Linear Systems (034032)
lecture no. 4

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Outline

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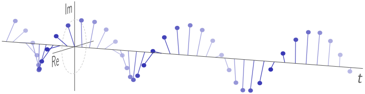
Discrete harmonic signal

Signal $\alpha = a \exp_{e^{j\theta}} : \mathbb{Z} \to \mathbb{C}$,

$$\alpha[t] = a e^{j\theta t} = |a| e^{j(\theta t + \arg(a))} =$$



for $a \in \mathbb{C}$ and $\theta \in \mathbb{R}$ is called harmonic signal with frequency θ, amplitude $|a|$, and initial phase $\arg(a)$. Given $T \in \mathbb{N}$,

$$\alpha[t+T] = a e^{j\theta(t+T)} = a e^{j\theta t + j\theta T} = a e^{j\theta t} e^{j\theta T} = \alpha[t]\, e^{j\theta T}$$

so that

$$\alpha[t+T] = \alpha[t] \iff e^{j\theta T} = 1 \iff |\theta| T = 2\pi k \quad \text{for } k \in \mathbb{N}$$

and this condition is independent of t. Therefore,

- α is T-periodic iff $\exists k \in \mathbb{N}$ such that $|\theta| = \dfrac{2\pi k}{T}$,

which happens iff $2\pi/|\theta| \in \mathbb{Q}$.

Discrete harmonic signal (contd)

Because

$$e^{j(\theta+2\pi)t} = e^{j\theta t} e^{j2\pi t} = e^{j\theta t},$$

shifting θ by a multiple of 2π does *not* change the harmonic signal. Hence,

– discrete frequencies only make sense on intervals of the length 2π.

Thereafter we use the convention that $\theta \in [-\pi, \pi]$, which is a choice.

The normalized change at each step of the harmonic signal,

$$\frac{|\alpha[t+1]-\alpha[t]|}{|\alpha[t]|} = \left|\frac{a e^{j\theta(t+1)}}{a e^{j\theta t}} - 1\right| = |e^{j\theta} - 1| = 2\left|\sin\frac{\theta}{2}\right|,$$

is independent of t and increases with $|\theta|$ (provided $\theta \in [-\pi, \pi]$, of course). It is thus still justifiable to say that

– $a_1 \exp_{e^{j\theta_1}}$ is faster (slower) than $a_2 \exp_{e^{j\theta_2}}$ if $|\theta_1| > |\theta_2|$ ($|\theta_1| < |\theta_2|$).

The fastest harmonic signal is that for $\theta = \pm\pi$, for which $\alpha[t] = a(-1)^t$.

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Discrete-time Fourier transform

The discrete-time Fourier transform (DTFT) of a discrete signal $x : \mathbb{Z} \to \mathbb{R}$ is the signal $X : [-\pi, \pi] \to \mathbb{C}$ such that

$$X(\mathrm{e}^{\mathrm{j}\theta}) = (\mathfrak{F}\{x\})(\mathrm{e}^{\mathrm{j}\theta}) := \sum_{t \in \mathbb{Z}} x[t]\mathrm{e}^{-\mathrm{j}\theta t}$$

where θ is called the angular frequency (in radians per step). The DTFT is well defined (uniform convergence, continuous X) if $x \in \ell_1$ and then

$$x[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{j}\theta})\mathrm{e}^{\mathrm{j}\theta t}\mathrm{d}\theta =: (\mathfrak{F}^{-1}\{X\})(\mathrm{e}^{\mathrm{j}\theta})$$

If the domain of X is extended to $\mathbb{R}$, then X is a 2π-periodic function of θ.

If $x \in \ell_2$, then the DTFT sum might converge only in the (weaker) ℓ_2-norm sense, so that X is defined almost everywhere and might not be continuous or even bounded. The use of distributions facilitates extending the DTFT to yet wider classes of signals.

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Meaning

Similarly to the continuous-time case, the relation

$$x[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{j}\theta}) \mathrm{e}^{\mathrm{j}\theta t} \mathrm{d}\theta$$

means that

- x is a superposition of elementary harmonics $\frac{1}{2\pi} X(\mathrm{e}^{\mathrm{j}\theta})\,\exp_{\mathrm{e}^{\mathrm{j}\theta}}$,

with a continuum of frequencies θ, although now in the final range $[-\pi, \pi]$.

The signal X is then called the spectrum or frequency-domain representation of x, with the amplitude spectrum $|X|$ and the phase spectrum $\arg(X)$.

The value $X(\mathrm{e}^{\mathrm{j}0}) = X(1)$ is the average of x over all its domain, i.e. $\mathbb{Z}$.

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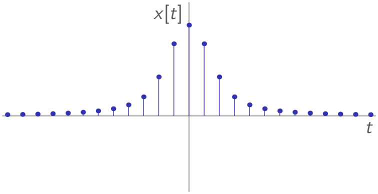
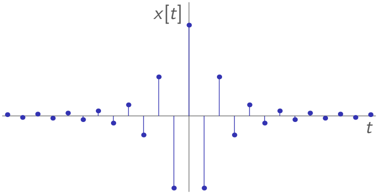
Example

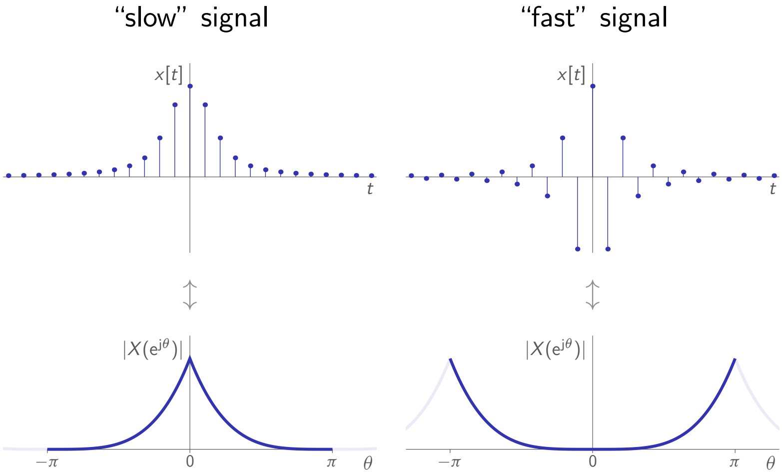
“slow” signal



“fast” signal



Example



Basic properties

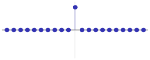
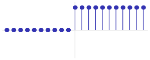
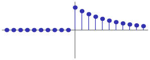
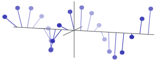
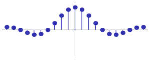
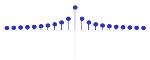
property	time domain	frequency domain
linearity	$x = a_1 x_1 + a_2 x_2$	$X(e^{j\theta}) = a_1 X_1(e^{j\theta}) + a_2 X_2(e^{j\theta})$
time shift	$y = \mathbb{S}_\tau x$	$Y(e^{j\theta}) = e^{j\theta\tau} X(e^{j\theta})$
time reversal	$y = \mathbb{P}_{-1} x$	$Y(e^{j\theta}) = X(e^{-j\theta})$
conjugation	$y = \overline{x}$	$Y(e^{j\theta}) = \overline{X(e^{-j\theta})}$
modulation	$y = x \exp_{e^{j\theta_0}}$	$Y(e^{j\theta}) = X(e^{j(\theta-\theta_0)})$
convolution	$z = x * y$	$Z(e^{j\theta}) = X(e^{j\theta}) Y(e^{j\theta})$

Parseval: if $x \in \ell_2$, then

$$\|x\|_2^2 = \sum_{t \in \mathbb{Z}} |x[t]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\theta})|^2 d\theta = \frac{1}{2\pi} \|X\|_2^2$$

Remember, $(\mathbb{S}_\tau x)[t] = x[t+\tau]$ and $(\mathbb{P}_{-1} x)[t] = x[-t]$.

DTFTs of some discrete signals

$x[t]$	$X(e^{j\theta}),\ \theta \in [-\pi, \pi]$	condition
$\delta[t] =$ 	1	
$\mathbb{1}[t] =$ 	$\dfrac{1}{1 - e^{-j\theta}} + \pi\delta(\theta)$	
$\lambda^t \mathbb{1}[t] =$ 	$\dfrac{1}{1 - \lambda e^{-j\theta}}$	$\lvert\lambda\rvert < 1$
$e^{j\theta_0 t} =$ 	$2\pi\ \mathrm{rect}_{2\pi}(\theta) \sum_{i \in \mathbb{Z}} \delta(\theta - \theta_0 - 2\pi i)$	$\lvert\theta_0\rvert \leq \pi$
$\mathrm{sinc}[\pi\eta t] =$ 	$\dfrac{1}{\eta}\mathrm{rect}_{2\pi\eta}[\theta]$	$0 < \eta \leq 1$
$\dfrac{1}{1 + \lvert t\rvert} =$ 	$-\ln(2 - 2\cos\theta)\cos\theta + (\pi - \lvert\theta\rvert)\sin\lvert\theta\rvert - 1$	

Discrete Fourier transform

The discrete counterpart of the Fourier series is called the discrete Fourier transform (DFT). Namely, if $x : \mathbb{Z} \to \mathbb{R}$ is T-periodic, then at every t

$$x[t] = \frac{1}{T} \sum_{k=0}^{T-1} X[k] \mathrm{e}^{\mathrm{j}2\pi t\, k/T},$$

where

$$X[k] = \sum_{t=0}^{T-1} x[t] \mathrm{e}^{-\mathrm{j}2\pi k\, t/T}, \quad k \in \mathbb{Z}_{0..T-1}$$

Note that each harmonic in the DFT, $\mathrm{e}^{\mathrm{j}2\pi t\, k/T}$, is a T-periodic function of t. The DFT plays an important role in

- numerical algorithms (including numerical Fourier transforms, via FFT)
- image processing
- ...

but is less prominent in the context of linear systems studied in this course.

Outline

Discrete-time frequency-domain analysis

A/D conversion in the frequency domain

D/A conversion in the frequency domain

The Sampling Theorem

The ideal sampler: what do we know

It maps continuous-time signals $x : \mathbb{R} \to \mathbb{R}$ to discrete signals $\bar{x} : \mathbb{Z} \to \mathbb{R}$ as

$$\bar{x}[i] = x(ih), \quad \forall i$$

for a given sampling period $h > 0$ (we assume periodic sampling hereafter). We may also think in terms of the sampling frequency $\omega_s := 2\pi/h$ (radians per time unit).

Sometimes it's convenient to think of $\bar{x}$ not over the abstract $\mathbb{Z}$, but rather over $\{\ldots, -2h, -h, 0, h, 2h, \ldots\}$, i.e. synchronized with the analog time t:

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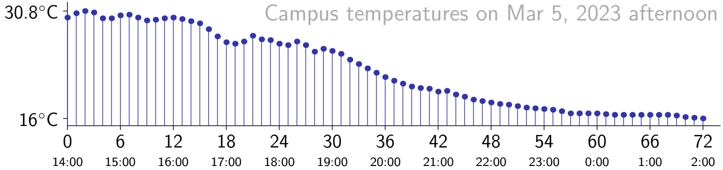
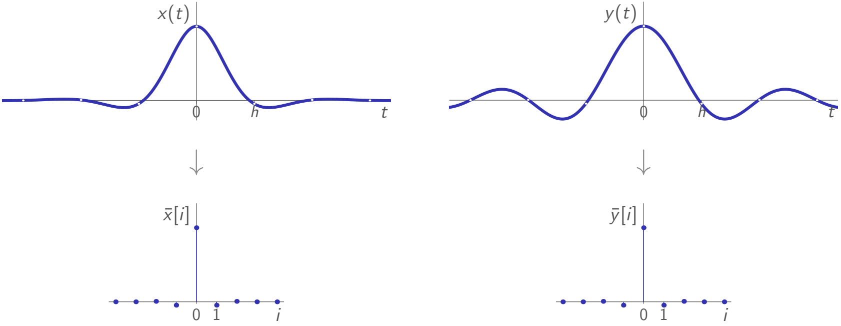


Sampling: a key question



The question of

- what is lost in transforming the domain $\mathbb{R}$ to the domain $\mathbb{Z}$

might not be straightforward to address in the time domain. Is it apparent that sampling x is problematic, whereas in sampling y we loose nothing for the very same sampling period h?

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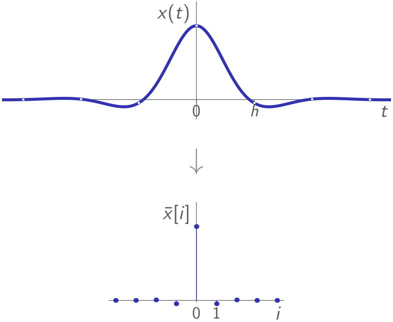

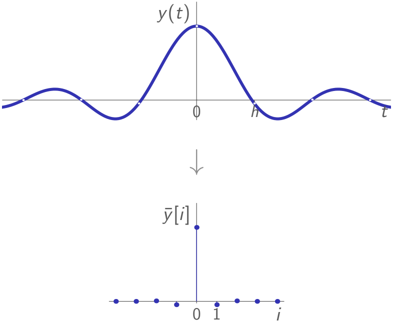

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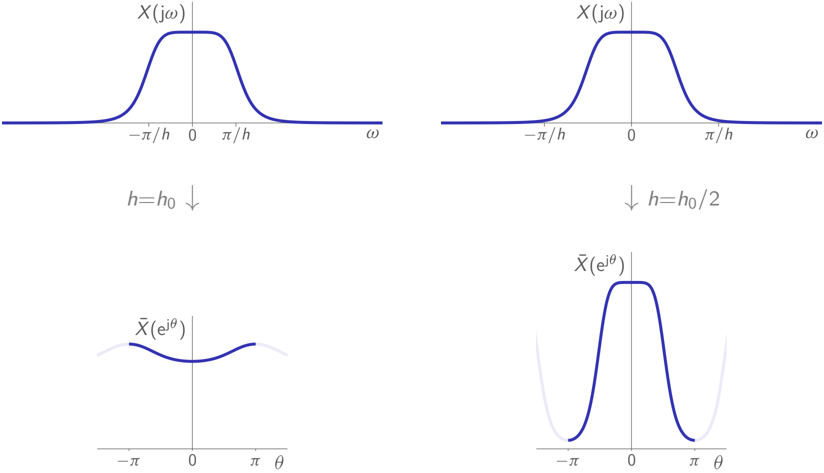
Sampling: a key question (contd)



Let's try to understand

– what is lost in transforming (squeezing) the domain $\mathbb{R}$ to $[-\pi, \pi]$.

Periodic summation

Consider a function $x : \mathbb{R} \to \mathbb{F}$, its periodic summation with period T is

$$x_T := \sum_{i \in \mathbb{Z}} \mathbb{S}_{iT} x \quad \Longrightarrow \quad x_T(t) = \sum_{i \in \mathbb{Z}} x(t + iT)$$

The function $x_T : \mathbb{R} \to \mathbb{F}$ is T-periodic, because

$$x_T(t + T) = \sum_{i \in \mathbb{Z}} x(t + iT + T)\Big|_{j=i+1} = \sum_{j \in \mathbb{Z}} x(t + jT) = x_T(t)$$

Example: if $x =$ tent, then

$$x_T = \begin{cases} & \text{if } T = \frac{4}{3} \\ & \text{if } T = \frac{5}{3} \end{cases}$$

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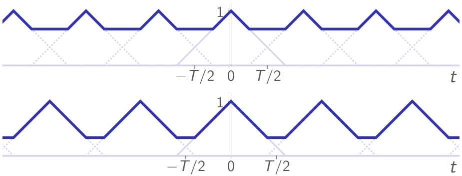

if $T = \frac{4}{3}$

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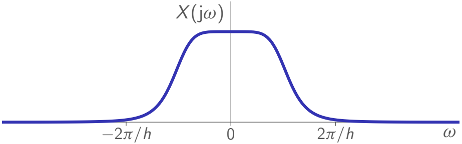
Periodic summation of continuous-time spectra

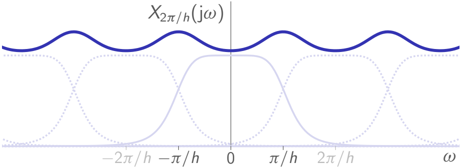
Let x be a continuous-time signal with the frequency response X, say

$$X(\mathrm{j}\omega) =$$



and consider its periodic summation with the period $T = 2\pi/h$,

$$X_{2\pi/h}(\mathrm{j}\omega) = \sum_{i\in\mathbb{Z}} X(\mathrm{j}(\omega + \tfrac{2\pi}{h} i)) =$$



Because this function is periodic, it can be expanded into the Fourier series $X_{2\pi/h}(\mathrm{j}\omega) = \sum_{k\in\mathbb{Z}} X_{2\pi/h}[k]\mathrm{e}^{\mathrm{j}hk\omega}$ (the fundamental frequency is $\omega_0 = h$).

Fourier coefficients of $X_{2\pi/h}(\mathrm{j}\omega)$

By the definition of Fourier coefficient and taking into account that $\omega_0 = h$,

$$
\begin{aligned}
X_{2\pi/h}[k] &= \frac{1}{2\pi/h}\int_{-\pi/h}^{\pi/h} X_{2\pi/h}(\omega)\mathrm{e}^{-\mathrm{j}hk\omega}\,\mathrm{d}\omega \\
&= \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}\sum_{i\in\mathbb{Z}} X(\mathrm{j}(\omega + \tfrac{2\pi}{h}i))\mathrm{e}^{-\mathrm{j}kh\omega}\,\mathrm{d}\omega\, \underbrace{\mathrm{e}^{-\mathrm{j}2\pi ki}}_{=1} \\
&= \frac{h}{2\pi}\sum_{i\in\mathbb{Z}}\int_{-\pi/h}^{\pi/h} X(\mathrm{j}(\omega + \tfrac{2\pi}{h}i))\mathrm{e}^{-\mathrm{j}kh(\omega+2\pi i/h)}\,\mathrm{d}\omega\Big|_{\eta=\omega+2\pi i/h} \\
&= \frac{h}{2\pi}\sum_{i\in\mathbb{Z}}\int_{(2i-1)\pi/h}^{(2i+1)\pi/h} X(\mathrm{j}\eta)\mathrm{e}^{-\mathrm{j}\eta kh}\,\mathrm{d}\eta\Big|_{\omega=\eta} \\
&= \frac{h}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega(-kh)}\,\mathrm{d}\omega \qquad \text{cf. } x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega \\
&= hx(-kh)
\end{aligned}
$$

Fourier series of $X_{2\pi/h}(j\omega)$

Thus

$$X_{2\pi/h}(j\omega) = \sum_{k\in\mathbb{Z}} hx(-kh)e^{jhk\omega}\Big|_{i=-k} = h\sum_{i\in\mathbb{Z}} x(ih)e^{-j(h\omega)i}$$

At the same time, the DTFT of $\bar{x}$ such that if $\bar{x}[i] = x(ih)$ satisfies

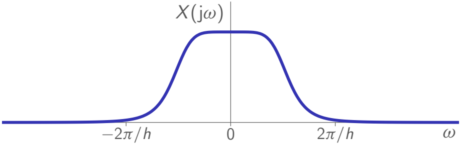
$$\bar{X}(e^{j\theta}) = \sum_{i\in\mathbb{Z}} \bar{x}[i]e^{-j\theta i} = \sum_{i\in\mathbb{Z}} x(ih)e^{-j\theta i}$$

Hence, the DTFT of the sampled signal $\bar{x}$

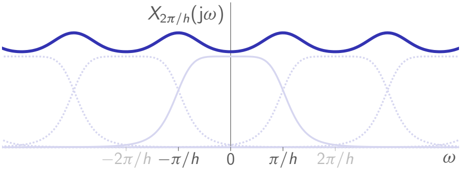
$$\bar{X} = \frac{1}{h}\mathbb{P}_{1/h}X_{2\pi/h} \quad \Longrightarrow \quad \bar{X}(e^{j\theta}) = \frac{1}{h}X_{2\pi/h}(j\tfrac{\theta}{h}) = \frac{1}{h}\sum_{i\in\mathbb{Z}} X(j\tfrac{\theta+2\pi i}{h})$$

i.e. it's the periodic summation, whose period equals the sampling frequency ω_s, of the spectrum of the continuous-time x, scaled by the factor $1/h$ both in amplitude and in frequency.

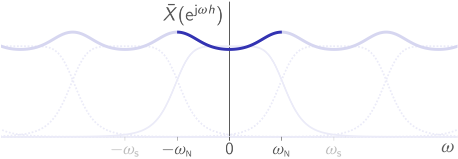
Spectrum of sampled signal



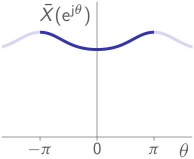
$\downarrow$ periodic summation



$\downarrow$ scaling by $\frac{1}{h}$



$\xrightarrow{\theta=\omega h}$



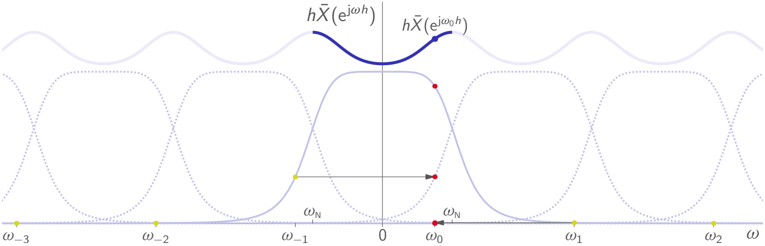
spectrum of $\bar{x}$:

$$\bar{X}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} X\left(\mathrm{j}\frac{\theta + 2\pi i}{h}\right)$$

Nyquist frequency

$$\omega_\mathrm{N} := \frac{\pi}{h} = \frac{\omega_\mathrm{s}}{2}$$

Spectrum of sampled signal: aliasing



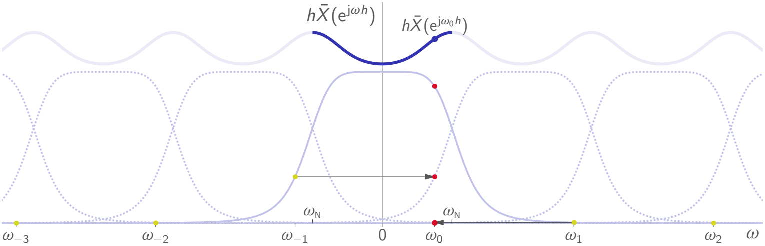
The spectrum of $\bar{x}$ at each discrete frequency $\theta_0 = \omega_0 h$ is a

– blend of analog spectra at all analog frequencies $\omega_i := \omega_0 + \omega_s i$, $i \in \mathbb{Z}$.

In this process every ω_i effectively tries to *alias* as θ_0. This phenomenon is then dubbed aliasing, with respect to the base frequency ω_0.

Aliasing means information loss, we can no longer tell $X(j\omega_i)$ from $X(j\omega_j)$ in their effect on $\bar{X}(e^{j\omega_0 h})$, unless we know how they depend on each other. For instance, in this particular case $\bar{X}(e^{j\omega_0 h})$ has significant contributions of both $X(j\omega_0)$ and $X(j\omega_{-1})$ for most $\omega_0 > 0$, which is problematic indeed.

Spectrum of sampled signal: aliasing



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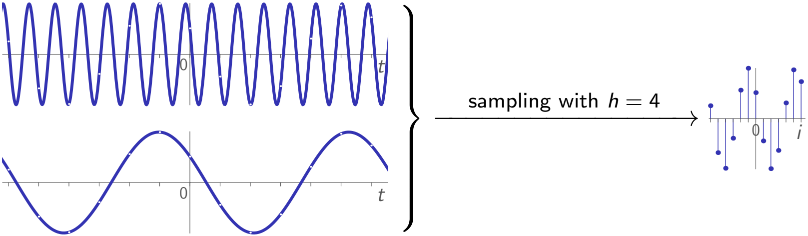
– blend of analog spectra at all analog frequencies $\omega_i := \omega_0 + \omega_s i$, $i \in \mathbb{Z}$.

In this process every ω_i effectively tries to *alias* as θ_0. This phenomenon is then dubbed aliasing, with respect to the base frequency ω_0.

Aliasing means information loss, we can no longer tell $X(j\omega_i)$ from $X(j\omega_j)$ in their effect on $\bar{X}(e^{j\omega_0 h})$, unless we know how they depend on each other. For instance, in this particular case $\bar{X}(e^{j\omega_0 h})$ has significant contributions of both $X(j\omega_0)$ and $X(j\omega_{-1})$ for most $\omega_0 > 0$, which is problematic indeed.

Aliasing: examples

Two sine waves: Remember (Lect. 2, Slide 35)

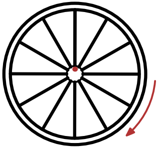


They are both of the form $\sin(\omega_i t + 2.6)$ for $\omega_1 = \frac{2\pi+1}{4}$ and $\omega_2 = \frac{1}{4}$. With $h = 4$ the dominant frequencies of their spectra alias as $\theta_0 = \omega_0 h = 1$:

$$\omega_1 = \omega_0 + 1 \cdot \frac{2\pi}{h} \quad \text{and} \quad \omega_2 = \omega_0 + 0 \cdot \frac{2\pi}{h}$$

That's why sampled versions of both these signals are the same, $\sin[t + 2.6]$.

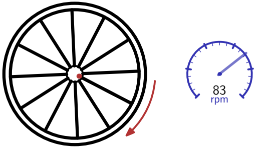
Aliasing: examples (contd)

Wagon-wheel effect: 


(shot with 12 FPS frame rate)

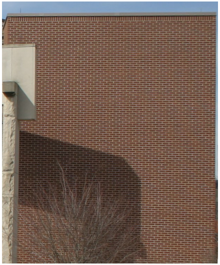
Aliasing: examples (contd)

Wagon-wheel effect:



(shot with 12 FPS frame rate)

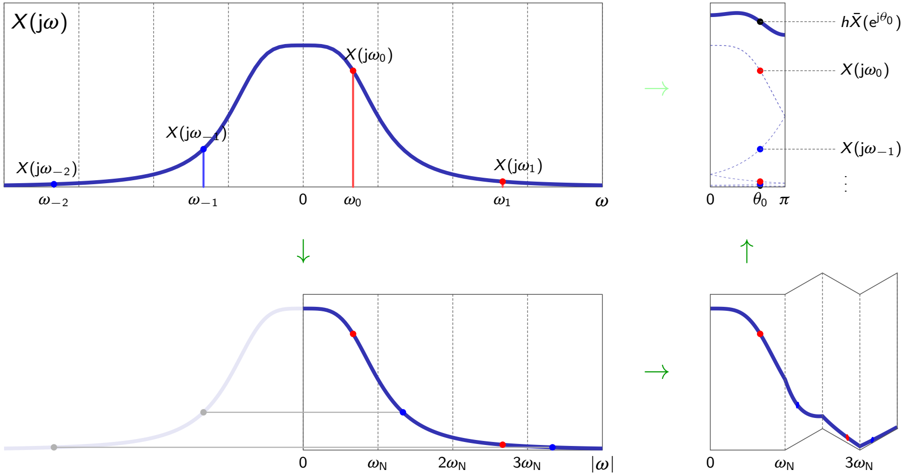
Moiré pattern:



downsampling ⟶

Frequency folding

If x is real-valued and even, then so is X and $\bar{X}$ can be also constructed via the folding procedure:



Outline

Discrete-time frequency-domain analysis

A/D conversion in the frequency domain

D/A conversion in the frequency domain

The Sampling Theorem

The zero-order-hold: what do we know

It maps discrete signals $\bar{x} : \mathbb{Z} \to \mathbb{R}$ to continuous-time signals $x : \mathbb{R} \to \mathbb{R}$ as

$$x(t) = \bar{x}[i], \quad \forall t \in (ih, (i+1)h)$$

for a given sampling period $h > 0$.

Spectrum of x

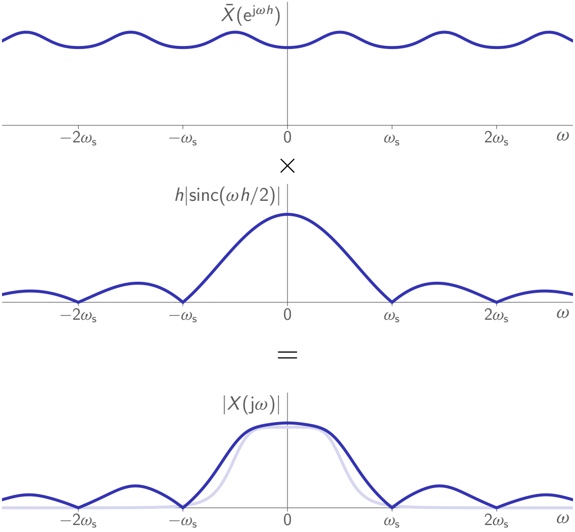
By the definition of the Fourier transform,

$$\begin{aligned}
X(\mathrm{j}\omega) &= \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \sum_{i\in\mathbb{Z}} \int_{ih}^{(i+1)h} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \\
&= \sum_{i\in\mathbb{Z}} \int_{ih}^{(i+1)h} \bar{x}[i]\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \sum_{i\in\mathbb{Z}} \bar{x}[i] \int_{ih}^{(i+1)h} \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \\
&= \sum_{i\in\mathbb{Z}} \bar{x}[i] \left(\frac{\mathrm{e}^{-\mathrm{j}\omega t}}{-\mathrm{j}\omega} \right) \Bigg|_{ih}^{(i+1)h} = \sum_{i\in\mathbb{Z}} \bar{x}[i] \frac{\mathrm{e}^{-\mathrm{j}\omega hi} - \mathrm{e}^{-\mathrm{j}\omega h(i+1)}}{\mathrm{j}\omega} \\
&= \sum_{i\in\mathbb{Z}} \bar{x}[i]\mathrm{e}^{-\mathrm{j}\omega hi} \frac{1-\mathrm{e}^{-\mathrm{j}\omega h}}{\mathrm{j}\omega} \qquad \text{cf. } \bar{X}(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{i\in\mathbb{Z}} \bar{x}[i]\mathrm{e}^{-\mathrm{j}\theta i} \\
&= h\frac{1-\mathrm{e}^{-\mathrm{j}\omega h}}{\mathrm{j}\omega h} \bar{X}(\mathrm{e}^{\mathrm{j}\omega h}) = h \operatorname{sinc}\left(\frac{\omega h}{2}\right) \mathrm{e}^{-\mathrm{j}\omega h/2} \bar{X}(\mathrm{e}^{\mathrm{j}\omega h})
\end{aligned}$$

(note that $\bar{X}$ is ω_s-periodic as a function of ω).

Example

With $\bar{x}$ as above,



Outline

Discrete-time frequency-domain analysis

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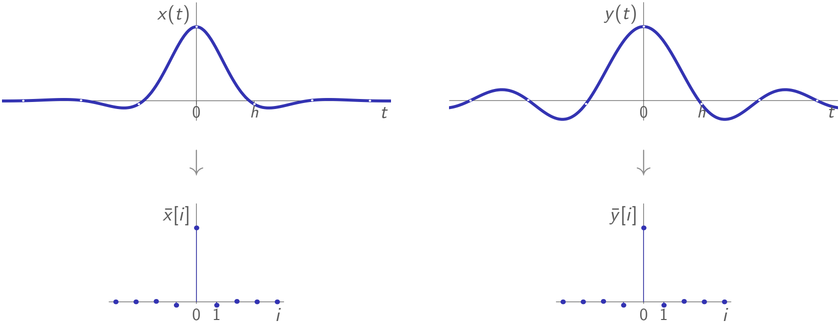
The Sampling Theorem

Sampling in the time domain (contd)

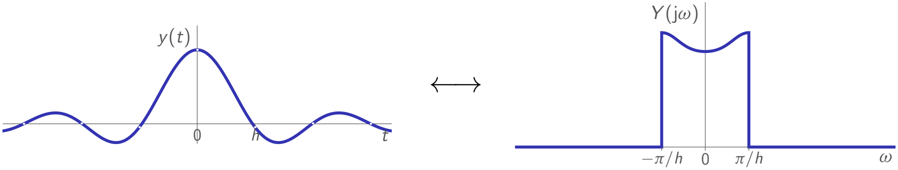


We saw why sampling of x is problematic (strong aliasing, $\bar{X}$ is qualitatively different from X). But

– why do we loose nothing in sampling y ?

Frequency-domain viewpoint

Spectrum of y is



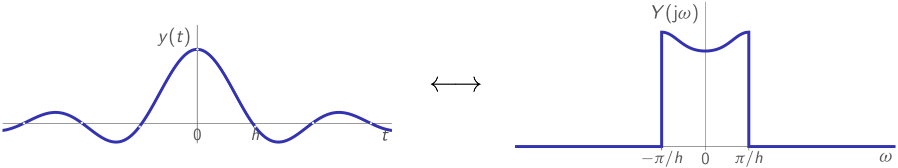
hence the spectrum of $\bar{y}$ is

has no contributions of Y at aliased frequencies ($Y(j\omega_i) = 0$ for all $i \neq 0$). As a result, $\bar{Y}$ is merely a scaled version of Y and sampling is lossless here. This conclusion obviously applies to

— all signals whose spectrum has support within $[-\omega_N, \omega_N]$ (bandlimited).

Frequency-domain viewpoint

Spectrum of y is



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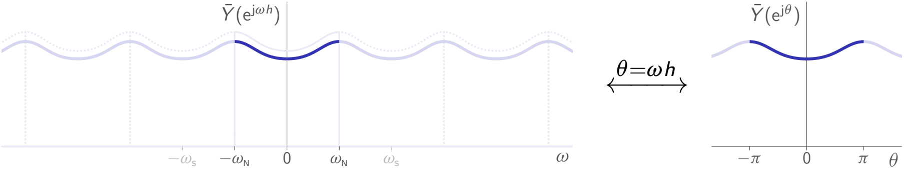

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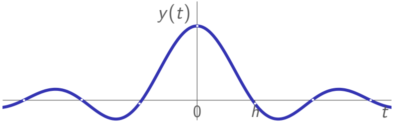
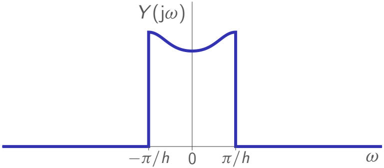


$\longleftrightarrow$



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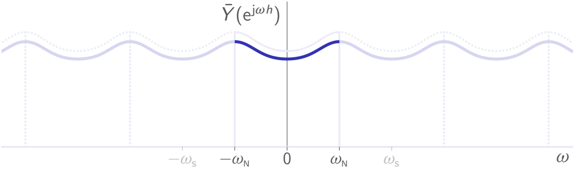
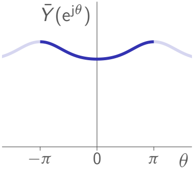


$\overset{\theta=\omega h}{\longleftrightarrow}$



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- all signals whose spectrum has support within $[-\omega_{\mathrm{N}}, \omega_{\mathrm{N}}]$ (bandlimited).

Reconstructing x from its lossless sampling

If $\mathrm{supp}(X) \subset [-\omega_\mathrm{N}, \omega_\mathrm{N}]$, then $\bar{X}(\mathrm{e}^{\mathrm{j}\omega h}) = \frac{1}{h} X(\mathrm{j}\omega)$ for all $\omega \in [-\omega_\mathrm{N}, \omega_\mathrm{N}]$ and

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega = \frac{h}{2\pi}\int_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \bar{X}(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \\
&= \frac{1}{2\omega_\mathrm{N}}\int_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \sum_{i\in\mathbb{Z}} \bar{x}[i]\mathrm{e}^{-\mathrm{j}\omega h i}\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \\
&= \frac{1}{2\omega_\mathrm{N}}\sum_{i\in\mathbb{Z}} \bar{x}[i]\int_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \mathrm{e}^{-\mathrm{j}\omega h i}\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega = \sum_{i\in\mathbb{Z}} \bar{x}[i]\frac{1}{2\omega_\mathrm{N}}\left(\frac{\mathrm{e}^{\mathrm{j}(t-ih)\omega}}{\mathrm{j}(t-ih)}\right)\Bigg|_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \\
&= \sum_{i\in\mathbb{Z}} \bar{x}[i]\,\frac{\mathrm{e}^{\mathrm{j}(t-ih)\omega_\mathrm{N}} - \mathrm{e}^{-\mathrm{j}(t-ih)\omega_\mathrm{N}}}{\mathrm{j}2(t-ih)\omega_\mathrm{N}} = \sum_{i\in\mathbb{Z}} \bar{x}[i]\,\frac{\sin((t-ih)\omega_\mathrm{N})}{(t-ih)\omega_\mathrm{N}} \\
&= \sum_{i\in\mathbb{Z}} \bar{x}[i]\,\mathrm{sinc}((t-ih)\omega_\mathrm{N})
\end{aligned}
$$

In other words, signals that are bandlimited within $[-\omega_\mathrm{N}, \omega_\mathrm{N}]$ are uniquely described by their periodic samples.

The Sampling Theorem

Theorem (Whittaker-Kotel'nikov-Shannon)

If $\operatorname{supp}(X) \subset [-\omega_N, \omega_N]$, *then* x *can be perfectly recovered from its sampled measurements as*

$$x = \sum_{i\in\mathbb{Z}} x(ih)\mathbb{S}_{-ih}\mathbb{P}_{\omega_N}\operatorname{sinc} \qquad \Longrightarrow \qquad x(t) = \sum_{i\in\mathbb{Z}} x(ih)\operatorname{sinc}((t-ih)\omega_N)$$

known as the sinc-*interpolator* (sinc *hold).*

The sinc-interpolator acts as

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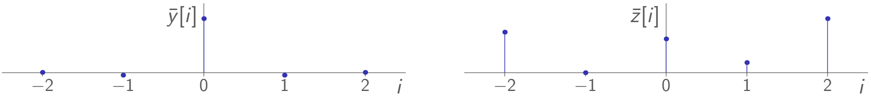

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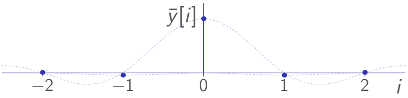

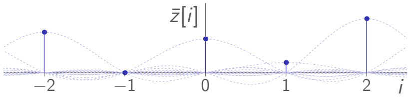

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