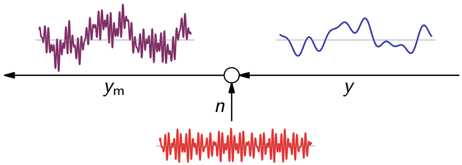
Linear Systems (034032)
lecture no. 3

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Example 0



A signal y is observed via an imperfect sensor, where the measured signal is

$$y_m = y + n$$

for (unknown) additive noise n reflecting sensor inaccuracies. The question then is

– how signal (y) can be recovered from its corrupt measurements (y_m)?

This might appear a tough task, for there is no way we can separate y from n in the time domain . . .

Outline

Fourier series

Fourier transform

Fourier transforms of some standard signals

Outline

Fourier series

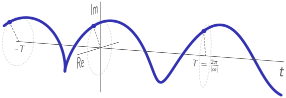
Fourier transform

Fourier transforms of some standard signals

Harmonic signal

Signal $\alpha = a\exp_{j\omega} : \mathbb{R} \to \mathbb{C}$,

$$\alpha(t) = ae^{j\omega t} = |a|e^{j(\omega t+\arg(a))} =$$



for $a \in \mathbb{C}$ and $\omega \in \mathbb{R}$ is called harmonic signal with frequency ω, amplitude $|a|$, and initial phase $\arg(a)$. Because

$$\alpha(t + \tfrac{2\pi}{\omega}k) = ae^{j\omega(t+2\pi k/\omega)} = ae^{j\omega t+j2\pi k} = ae^{j\omega t}e^{j2\pi k} = ae^{j\omega t} = \alpha(t)$$

for all $k \in \mathbb{Z}$, the harmonic signal is $\frac{2\pi}{|\omega|}$-periodic (ω may be negative).

Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$ can be used to connect the (complex) harmonic signal with real sinusoids:

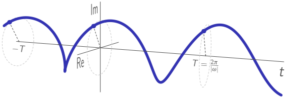
$$\sin(\omega t + \phi) = ae^{j\omega t} + \bar{a}e^{-j\omega t}, \quad a = 0.5e^{j(\phi-\pi/2)}$$

(harmonics with ω and $-\omega$ come together in real-valued signals).

Harmonic signal

Signal $\alpha = a \exp_{j\omega} : \mathbb{R} \to \mathbb{C}$,

$$\alpha(t) = a e^{j\omega t} = |a| e^{j(\omega t + \arg(a))} =$$



for $a \in \mathbb{C}$ and $\omega \in \mathbb{R}$ is called harmonic signal with frequency ω, amplitude $|a|$, and initial phase $\arg(a)$. Because

$$\alpha(t + \tfrac{2\pi}{\omega} k) = a e^{j\omega(t + 2\pi k/\omega)} = a e^{j\omega t + j2\pi k} = a e^{j\omega t} e^{j2\pi k} = a e^{j\omega t} = \alpha(t)$$

for all $k \in \mathbb{Z}$, the harmonic signal is $\frac{2\pi}{|\omega|}$-periodic (ω may be negative).

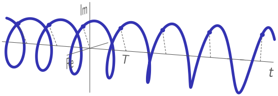
Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$ can be used to connect the (complex) harmonic signal with real sinusoids:

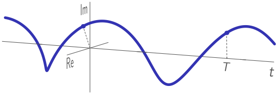
$$\sin(\omega t + \phi) = a e^{j\omega t} + \bar{a} e^{-j\omega t}, \quad a = 0.5 e^{j(\phi - \pi/2)}$$

(harmonics with ω and $-\omega$ come together in real-valued signals).

Harmonic signal (contd)

The signal $a_1 \exp_{j\omega_1}$ is said to be faster (slower) than $a_2 \exp_{j\omega_2}$ if $|\omega_1| > |\omega_2|$ ($|\omega_1| < |\omega_2|$). Indeed,

$x_1 =$ 

is faster than $x_2 =$ 

just because

$$\frac{|\dot{x}_1(t)|}{|x_1(t)|} = |\omega_1| > |\omega_2| = \frac{|\dot{x}_2(t)|}{|x_2(t)|}, \quad \forall t$$

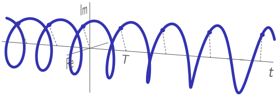
Note that there is no "the fastest" continuous-time signals.

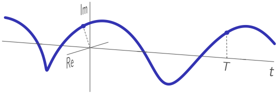
Power

$$P_{a\exp_{j\omega}} = \frac{1}{T}\int_0^T |a e^{j\omega t}|^2 dt = \frac{|a|^2}{T}\int_0^T dt = |a|^2$$

Harmonic signal (contd)

The signal $a_1 \exp_{j\omega_1}$ is said to be faster (slower) than $a_2 \exp_{j\omega_2}$ if $|\omega_1| > |\omega_2|$ ($|\omega_1| < |\omega_2|$). Indeed,

$x_1 =$ 

is faster than $x_2 =$ 

just because

$$\frac{|\dot{x}_1(t)|}{|x_1(t)|} = |\omega_1| > |\omega_2| = \frac{|\dot{x}_2(t)|}{|x_2(t)|}, \quad \forall t$$

Note that there is no "the fastest" continuous-time signals.

Power

$$P_{a \exp_{j\omega}} = \frac{1}{T} \int_0^T |a e^{j\omega t}|^2 dt = \frac{|a|^2}{T} \int_0^T dt = |a|^2$$

Fourier series of continuous signals

If $x : \mathbb{R} \to \mathbb{R}$ is T-periodic and continuous, then at every t

$$x(t) = \sum_{k\in\mathbb{Z}} X[k]\mathrm{e}^{\mathrm{j}\omega_0 kt} = \sum_{k\in\mathbb{Z}} X[k]\mathrm{e}^{\mathrm{j}(2\pi/T)kt},$$

where

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 kt}\mathrm{d}t = \frac{1}{T}\int_{a}^{a+T} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 kt}\mathrm{d}t \in \mathbb{C}$$

are known as Fourier coefficients of x and

$$\omega_0 := \frac{2\pi}{T}$$

is known as its fundamental frequency. The T-periodic $x_N : \mathbb{R} \to \mathbb{R}$ with

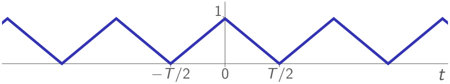
$$x_N(t) := \sum_{k=-N}^{N} X[k]\mathrm{e}^{\mathrm{j}\omega_0 kt}$$

is known as a partial sum of the Fourier series of x.

Example 1

Let

$$x = \sum_{i \in \mathbb{Z}} \mathbb{S}_{iT}\, \mathrm{tent}_{T/2} =$$



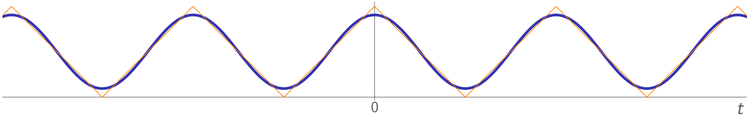
Fourier coefficients:

$$
\begin{aligned}
X[k] &= \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{tent}_{T/2}(t)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\,\mathrm{d}t = \frac{1}{T}\int_{-T/2}^{T/2}\left(1-\frac{2|t|}{T}\right)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\,\mathrm{d}t \\
&= \frac{1}{T}\left(\int_{-T/2}^{0}\left(1+\frac{2t}{T}\right)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\,\mathrm{d}t + \int_{0}^{T/2}\left(1-\frac{2t}{T}\right)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\,\mathrm{d}t\right) \\
&= \frac{1}{T}\left(\frac{(1-\mathrm{e}^{\mathrm{j}\pi k}+\mathrm{j}\pi k)T}{2\pi^2 k^2} + \frac{(1-\mathrm{e}^{-\mathrm{j}\pi k}-\mathrm{j}\pi k)T}{2\pi^2 k^2}\right) \\
&= \frac{1-\cos(\pi k)}{\pi^2 k^2} = \frac{1-(-1)^k}{\pi^2 k^2} = \begin{cases} 1/2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \text{ and is even} \\ 2/(\pi k)^2 & \text{if } k \text{ is odd} \end{cases}
\end{aligned}
$$

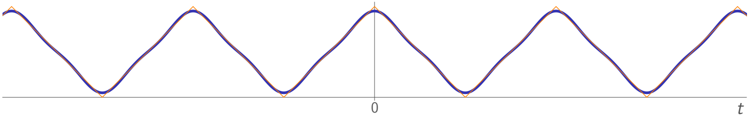
Example 1 (contd)

Partial sums:

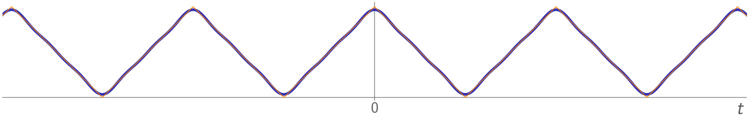
$N = 1:$



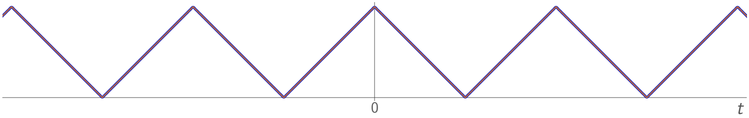
$N = 3:$



$N = 5:$



$N = 51:$



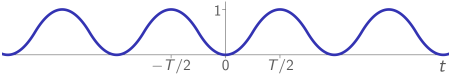
Example 2

Let

$$s(t) = \begin{cases} 8t^2 & \text{if } 0 \le t \le 1/4 \\ -8t^2 + 8t - 1 & \text{if } 1/4 \le t \le 3/4 \\ 8(t-1)^2 & \text{if } 3/4 \le t \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

and define T-periodic

$$x = \sum_{i \in \mathbb{Z}} \mathbb{S}_{iT} \mathbb{P}_{1/T} s =$$



Fourier coefficients:

$$X[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \mathrm{e}^{-\mathrm{j}\omega_0 k t} \mathrm{d}t = \frac{1}{T} \int_0^T s(t/T) \mathrm{e}^{-\mathrm{j}\omega_0 k t} \mathrm{d}t$$

$$= \frac{8(-1)^k \sin(\pi k/2)}{\pi^3 k^3} = \begin{cases} 1/2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \text{ and is even} \\ 8(-1)^{(k+1)/2}/(\pi k)^3 & \text{if } k \text{ is odd} \end{cases}$$

Example 2 (contd)

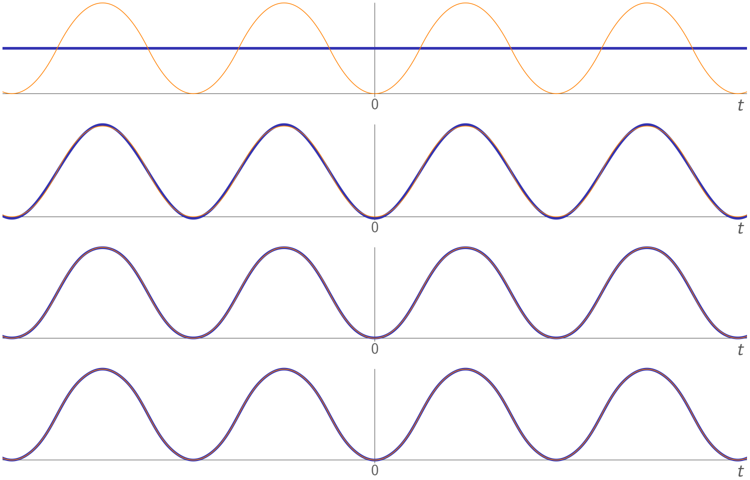
Partial sums:

$N = 0$:

$N = 1$:

$N = 3$:

$N = 5$:



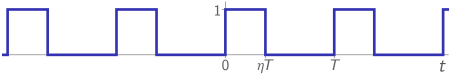
Piecewise-smooth signals

A T-periodic signal x is said to be piecewise smooth if it's differentiable and has continuous derivative everywhere on $[0, T]$ with a possible exception of a finite number of distinct t_i, at which $x(t_i^+)$, $x(t_i^-)$, $\dot{x}(t_i^+)$ and $\dot{x}(t_i^-)$ exist in the sense of the existence of

$$x(t_i^+) := \lim_{h\downarrow 0} x(t_i + h), \quad x(t_i^-) := \lim_{h\downarrow 0} x(t_i - h), \quad \text{etc}$$

Example:

$$x = \sum_{i\in\mathbb{Z}} \mathbb{S}_{(i-\eta/2)T} \,\mathrm{rect}_{\eta T} =$$



for $\eta \in (0, 1)$.

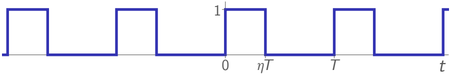
Piecewise-smooth signals

A T-periodic signal x is said to be piecewise smooth if it's differentiable and has continuous derivative everywhere on $[0, T]$ with a possible exception of a finite number of distinct t_i, at which $x(t_i^+)$, $x(t_i^-)$, $\dot{x}(t_i^+)$ and $\dot{x}(t_i^-)$ exist in the sense of the existence of

$$x(t_i^+) := \lim_{h\downarrow 0} x(t_i + h), \quad x(t_i^-) := \lim_{h\downarrow 0} x(t_i - h), \quad \text{etc}$$

Example:

$$x = \sum_{i\in\mathbb{Z}} \mathbb{S}_{(i-\eta/2)T} \operatorname{rect}_{\eta T} =$$



for $\eta \in (0, 1)$.

Remark The notion extends to non-periodic functions, which are ∞-periodic, in a sense. In that case we need continuous derivatives everywhere except a counted number points t_i, at which $x(t_i^+)$, $x(t_i^-)$, $\dot{x}(t_i^+)$ and $\dot{x}(t_i^-)$ exist.

Fourier series of piecewise-smooth signals

Everything you always wanted to know about Fourier peculiarities* (*but were afraid to ask)

If $x : \mathbb{R} \to \mathbb{R}$ is T-periodic and piecewise smooth, then at every t

$$\frac{x(t^+) + x(t^-)}{2} = \sum_{k \in \mathbb{Z}} X[k] \mathrm{e}^{\mathrm{j}\omega_0 k t}$$

Let $X : \mathbb{Z} \to \mathbb{C}$ be the signal comprising the Fourier coefficients. If

$X \in \ell_1$, then

$$\lim_{N\to\infty} x_N(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} X[k] \mathrm{e}^{\mathrm{j}\omega_0 k t}$$

converges to $x(t)$ at every t and the limit is necessarily continuous

$X \notin \ell_1$, then the partial sum converges only in the ℓ_2 sense, i.e.

$$\lim_{N\to\infty} \|x - x_N\|_2 = 0$$

Fourier series of piecewise-smooth signals

Everything you always wanted to know about Fourier peculiarities★ (★but were afraid to ask)

If $x : \mathbb{R} \to \mathbb{R}$ is T-periodic and piecewise smooth, then at every t

$$\frac{x(t^+) + x(t^-)}{2} = \sum_{k \in \mathbb{Z}} X[k] e^{j\omega_0 kt}$$

Let $X : \mathbb{Z} \to \mathbb{C}$ be the signal comprising the Fourier coefficients. If

$X \in \ell_1$, then

$$\lim_{N\to\infty} x_N(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} X[k] e^{j\omega_0 kt}$$

converges to $x(t)$ at every t and the limit is necessarily continuous

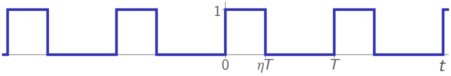
$X \notin \ell_1$, then the partial sum converges only in the ℓ_2 sense, i.e.

$$\lim_{N\to\infty} \|x - x_N\|_2 = 0$$

Example 3

Let

$$x = \sum_{i \in \mathbb{Z}} \mathbb{S}_{(i-\eta/2)T}\, \mathrm{rect}_{\eta T} =$$



Fourier coefficients:

$$\begin{aligned} X[k] &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\,\mathrm{d}t = \frac{1}{T}\int_{0}^{T} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\,\mathrm{d}t \\ &= \frac{1}{T}\int_{0}^{\eta T} \mathrm{e}^{-\mathrm{j}\omega_0 k t}\,\mathrm{d}t \\ &= \frac{\sin(2\pi\eta k)}{2\pi k} - \mathrm{j}\,\frac{1-\cos(2\pi\eta k)}{2\pi k} \end{aligned}$$

(with $X[0] = \eta$).

Example 3 (contd)

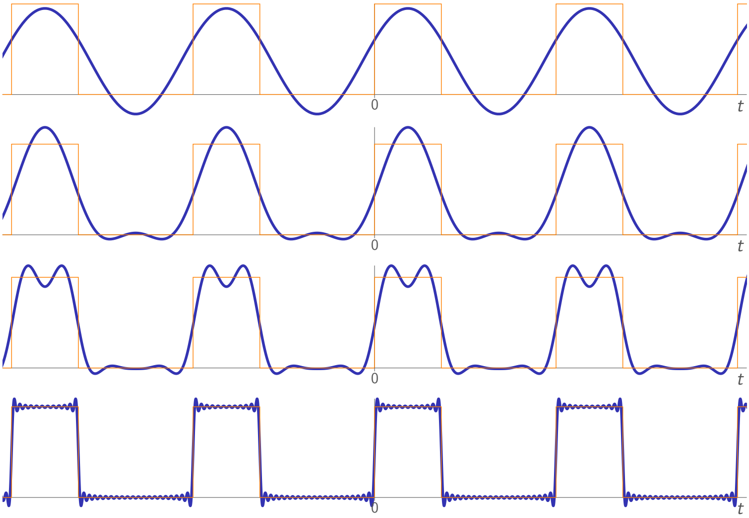
Partial sums:

$N = 1$:

$N = 2$:

$N = 5$:

$N = 33$:



Gibbs phenomenon

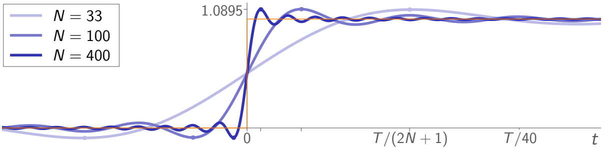
Near every discontinuity point of x,

$$x_N\left(t+\tfrac{T}{2N+1}\right)-x(t^+)\xrightarrow{N\to\infty}\underbrace{\left(\frac{1}{\pi}\int_0^{\pi}\operatorname{sinc}(t)\mathrm{d}t-\frac{1}{2}\right)}_{\approx 0.089549}\left(x(t^+)-x(t^-)\right)$$

and

$$x_N\left(t-\tfrac{T}{2N+1}\right)-x(t^-)\xrightarrow{N\to\infty}\left(\frac{1}{\pi}\int_0^{\pi}\operatorname{sinc}(t)\mathrm{d}t-\frac{1}{2}\right)\left(x(t^-)-x(t^+)\right)$$

like in



Some properties

property	signal	Fourier coefficients
linearity	$x = a_1 x_1 + a_2 x_2$	$X[k] = a_1 X_1[k] + a_2 X_2[k]$
time shift	$y = \mathbb{S}_\tau x$	$Y[k] = e^{j\omega_0 \tau k} X[k]$
time reversal	$y = \mathbb{P}_{-1} x$	$Y[k] = X[-k]$
conjugation	$y = \overline{x}$	$Y[k] = \overline{X[-k]}$
convolution	$z = x * y$	$Z[k] = X[k] Y[k]$

Consequently, if x is real-valued (codomain is $\mathbb{R}$), then $X[k] = \overline{X[-k]}$

- i.e. $|X[k]| = |X[-k]|$ and $\arg(X[k]) = -\arg(X[k])$

and all information is in $X[k]$ for $k \in \mathbb{Z}_+$.

Remember, $(\mathbb{S}_\tau x)(t) = x(t + \tau)$ and $(\mathbb{P}_\varsigma x)(t) = x(\varsigma t)$.

Meaning

The expansion

$$x(t) = \sum_{k \in \mathbb{Z}} X[k] e^{j\omega_0 k t}$$

means that every T-periodic x is a superposition of elementary harmonics

$$\alpha_{x,k}(t) := X[k] e^{j\omega_0 k t}$$

whose frequencies are multiples of the fundamental frequency $\omega_0 = 2\pi/T$.

The discrete signal $X : \mathbb{Z} \to \mathbb{C}$ comprised of Fourier coefficients is known as the (line) spectrum of x. In other words,

$\implies$ T-periodic x can be equivalently represented by the discrete-time X,

known as the frequency-domain representation of x.

Meaning

The expansion

$$x(t) = \sum_{k \in \mathbb{Z}} X[k] e^{j\omega_0 k t}$$

means that every T-periodic x is a superposition of elementary harmonics

$$\alpha_{x,k}(t) := X[k] e^{j\omega_0 k t}$$

whose frequencies are multiples of the fundamental frequency $\omega_0 = 2\pi/T$.

The discrete signal $X : \mathbb{Z} \to \mathbb{C}$ comprised of Fourier coefficients is known as the (line) spectrum of x. In other words,

– T-periodic x can be equivalently represented by the discrete-time X,

known as the frequency-domain representation of x.

Parseval identity

Theorem

If T-periodic x is piecewise smooth, then

$$P_x = \sum_{k\in\mathbb{Z}} |X[k]|^2.$$

Proof: Because $|c|^2 = c\overline{c}$,

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\overline{x(t)}\,\mathrm{d}t = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\overline{\Big(\sum_{k\in\mathbb{Z}} X[k]\mathrm{e}^{\mathrm{j}\omega_0 kt}\Big)}\,\mathrm{d}t$$

$$= \sum_{k\in\mathbb{Z}} \left(\frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 kt}\right)\mathrm{d}t\,\overline{X[k]}$$

$$= \sum_{k\in\mathbb{Z}} X[k]\overline{X[k]} = \sum_{k\in\mathbb{Z}} |X[k]|^2$$

□

Parseval identity: meaning

The Parseval identity effectively says that

$$P_x = \sum_{k \in \mathbb{Z}} P_{\alpha_{x,k}}$$

i.e. the power of x equals the sum of powers of all its harmonic components in the expansion $x = \sum_k \alpha_{x,k}$. This, in turn, implies that

– harmonics with largest amplitudes dominate the behavior of x
  – if dominating harmonics are low, x is slow

$$\longleftrightarrow$$

  – if dominating harmonics are high, x is fast

$$\longleftrightarrow$$

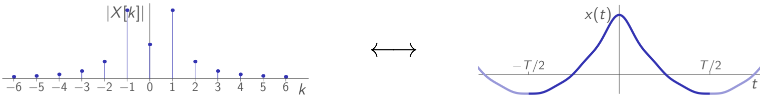
Parseval identity: meaning

The Parseval identity effectively says that

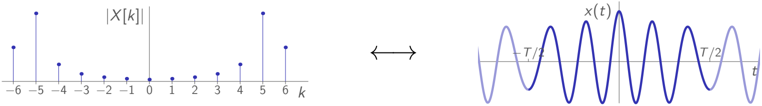
$$P_x = \sum_{k \in \mathbb{Z}} P_{\alpha_{x,k}}$$

i.e. the power of x equals the sum of powers of all its harmonic components in the expansion $x = \sum_k \alpha_{x,k}$. This, in turn, implies that

- harmonics with largest amplitudes dominate the behavior of x
  - if dominating harmonics are low, x is slow

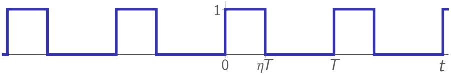


  - if dominating harmonics are high, x is fast

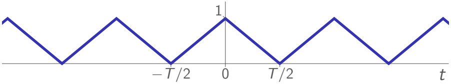


Decay of Fourier coefficients

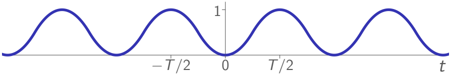
In general, the faster $|X[k]|$ decays as $|k|$ grows, the smoother x is, because faster harmonics have then smaller effect on x, cf.



$$\longrightarrow \quad |X[k]| = \frac{\sin(\pi\eta k)}{\pi|k|}$$



$$\longrightarrow \quad |X[k]| = \frac{1-\cos(\pi k)}{\pi^2 k^2}$$



$$\longrightarrow \quad |X[k]| = \frac{8|\sin(\pi k/2)|}{\pi^3|k|^3}$$

Outline

Definition

The Fourier transform $\mathfrak{F}\{x\}$ of a signal x is the signal $X : \mathbb{R} \to \mathbb{C}$ such that

$$X(\mathrm{j}\omega) = (\mathfrak{F}\{x\})(\mathrm{j}\omega) := \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$$

('jω' is used for a technical reason to be clarified one day). It is well defined if $x \in L_1$ (plus some mild technical assumptions) and then

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega =: (\mathfrak{F}^{-1}\{X\})(t)$$

at every t. The latter integral is referred to as the inverse Fourier transform.

If $x \in L_2$, then the Fourier integral converges only in the (weaker) L_2-norm sense, $\|x - \mathfrak{F}^{-1}\{X_N\}\|_2 \to 0$. We can then still extend the transform to L_2 signals. The use of distributions facilitates extending the Fourier transform to yet wider classes of signals.

Meaning

Similarly to the Fourier series, the relation

$$x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} X(\mathrm{j}\omega) \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$

means that

- aperiodic x is also a superposition of elementary harmonics $\alpha_{x,\omega}$, with $\alpha_{x,\omega}(t) = \frac{1}{2\pi} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}$,

now with a continuum of frequencies ω (measured in radians per time unit).

The signal X is then called the spectrum or frequency-domain representation of x, with the amplitude spectrum $|X|$ and the phase spectrum $\arg(X)$.

The value $X(\mathrm{j}0) = X(0)$ is the average of x over all its domain, i.e. $\mathbb{R}$.

Meaning

Similarly to the Fourier series, the relation

$$x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} X(\mathrm{j}\omega) \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$

means that

- aperiodic x is also a superposition of elementary harmonics $\alpha_{x,\omega}$, with $\alpha_{x,\omega}(t) = \frac{1}{2\pi} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}$,

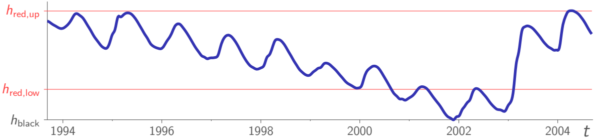
now with a continuum of frequencies ω (measured in radians per time unit).

The signal X is then called the spectrum or frequency-domain representation of x, with the amplitude spectrum $|X|$ and the phase spectrum $\arg(X)$.

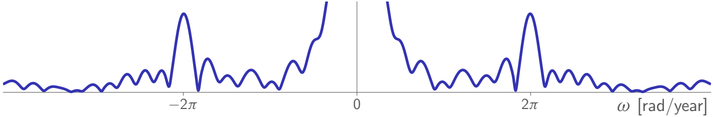
The value $X(\mathrm{j}0) = X(0)$ is the average of x over all its domain, i.e. $\mathbb{R}$.

Example 1

Kinneret water level from Sep 1993 to Sep 2004



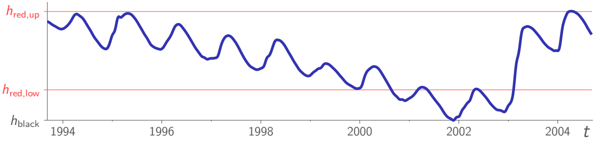
and its amplitude spectrum (with $h_{\text{red,up}}$ taken as zero)



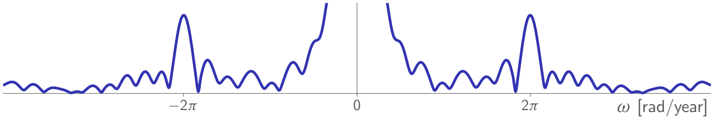
Peaks at $\omega = \pm 2\pi$ [rad/year] reflect the annual cycle.

Example 1

Kinneret water level from Sep 1993 to Sep 2004



and its amplitude spectrum (with $h_{\text{red,up}}$ taken as zero)



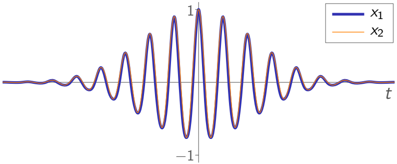
Peaks at $\omega = \pm 2\pi$ [rad/year] reflect the annual cycle.

Example 2

Two L_1 signals x_1 and x_2, with

$$x_1(t) = \frac{8\cos(2\pi t) + \cos(4\pi t)}{9} e^{-t^2/10}$$
$$x_2(t) = \frac{8\cos(2\pi t) + 1}{9} e^{-t^2/10}$$

=

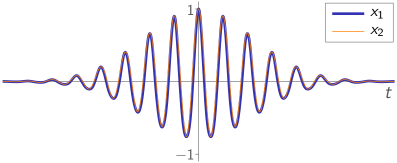


are hardly distinguishable in the time domain. But they are quite different, e.g. the average $\int_{\mathbb{R}} x_1(t)\mathrm{d}t \approx 0 \neq 0.62 \approx \int_{\mathbb{R}} x_2(t)\mathrm{d}t$. These differences are visible in their spectra

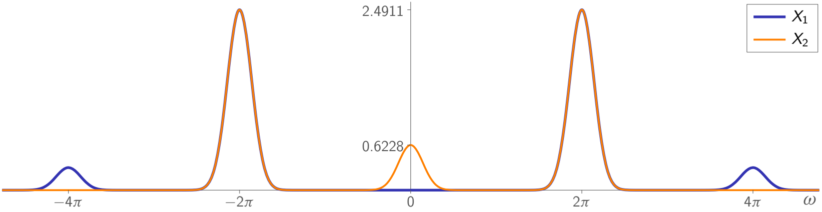
Example 2

Two L_1 signals x_1 and x_2, with

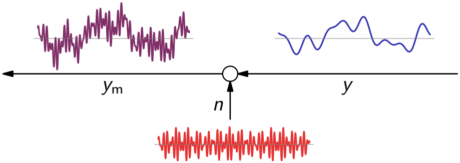
$$x_1(t) = \frac{8\cos(2\pi t) + \cos(4\pi t)}{9} e^{-t^2/10}$$
$$x_2(t) = \frac{8\cos(2\pi t) + 1}{9} e^{-t^2/10}$$
$=$



are hardly distinguishable in the time domain. But they are quite different, e.g. the average $\int_{\mathbb{R}} x_1(t)\mathrm{d}t \approx 0 \neq 0.62 \approx \int_{\mathbb{R}} x_2(t)\mathrm{d}t$. These differences are visible in their spectra



Example 0 (contd)



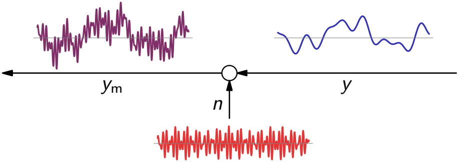
The question was

- how signal (y) can be recovered from its corrupt measurements (y_{m})?

This might appear a tough task, for there is no way we can separate y from n in the time domain . . . But in the frequency domain these signals are well separated,

suggesting that the frequency-domain viewpoint is valuable[1].

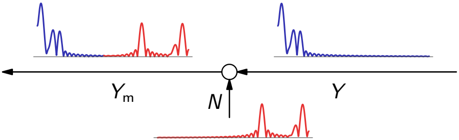
Example 0 (contd)



The question was

- how signal (y) can be recovered from its corrupt measurements (y_m)?

This might appear a tough task, for there is no way we can separate y from n in the time domain . . . But in the frequency domain these signals are well separated,



suggesting that the frequency-domain viewpoint is valuable[1].

[1] However, we yet to learn how to fully exploit it efficiently.

Basic properties

property	time domain	frequency domain
linearity	$x = a_1x_1 + a_2x_2$	$X(\mathrm{j}\omega) = a_1X_1(\mathrm{j}\omega) + a_2X_2(\mathrm{j}\omega)$
duality	$y = X\|_{\omega=t}$	$Y(\mathrm{j}\omega) = 2\pi x(-\omega)$
time shift	$y = \mathbb{S}_\tau x$	$Y(\mathrm{j}\omega) = \mathrm{e}^{\mathrm{j}\omega\tau}X(\mathrm{j}\omega)$
time scaling	$y = \mathbb{P}_\varsigma x$	$Y(\mathrm{j}\omega) = \frac{1}{\|\varsigma\|}X(\mathrm{j}\,\omega/\varsigma)$
conjugation	$y = \overline{x}$	$Y(\mathrm{j}\omega) = \overline{X(-\mathrm{j}\omega)}$
modulation	$y = x\exp_{\mathrm{j}\omega_0}$	$Y(\mathrm{j}\omega) = X(\mathrm{j}(\omega - \omega_0))$
differentiation[2]	$y = \dot{x}$	$Y(\mathrm{j}\omega) = \mathrm{j}\omega X(\mathrm{j}\omega)$
convolution	$z = x * y$	$Z(\mathrm{j}\omega) = X(\mathrm{j}\omega)Y(\mathrm{j}\omega)$

. . . provided all involved transforms exist

[2] If $\lim_{t\to\pm\infty} x(t) = 0$, which is normally the case for $x \in L_1$.

Basic properties: duality

If $y = X|_{\omega=t}$, i.e. $y(t) = X(\mathrm{j}t)$, where $X = \mathfrak{F}\{x\}$, then

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int_{\mathbb{R}} X(\mathrm{j}t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t\Big|_{t=\tilde{\omega}} = 2\pi\left(\frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\tilde{\omega})\mathrm{e}^{\mathrm{j}\tilde{\omega}(-\omega)}\mathrm{d}\tilde{\omega}\right) \\ &= 2\pi x(-\omega) \end{aligned}$$

because

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega$$

Basic properties: time shift

If $y = \mathbb{S}_\tau x$, i.e. $y(t) = x(t+\tau)$, then

$$
\begin{aligned}
Y(\mathrm{j}\omega) &= \int_{\mathbb{R}} x(t+\tau)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t\Big|_{s=t+\tau} = \int_{\mathbb{R}} x(s)\mathrm{e}^{-\mathrm{j}\omega(s-\tau)}\mathrm{d}s \\
&= \mathrm{e}^{\mathrm{j}\omega\tau}\int_{\mathbb{R}} x(s)\mathrm{e}^{-\mathrm{j}\omega s}\mathrm{d}s \\
&= \mathrm{e}^{\mathrm{j}\omega\tau}X(\mathrm{j}\omega)
\end{aligned}
$$

Basic properties: time scaling

If $y = \mathbb{P}_\varsigma x$, i.e. $y(t) = x(\varsigma t)$, then

$$Y(\mathrm{j}\omega) = \int_{\mathbb{R}} x(\varsigma t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t\Big|_{s=\varsigma t} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} x(s)\mathrm{e}^{-\mathrm{j}\omega s/\varsigma}\mathrm{d}(s/\varsigma) & \text{if } \varsigma > 0 \\ \displaystyle\int_{\infty}^{-\infty} x(s)\mathrm{e}^{-\mathrm{j}\omega s/\varsigma}\mathrm{d}(s/\varsigma) & \text{if } \varsigma < 0 \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{\varsigma}\int_{-\infty}^{\infty} x(s)\mathrm{e}^{-\mathrm{j}(\omega/\varsigma)s}\mathrm{d}s & \text{if } \varsigma > 0 \\ \displaystyle-\frac{1}{\varsigma}\int_{-\infty}^{\infty} x(s)\mathrm{e}^{-\mathrm{j}(\omega/\varsigma)s}\mathrm{d}s & \text{if } \varsigma < 0 \end{cases}$$

$$= \frac{1}{|\varsigma|}\int_{\mathbb{R}} x(s)\mathrm{e}^{-\mathrm{j}(\omega/\varsigma)s}\mathrm{d}s$$

$$= \frac{1}{|\varsigma|}X(\mathrm{j}\omega/\varsigma)$$

Basic properties: conjugation

If $y = \overline{x}$ (complex conjugate), then

$$Y(\mathrm{j}\omega) = \int_{\mathbb{R}} \overline{x(t)} e^{-\mathrm{j}\omega t} dt = \int_{\mathbb{R}} \overline{x(t)\overline{e^{-\mathrm{j}\omega t}}} dt = \overline{\int_{\mathbb{R}} x(t) e^{\mathrm{j}\omega t} dt}$$
$$= \overline{\int_{\mathbb{R}} x(t) e^{-\mathrm{j}(-\omega) t} dt}$$
$$= \overline{X(-\mathrm{j}\omega)}$$

If $x : \mathbb{R} \to \mathbb{R}$ (real valued), then $y = \overline{x} = x$, so that

$$X(\mathrm{j}\omega) = \overline{X(-\mathrm{j}\omega)},$$

i.e. we may look only at the spectrum over $\omega \geq 0$. If, in addition, $x = \mathbb{P}_{-1}x$ (such functions are called *even*), then by the time scaling property

$$X(\mathrm{j}\omega) = X(-\mathrm{j}\omega) = \overline{X(-\mathrm{j}\omega)}.$$

Hence, if x is real valued and even, then X is real valued and even as well.

Basic properties: modulation

If $y = x \exp_{j\omega_0}$, i.e. $y(t) = x(t)e^{j\omega_0 t}$, then

$$\begin{aligned} Y(j\omega) &= \int_{\mathbb{R}} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{\mathbb{R}} x(t)e^{-j(\omega-\omega_0)t}dt \\ &= X(j(\omega-\omega_0)) \end{aligned}$$

By linearity and modulation, if

$$y(t) = \sin(\omega_0 t + \phi)x(t) = ae^{j\omega_0 t}x(t) + \bar{a}e^{-j\omega_0 t}x(t),$$

where $a := 0.5e^{j(\phi-\pi/2)}$, then

$$Y(j\omega) = aX(j(\omega-\omega_0)) + \bar{a}X(j(\omega+\omega_0))$$

Basic properties: differentiation

If $y = \dot{x}$, i.e. $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, and $\lim_{t\to\pm\infty} x(t) = 0$, then

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int_{\mathbb{R}} \dot{x}(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = x(t)\mathrm{e}^{-\mathrm{j}\omega t}\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} x(s)\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\mathrm{j}\omega t}\right)\mathrm{d}t \\ &= \mathrm{j}\omega \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \\ &= \mathrm{j}\omega\, X(\mathrm{j}\omega) \end{aligned}$$

because

$$\int_a^b \dot{f}(t)g(t)\mathrm{d}t = f(t)g(t)\Big|_a^b - \int_a^b f(t)\dot{g}(t)\mathrm{d}t$$

(integration by parts).

Similar arguments apply to the inverse Fourier, so

$$y(t) = -\mathrm{j}tx(t) \iff Y(\mathrm{j}\omega) = \tfrac{\mathrm{d}}{\mathrm{d}\omega}X(\mathrm{j}\omega)$$

Basic properties: convolution

If $z = x * y$, then

$$\begin{aligned} Z(\mathrm{j}\omega) &= \int_{\mathbb{R}} \int_{\mathbb{R}} x(t-s)y(s)\mathrm{d}s\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}} x(t-s)y(s)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t\,\mathrm{d}s \\ &= \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} x(t-s)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t}_{\mathfrak{F}\{\mathbb{S}_{-s}x\}}\, y(s)\mathrm{d}s \\ &= \int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega s} y(s)\mathrm{d}s \qquad \text{(by the time shift property)} \\ &= X(\mathrm{j}\omega)\int_{\mathbb{R}} y(s)\mathrm{e}^{-\mathrm{j}\omega s}\mathrm{d}s \\ &= X(\mathrm{j}\omega)Y(\mathrm{j}\omega) \end{aligned}$$

the (unproven) fact that $x, y \in L_1 \implies x * y \in L_1$ is required in this proof.

Parseval's theorem

Theorem

If $x \in L_1 \cap L_2$, then

$$\|x\|_2^2 = \int_{\mathbb{R}} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{\mathbb{R}} |X(\mathrm{j}\omega)|^2 \mathrm{d}\omega = \frac{1}{2\pi} \|X\|_2^2$$

In other words,

- the energy of x equals the energy of $\mathfrak{F}\{x\}$, modulo the factor $1/(2\pi)$.

Outline

Fourier series

Fourier transform

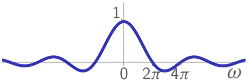
Fourier transforms of some standard signals

Rectangular pulse

If $x = \mathrm{rect}$, then

$$X(\mathrm{j}\omega) = \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{\mathrm{e}^{-\mathrm{j}\omega t}}{-\mathrm{j}\omega}\bigg|_{-1/2}^{1/2} = \frac{\mathrm{e}^{\mathrm{j}\omega/2} - \mathrm{e}^{-\mathrm{j}\omega/2}}{\mathrm{j}\omega}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\omega/2} - \mathrm{e}^{-\mathrm{j}\omega/2}}{2\mathrm{j}\,\omega/2} = \frac{\sin(\omega/2)}{\omega/2} = \mathrm{sinc}(\omega/2) =$$

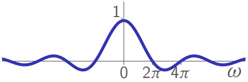


Rectangular pulse

If $x = \mathrm{rect}$, then

$$X(\mathrm{j}\omega) = \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t = \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t = \frac{\mathrm{e}^{-\mathrm{j}\omega t}}{-\mathrm{j}\omega}\bigg|_{-1/2}^{1/2} = \frac{\mathrm{e}^{\mathrm{j}\omega/2} - \mathrm{e}^{-\mathrm{j}\omega/2}}{\mathrm{j}\omega}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\omega/2} - \mathrm{e}^{-\mathrm{j}\omega/2}}{2\mathrm{j}\,\omega/2} = \frac{\sin(\omega/2)}{\omega/2} = \mathrm{sinc}(\omega/2) =$$



Consequences:

– by time scaling, if $y = \mathrm{rect}_a$ for some $a > 0$, then

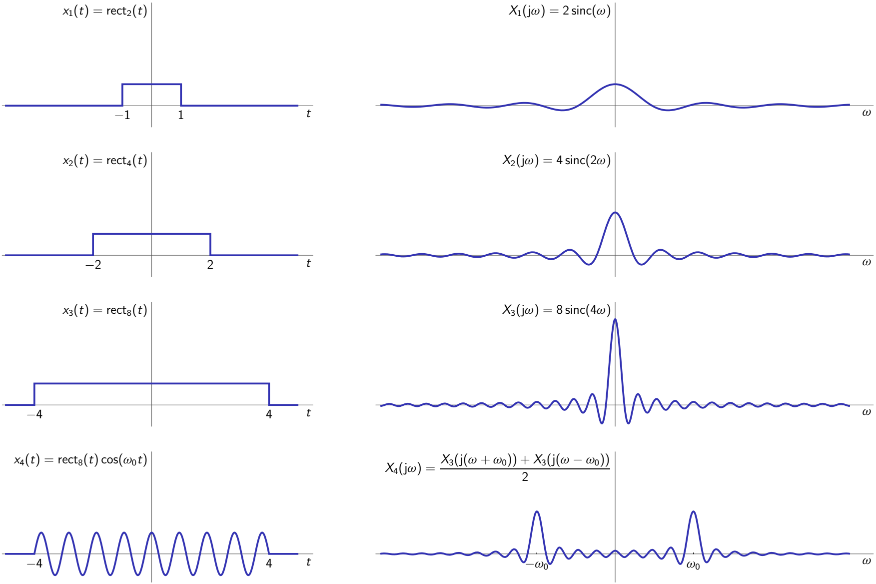
$$Y(\mathrm{j}\omega) = (\mathfrak{F}\{\mathbb{P}_{1/a}\,\mathrm{rect}\})(\omega) = a\,\mathrm{sinc}\left(\tfrac{a}{2}\omega\right) = a(\mathbb{P}_{1/2}\,\mathrm{sinc})(\omega)$$

– by duality and time scaling, if $y = \mathbb{P}_{1/2}\,\mathrm{sinc}$ and $z = \mathbb{P}_{\varsigma}\,\mathrm{sinc}$, then

$$Y(\mathrm{j}\omega) = 2\pi\,\mathrm{rect}(\omega) \quad \text{and} \quad Z(\mathrm{j}\omega) = \frac{\pi}{\varsigma}\,\mathrm{rect}_{2\varsigma}(\omega)$$

(mind that $\mathrm{rect}(-\omega) = \mathrm{rect}(\omega)$ and $\mathbb{P}_{\varsigma}\,\mathrm{sinc} = \mathbb{P}_{2\varsigma}\mathbb{P}_{1/2}\,\mathrm{sinc}$).

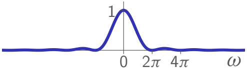
Rectangular pulse: examples



Triangular pulse

Remember (Lect. 2, Slide 12) that $\mathrm{tent} = \mathrm{rect} * \mathrm{rect}$. So if $x = \mathrm{tent}$, then

$$X(\mathrm{j}\omega) = \big(\mathfrak{F}\{\mathrm{rect} * \mathrm{rect}\}\big)(\mathrm{j}\omega) = \mathrm{sinc}^2(\omega/2) =$$



by the convolution property of the Fourier transform.

Consequence:

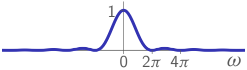
– by time scaling, if $y = \mathrm{tent}_a$ for some $a > 0$, then

$$Y(\mathrm{j}\omega) = \big(\mathfrak{F}\{\mathbb{P}_{1/a}\,\mathrm{tent}\}\big)(\omega) = a\,\mathrm{sinc}^2\big(\tfrac{a}{2}\omega\big)$$

Triangular pulse

Remember (Lect. 2, Slide 12) that tent = rect ∗ rect. So if $x = \text{tent}$, then

$$X(\mathrm{j}\omega) = \big(\mathfrak{F}\{\text{rect} * \text{rect}\}\big)(\mathrm{j}\omega) = \text{sinc}^2(\omega/2) =$$



by the convolution property of the Fourier transform.

Consequence:

– by time scaling, if $y = \text{tent}_a$ for some $a > 0$, then

$$Y(\mathrm{j}\omega) = (\mathfrak{F}\{\mathbb{P}_{1/a}\,\text{tent}\})(\omega) = a\,\text{sinc}^2\left(\tfrac{a}{2}\omega\right)$$

Exponent with support in $\mathbb{R}_+$

If $x = \exp_\lambda \mathbb{1}$ for $\lambda \in \mathbb{R}$, then $x \in L_1$ iff $\lambda < 0$ and in that case

$$X(\mathrm{j}\omega) = \int_{\mathbb{R}} \mathrm{e}^{\lambda t} \mathbb{1}(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t = \int_{\mathbb{R}_+} \mathrm{e}^{(\lambda - \mathrm{j}\omega)t} \mathrm{d}t = \frac{\mathrm{e}^{(\lambda-\mathrm{j}\omega)t}}{\lambda - \mathrm{j}\omega}\bigg|_0^\infty = \frac{1}{\mathrm{j}\omega - \lambda}$$

The same formula holds if $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda < 0$.

Consequences:

- by differentiation with respect to frequency,

$$y(t) = \frac{t^n}{n!} \mathrm{e}^{\lambda t} \mathbb{1}(t) \iff Y(\mathrm{j}\omega) = \frac{1}{(\mathrm{j}\omega - \lambda)^{n+1}}$$

- by linearity and modulation, if $y(t) = \sin(\omega_0 t + \phi) \mathrm{e}^{\lambda t} \mathbb{1}(t)$, then

$$Y(\mathrm{j}\omega) = \frac{\omega_0 \cos\phi + (\mathrm{j}\omega - \lambda)\sin\phi}{(\mathrm{j}\omega - \lambda)^2 + \omega_0^2}$$

Exponent with support in $\mathbb{R}_+$

If $x = \exp_\lambda \mathbb{1}$ for $\lambda \in \mathbb{R}$, then $x \in L_1$ iff $\lambda < 0$ and in that case

$$X(\mathrm{j}\omega) = \int_{\mathbb{R}} \mathrm{e}^{\lambda t}\mathbb{1}(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{\mathbb{R}_+} \mathrm{e}^{(\lambda-\mathrm{j}\omega)t}\mathrm{d}t = \frac{\mathrm{e}^{(\lambda-\mathrm{j}\omega)t}}{\lambda-\mathrm{j}\omega}\Big|_0^\infty = \frac{1}{\mathrm{j}\omega-\lambda}$$

The same formula holds if $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda < 0$.

Consequences:

- by differentiation with respect to frequency,

$$y(t) = \frac{t^n}{n!}\mathrm{e}^{\lambda t}\mathbb{1}(t) \iff Y(\mathrm{j}\omega) = \frac{1}{(\mathrm{j}\omega-\lambda)^{n+1}}$$

- by linearity and modulation, if $y(t) = \sin(\omega_0 t + \phi)\mathrm{e}^{\lambda t}\mathbb{1}(t)$, then

$$Y(\mathrm{j}\omega) = \frac{\omega_0\cos\phi + (\mathrm{j}\omega-\lambda)\sin\phi}{(\mathrm{j}\omega-\lambda)^2+\omega_0^2}$$

Dirac delta (not quite rigorous, still true)

If $x = \delta$, then by the sifting property

$$X(\mathrm{j}\omega) = \int_{\mathbb{R}} \delta(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \mathrm{e}^{-\mathrm{j}\omega t}\big|_{t=0} = 1$$

i.e. $\mathfrak{F}\{\delta\}$ contains all harmonics with equal weights, $1/(2\pi)$.

Consequences:

- by duality and the fact that $\delta(\omega) = \delta(-\omega)$, if $y = 1$ (constant), then

$$Y(\mathrm{j}\omega) = 2\pi\delta(\omega)$$

- by modulation, if $y = \exp_{\mathrm{j}\omega_0}$, then

$$Y(\mathrm{j}\omega) = 2\pi\delta(\omega - \omega_0)$$

- by linearity, if $y(t) = \sin(\omega_0 t + \phi)$, then

$$Y(\mathrm{j}\omega) = \pi e^{\mathrm{j}(\pi/2-\phi)}\delta(\omega + \omega_0) - \pi e^{\mathrm{j}(\pi/2+\phi)}\delta(\omega - \omega_0)$$

Dirac delta (not quite rigorous, still true)

If $x = \delta$, then by the sifting property

$$X(\mathrm{j}\omega) = \int_{\mathbb{R}} \delta(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \mathrm{e}^{-\mathrm{j}\omega t}\big|_{t=0} = 1$$

i.e. $\mathfrak{F}\{\delta\}$ contains all harmonics with equal weights, $1/(2\pi)$.

Consequences:

– by duality and the fact that $\delta(\omega) = \delta(-\omega)$, if $y = 1$ (constant), then

$$Y(\mathrm{j}\omega) = 2\pi\delta(\omega)$$

– by modulation, if $y = \exp_{\mathrm{j}\omega_0}$, then

$$Y(\mathrm{j}\omega) = 2\pi\delta(\omega - \omega_0)$$

– by linearity, if $y(t) = \sin(\omega_0 t + \phi)$, then

$$Y(\mathrm{j}\omega) = \pi\,\mathrm{e}^{\mathrm{j}(\pi/2-\phi)}\delta(\omega + \omega_0) - \pi\,\mathrm{e}^{\mathrm{j}(\pi/2+\phi)}\delta(\omega - \omega_0)$$

Step (no proofs, still true)

If $x = \mathbb{1}$, then

$$X(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)$$

(think of $\mathbb{1} = \lim_{\lambda \uparrow 0} \exp_\lambda \mathbb{1}$ and $1 = \mathbb{1} + \mathbb{P}_{-1}\mathbb{1}$ as a kind of weak rationale).

Consequence:

- if $y(t) = \int_{-\infty}^{t} x(t)\,\mathrm{d}t$, then

$$Y(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega} X(\mathrm{j}\omega) + \pi X(0)\delta(\omega)$$

  because $y = \mathbb{1} * x$ (Lect. 2, Slide 12), the convolution property of the Fourier transform, and the fact that $x\delta = x(0)\delta$ (Lect. 2, Slide 16).

Step (no proofs, still true)

If $x = \mathbb{1}$, then

$$X(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)$$

(think of $\mathbb{1} = \lim_{\lambda \uparrow 0} \exp_\lambda \mathbb{1}$ and $1 = \mathbb{1} + \mathbb{P}_{-1}\mathbb{1}$ as a kind of weak rationale).

Consequence:

– if $y(t) = \int_{-\infty}^{t} x(t)\mathrm{d}t$, then

$$Y(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega} X(\mathrm{j}\omega) + \pi X(0)\delta(\omega)$$

because $y = \mathbb{1} * x$ (Lect. 2, Slide 12), the convolution property of the Fourier transform, and the fact that $x\delta = x(0)\delta$ (Lect. 2, Slide 16).