = \sum_{t\in\mathbb{Z}_+} z^{-t} = \lim_{T\to\infty} \sum_{t=0}^{T} z^{-t} = \lim_{T\to\infty} \frac{z - z^{-T}}{z-1}$$

Cases:

- if $|z| < 1$, then $\lim_{T\to\infty}|z^{-T}| = \lim_{T\to\infty}|z|^{-T} = \infty$
- if $z = 1$, then $\lim_{z\to 1} \frac{z-z^{-T}}{z-1} = T+1$ diverge as $T\to\infty$
- if $z = \mathrm{e}^{\mathrm{j}\theta}$ for $\theta \neq 0$, then $\mathrm{e}^{-\mathrm{j}\theta T}$ doesn't converge as $T\to\infty$
- if $|z| > 1$, then $\lim_{T\to\infty}|z^{-T}| = \lim_{T\to\infty}|z|^{-T} = 0$

Thus, the sum converges iff $|z| > 1$ and then

$$X(z) = \frac{z}{z-1}$$

and $\mathrm{RoC} = \{z\in\mathbb{C} \mid |z| > 1\}$. If treated as a signal $\mathbb{C}\to\mathbb{C}$, this X has a singularity (pole) at $z = 1$.

z-transform of exponential

If $x = \exp_\lambda \mathbb{1}$ for $\lambda\in\mathbb{C}$, then

$$X(z) = \sum_{t\in\mathbb{Z}} x[t]z^{-t} = \sum_{t\in\mathbb{Z}_+} \lambda^t z^{-t} = \lim_{T\to\infty}\sum_{t=0}^{T}\left(\frac{z}{\lambda}\right)^{-t}$$
$$= \lim_{T\to\infty}\frac{z/\lambda - (z/\lambda)^{-T}}{z/\lambda - 1} = \lim_{T\to\infty}\frac{z - \lambda(z/\lambda)^{-T}}{z-\lambda}$$

and by already familiar arguments

$$X(z) = \frac{z}{z-\lambda}$$

and $\mathrm{RoC} = \{z\in\mathbb{C} \mid |z| > |\lambda|\}$, where $|z/\lambda| > 1$.

If treated as a signal $\mathbb{C}\to\mathbb{C}$, this X has a singularity (pole) at $z = \lambda$.

z-transform of sine wave

If $x[t] = \sin[\theta_x t + \phi]\mathbb{1}[t]$ for $\theta_x, \phi\in\mathbb{R}$, then (see Lect. 3, Slide 4)

$$x[t] = \left(\frac{\mathrm{e}^{\mathrm{j}(\phi-\pi/2)}}{2}(\mathrm{e}^{\mathrm{j}\theta_x})^t + \frac{\mathrm{e}^{-\mathrm{j}(\phi-\pi/2)}}{2}(\mathrm{e}^{-\mathrm{j}\theta_x})^t\right)\mathbb{1}[t]$$

Hence, by linearity and the transform of the exponential,

$$X(z) = \frac{\mathrm{e}^{\mathrm{j}(\phi-\pi/2)}}{2(z-\mathrm{e}^{\mathrm{j}\theta_x})} + \frac{\mathrm{e}^{-\mathrm{j}(\phi-\pi/2)}}{2(z-\mathrm{e}^{-\mathrm{j}\theta_x})} = \frac{-\mathrm{j}\mathrm{e}^{\mathrm{j}\phi}z}{2(z-\mathrm{e}^{\mathrm{j}\theta_x})} + \frac{\mathrm{j}\mathrm{e}^{-\mathrm{j}\phi}z}{2(z-\mathrm{e}^{-\mathrm{j}\theta_x})}$$
$$= \frac{z(z\sin\phi + \sin(\theta_x-\phi))}{z^2 - 2\cos\theta_x z + 1}$$

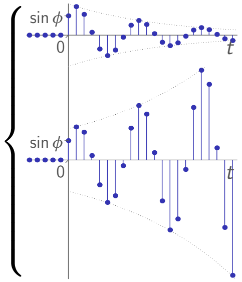
and $\mathrm{RoC} = \{z\in\mathbb{C} \mid |z| > 1\}$.

If treated as a signal $\mathbb{C}\to\mathbb{C}$, this X has singularities (poles) at $z = \mathrm{e}^{\pm\mathrm{j}\theta_x}$.

z-transform of modulated sine wave

If

$$x[t] = \lambda^t\sin[\theta_x t + \phi]\mathbb{1}[t] = \begin{cases} \text{(plot)} & \text{if } |\lambda| < 1 \\ \text{(plot)} & \text{if } |\lambda| > 1 \end{cases}$$



for $\theta_x, \phi, \lambda\in\mathbb{R}$, then by modulation

$$X(z) = \frac{z(z\sin\phi + \lambda\sin(\theta_x-\phi))}{z^2 - 2\lambda\cos\theta_x\, z + \lambda^2}$$

and $\mathrm{RoC} = \{z\in\mathbb{C} \mid |z| > |\lambda|\}$.

If treated as a signal $\mathbb{C}\to\mathbb{C}$, this X has singularities (poles) at $z = \lambda\mathrm{e}^{\pm\mathrm{j}\theta_x}$.

The final and initial value theorems

Theorem

If $x : \mathbb{Z} \to \mathbb{F}$ *with* $\operatorname{supp}(x) \subset \mathbb{Z}_+$ *is converging, then*

$$\lim_{t\to\infty} x[t] = \lim_{z\to 1}(z-1)X(z) = \operatorname{Res}(X,1)$$

Theorem

If $x : \mathbb{Z} \to \mathbb{F}$ *with* $\operatorname{supp}(x) \subset \mathbb{Z}_+$ *is such that* $x[0]$ *exists, then*

$$x[0] = \lim_{z\in\mathbb{R}, z\to\infty} X(z).$$

Proof: follows directly from $X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$ □

Outline

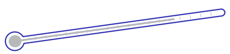
Background: (rudimentary) complex functions

(Bilateral) Laplace transform

(Bilateral) z transform

Solving differential equations with the Laplace transform

Thermometer model



A mercury thermometer is a system mapping the ambient temperature into the temperature of the mercury inside it (which, in turn, changes its volume as a result). Assuming that

- the heat transfer coefficient is constant and
- there is no thermal radiation in heat transfer

the Newton's law of cooling yields that the mercury temperature θ satisfies

$$\tau\dot{\theta}(t) = \theta_{\text{amb}}(t) - \theta(t),$$

where θ_{amb} is the ambient temperature and $\tau > 0$ is a parameter dependent on the thermal properties of the thermometer. Assuming that $\theta(t) = \theta_0$ for all $t < 0$, the model in terms of deviations from θ_0 is

$$\tau\dot{\tilde{\theta}}(t) + \tilde{\theta}(t) = \tilde{\theta}_{\text{amb}}(t),$$

where $\tilde{\theta}(t) = \theta(t) - \theta_0$ and $\tilde{\theta}_{\text{amb}}(t) = \theta_{\text{amb}}(t) - \theta_0$ with $\operatorname{supp}(\tilde{\theta}) = \mathbb{R}_+$.

Thermometer model in the s-domain

By the linearity and differentiation properties of the Laplace transform, the model in the s-domain reads as the *algebraic* relation

$$\tau s\tilde{\Theta}(s) + \tilde{\Theta}(s) = \tilde{\Theta}_{\text{amb}}(s) \quad\Longleftrightarrow\quad \tilde{\Theta}(s) = \frac{\tilde{\Theta}_{\text{amb}}(s)}{\tau s + 1}.$$

Solution sequence:

1. partial fraction expansion of $\tilde{\Theta}$ with real rational elements
2. inverse Laplace transform of each simple fractions

   partial fractions are either first- or second-order real-rational elements, whose inverse Laplace transforms we already know; if a pole is not simple, then the t-modulation property of the Laplace transform shall be used

Thermometer model: solution for the step input

If $\tilde{\theta}_{\mathrm{amb}} = \tilde{\theta}_1 \mathbb{1}$ for some $\tilde{\theta}_1 \in \mathbb{R}$, then

$$\tilde{\Theta}(s) = \frac{\tilde{\theta}_1}{s(\tau s + 1)} = \frac{\operatorname{Res}(\tilde{\Theta}, 0)}{s} + \frac{\operatorname{Res}(\tilde{\Theta}, -1/\tau)}{s + 1/\tau},$$

because both poles are simple. Residues are

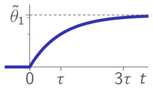
$$\operatorname{Res}(\tilde{\Theta}, 0) = \lim_{s\to 0} s\tilde{\Theta}(s) = \lim_{s \to 0} \frac{\tilde{\theta}_1}{\tau s + 1} = \tilde{\theta}_1$$

and

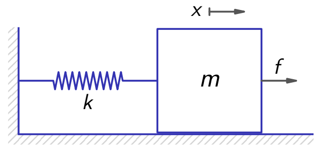
$$\operatorname{Res}(\tilde{\Theta}, -1/\tau) = \lim_{s \to -1/\tau} (s + 1/\tau)\tilde{\Theta}(s) = \lim_{s \to -1/\tau} \frac{\tilde{\theta}_1}{\tau s} = -\tilde{\theta}_1$$

Thus,

$$\tilde{\Theta}(s) = \frac{\tilde{\theta}_1}{s} - \frac{\tilde{\theta}_1}{s + 1/\tau} \quad \Longrightarrow \quad \tilde{\theta}(t) = \tilde{\theta}_1(1 - \mathrm{e}^{-t/\tau})\mathbb{1}(t) =$$



Mass-spring model



Assumptions:

- spring force is proportional to position differences (Hooke's law)
- we may *neglect* friction, force misalignment, etc

Supposing zero spring force at $x = 0$, by Newton's second law

$$m\ddot{x}(t) = f(t) + f_{\mathrm{spring}}(t) = f(t) - kx(t)$$

or, equivalently,

$$m\ddot{x}(t) + kx(t) = f(t).$$

Mass-spring model in the s domain

By the linearity and differentiation properties of the Laplace transform, the model in the s-domain reads as the *algebraic* relation

$$ms^2X(s) + kX(s) = F(s) \quad \iff \quad X(s) = \frac{F(s)}{ms^2 + k}.$$

The denominator polynomial has roots at

$$s_{1,2} = \pm \mathrm{j}\sqrt{\frac{k}{m}}$$

which are pure imaginary.

Mass-spring model: solution for the step input

If $f = f_1 \mathbb{1}$ for some $f_1 \in \mathbb{R}$, then

$$X(s) = \frac{f_1}{s(ms^2 + k)} = \frac{\operatorname{Res}(X, 0)}{s} + \overbrace{\left(\frac{f_1}{s(ms^2 + k)} - \frac{\operatorname{Res}(X, 0)}{s}\right)}^{\text{removable singularity at } s = 0},$$

(to avoid dealing with complex poles and residues). The residue at $s = 0$ is

$$\operatorname{Res}(X, 0) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{f_1}{ms^2 + k} = \frac{f_1}{k},$$

so that

$$X(s) = \frac{f_1}{ks} + \left(\frac{f_1}{s(ms^2 + k)} - \frac{f_1}{ks}\right) = \frac{f_1}{k}\left(\frac{1}{s} - \frac{s}{s^2 + k/m}\right)$$

Hence,

$$x(t) = \frac{f_1}{k}\left(1 - \cos\left(\sqrt{k/m}\, t\right)\right) \mathbb{1}(t) =$$