Linear Systems (034032)
lecture no. 2

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Outline

Continuous-time signals

Discrete-time signals

From continuous to discrete and back again

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Discrete-time signals

From continuous to discrete and back again

Basic definitions

Continuous-time signals are functions with domains in (a subset of) $\mathbb{R}$, like

$$\mathbb{R}, \quad \mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geq 0\}, \quad [a, b] := \{t \in \mathbb{R} \mid a \leq t \geq b\},$$

and where the independent variable is understood as the continuous time[1]. A subset of the domain in which a signal is nonzero is called its support,

$$\operatorname{supp}(x) := \{t \in \mathbb{R} \mid x(t) \neq 0\}$$

A signal x is said to be

- scalar-valued if the codomain is a scalar, like $\mathbb{R}$ or $\mathbb{C}$ (we use $\mathbb{F}$ if either)
- vector-valued if the codomain is a vector, like $\mathbb{R}^n$ or $\mathbb{C}^m$
- decaying if $\lim_{t\to\infty} x(t) = 0$
- converging if $\lim_{t\to\infty} x(t) = x_{ss}$ for some constant x_{ss} from its codomain

[1] Normally, denoted t, although this is not essential.

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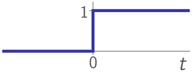
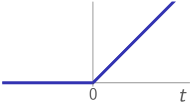
Elementary signals

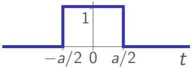
step: $\mathbb{1}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} =$



ramp: $\text{ramp}(t) = t\mathbb{1}(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} =$

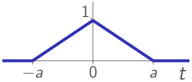


rectangular pulse: $\text{rect}_a(t) = \begin{cases} 1 & \text{if } |t| < a/2 \\ 0 & \text{if } |t| > a/2 \end{cases} =$



$a > 0$

triangular pulse: $\text{tent}_a(t) = \begin{cases} 1 - \frac{|t|}{a} & \text{if } |t| \leq a \\ 0 & \text{if } |t| > a \end{cases} =$



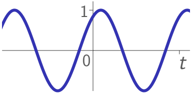
$a > 0$

We use shortcuts

$$\text{rect} := \text{rect}_1 \quad \text{and} \quad \text{tent} := \text{tent}_1.$$

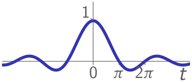
Elementary signals (contd)

sinusoidal: $\sin(\omega t + \phi) =$

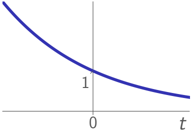


frequency $\omega \geq 0$, phase ϕ

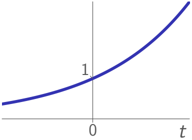
sine cardinal: $\operatorname{sinc}(t) = \dfrac{\sin(t)}{t} =$



exponential: $\exp_\lambda(t) = \exp(\lambda t) = e^{\lambda t} =$



if $\lambda < 0$



if $\lambda > 0$

Operations on signals: (amplitude) scaling

Given $x : \mathbb{R} \to \mathbb{F}$ and $a \in \mathbb{F}$, the signal $ax : \mathbb{R} \to \mathbb{F}$ (or $a \cdot x$) satisfies

$$(ax)(t) = ax(t), \quad \forall t \in \mathbb{R}$$

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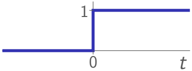
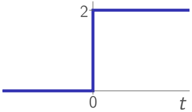
Examples

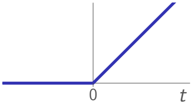
$\mathbb{1} =$



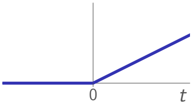
$\Longrightarrow$ $2\mathbb{1} =$



$\text{ramp} =$



$\Longrightarrow$ $\frac{1}{2}\text{ramp} =$



Operations on signals: addition

Given $x : \mathbb{R} \to \mathbb{F}$ and $y : \mathbb{R} \to \mathbb{F}$, the signal $x + y : \mathbb{R} \to \mathbb{F}$ satisfies

$$(x + y)(t) = x(t) + y(t), \quad \forall t \in \mathbb{R}$$

with $x - y = x + (-1 \cdot y)$.

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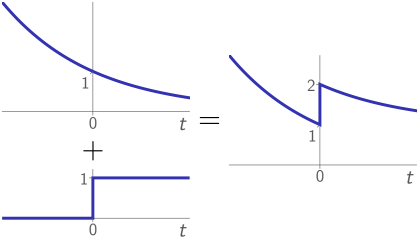
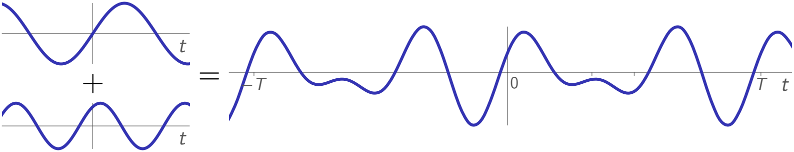
Examples

$\exp_\lambda + \mathbb{1} =$



$x + y =$



Operations on signals: multiplication

Given $x : \mathbb{R} \to \mathbb{F}$ and $y : \mathbb{R} \to \mathbb{F}$, the signal $xy : \mathbb{R} \to \mathbb{F}$ (or $x \cdot y$) satisfies

$$(xy)(t) = x(t)y(t), \quad \forall t \in \mathbb{R}$$

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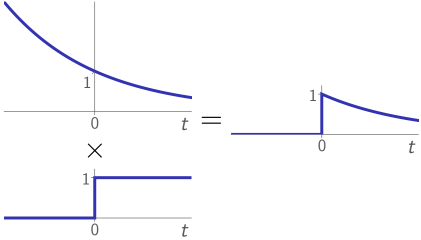
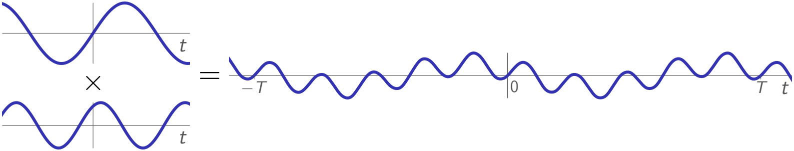


$xy =$



Operations on signals: time scale (pace change)

Given $x : \mathbb{R} \to \mathbb{F}$ and $\varsigma \in \mathbb{R} \setminus \{0\}$, the signal $\mathbb{P}_\varsigma x : \mathbb{R} \to \mathbb{F}$ satisfies

$$(\mathbb{P}_\varsigma x)(t) = x(\varsigma t), \quad \forall t \in \mathbb{R}$$

Commutativity property: $\mathbb{P}_{\varsigma_1}(\mathbb{P}_{\varsigma_2} x) = \mathbb{P}_{\varsigma_1 \varsigma_2} x = \mathbb{P}_{\varsigma_2}(\mathbb{P}_{\varsigma_1} x)$

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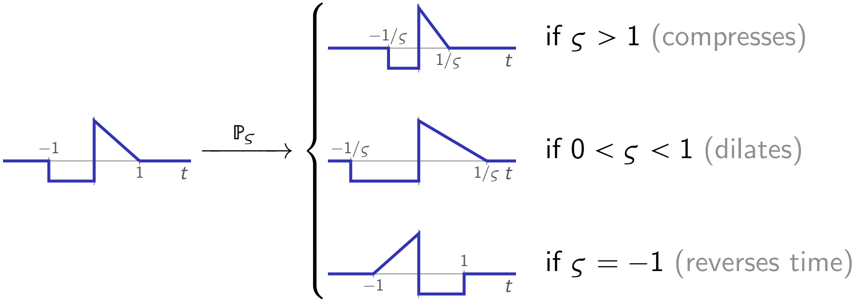
– 

– $\mathbb{P}_a \text{rect}_a = \text{rect} := \text{rect}_1$ and $\mathbb{P}_a \text{tent}_a = \text{tent} := \text{tent}_1$

Operations on signals: time shift

Given $x : \mathbb{R} \to \mathbb{F}$ and $\tau \in \mathbb{R}$, the signal $\mathbb{S}_\tau x : \mathbb{R} \to \mathbb{F}$ satisfies

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Commuting $\mathbb{S}_\tau$ and $\mathbb{P}_\varsigma$: $\mathbb{P}_\varsigma(\mathbb{S}_\tau x) = \mathbb{S}_{\tau/\varsigma}(\mathbb{P}_\varsigma x)$

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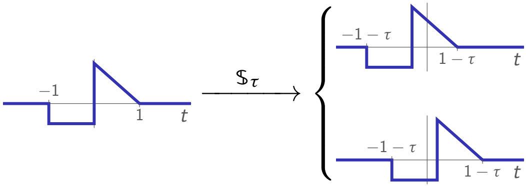
– 

if $\tau > 0$ (predicts)

if $\tau < 0$ (delays)

– $\mathbb{S}_{a/2}\mathbb{1} - \mathbb{S}_{-a/2}\mathbb{1} = \text{rect}_a$

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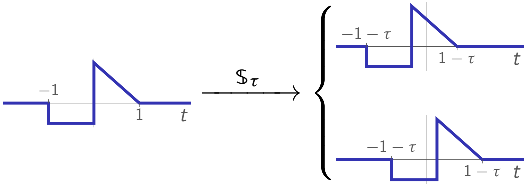
– 

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Operations on signals: convolution

Given $x : \mathbb{R} \to \mathbb{F}$ and $y : \mathbb{R} \to \mathbb{F}$, the signal $x * y : \mathbb{R} \to \mathbb{F}$ satisfies

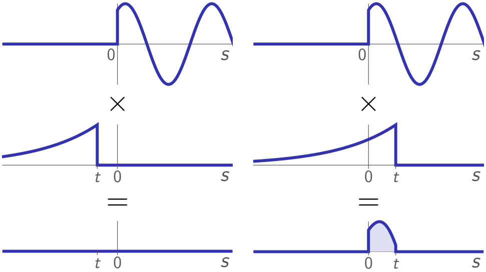
$$(x * y)(t) = \int_{\mathbb{R}} x(s)y(t-s)\mathrm{d}s = \int_{\mathbb{R}} x(t-s)y(s)\mathrm{d}s, \quad \forall t \in \mathbb{R}$$

like

$$x(s) =$$

$$y(t-s) = \begin{matrix} (\mathbb{P}_{-1}(\mathbb{S}_t y))(s) \\ (\mathbb{S}_{-t}(\mathbb{P}_{-1} y))(s) \end{matrix} =$$

$$x(s)y(t-s) =$$



Properties:

- $x * y = y * x$
- $(ax) * y = a(x * y)$
- $x * (y * z) = (x * y) * z$
- $(x + y) * z = x * z + y * z$

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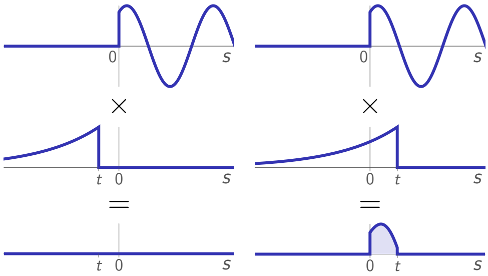

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Convolution: examples

Convolution with step:

$$(\mathbb{1} * x)(t) = \int_{\mathbb{R}} \mathbb{1}(t-s)x(s)\mathrm{d}s = \int_{-\infty}^{t} x(s)\mathrm{d}s$$

Convolution with rectangular pulse:

$$(\mathrm{rect}_a * x)(t) = \int_{\mathbb{R}} \mathrm{rect}_a(t-s)x(s)\mathrm{d}s = \int_{t-a/2}^{t+a/2} x(s)\mathrm{d}s$$

Convolution of rectangular pulses:

$$(\mathrm{rect}_a * \mathrm{rect}_a)(t) = \int_{t-a/2}^{t+a/2} \mathrm{rect}_a(s)\mathrm{d}s = \begin{cases} 0 & \text{if } |t| \geq a \\ \int_{-a/2}^{t+a/2} \mathrm{d}s & \text{if } -a < t \leq 0 \\ \int_{t-a/2}^{a/2} \mathrm{d}s & \text{if } 0 \leq t < a \end{cases}$$

$$= a\,\mathrm{tent}_a$$

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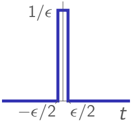
Dirac delta: naïve definition

Consider the family of signals d_ϵ such that

$$d_\epsilon(t) = \frac{1}{\epsilon}\mathrm{rect}_\epsilon(t) =$$



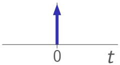
$$, \qquad \epsilon > 0$$

satisfying

$$\int_{\mathbb{R}} d_\epsilon(t)\mathrm{d}t = 1, \quad \forall\epsilon$$

Define now Dirac delta as

$$\delta := \lim_{\epsilon\downarrow 0} d_\epsilon =$$



(although this limit is mathematically problematic).

Dirac delta: integral and more formal definition

We already know that

$$\int_{\mathbb{R}} f(t) d_\epsilon(t) \mathrm{d}t = \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} f(t) \mathrm{d}t,$$

i.e. it equals the average value of f in the interval $t \in [-\epsilon/2, \epsilon/2]$. We may then expect that

$$\int_{\mathbb{R}} f(t) \delta(t) \mathrm{d}t = f(0)$$

whenever f is continuous at $t = 0$. This is actually a — defining property for the Dirac delta distribution (with some abuse of notation; a proper definition needs a measure notion).

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Dirac delta: sifting property

Immediately from the definition,

$$\int_{\mathbb{R}} f(t)\delta(t-t_0)\mathrm{d}t = \int_{\mathbb{R}} f(s+t_0)\delta(s)\mathrm{d}s = \int_{\mathbb{R}} (\mathbb{S}_{t_0} f)(s)\delta(s)\mathrm{d}s = (\mathbb{S}_{t_0} f)(0)$$
$$= f(t_0)$$

whenever f is continuous at $t = t_0$.

If x is continuous for all its domain, then

$$(x * \delta)(t) = \int_{\mathbb{R}} x(t-s)\delta(s)\mathrm{d}s = x(t), \quad \forall t$$

In other words,

$$x * \delta = x.$$

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In other words,

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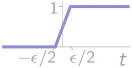
Dirac delta: more properties

- $\delta(t) = 0$ whenever $t \neq 0$
- given $a < b$, $\int_a^b \delta(t)\mathrm{d}t = \begin{cases} 0 & \text{if } a > 0 \vee b < 0 \\ 1 & \text{if } a < 0 \wedge b > 0 \end{cases}$
- $\mathbb{1}(t) = \int_{-\infty}^{t} \delta(s)\mathrm{d}s$, for all t
- $\delta = \dot{\mathbb{1}}$

think of $\mathbb{1} = \lim_{\epsilon \downarrow 0}$ 

- $a\delta$: $\int_{\mathbb{R}} f(t)(a\delta)(t)\mathrm{d}t = af(0)$ whenever f is continuous at $t = 0$
- $f\delta = f(0)\delta$ whenever f is continuous at $t = 0$

Size matters

We frequently need to decide on whether a signal is ‘large’ or ‘small’, think of

- measurements — how accurate a measurement is?
- precipitation level — was it a wet winter?
- blood sugar level — is it normal?
- . . .

Signal sizes are measured by norms, which are functions satisfying

1. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$ — positive definiteness
2. $\|ax\| = |a|\|x\|, \ \forall a \in \mathbb{F}$ — homogeneity
3. $\|x + y\| \leq \|x\| + \|y\|$ — triangle inequality

If the second condition of 1. does not hold, i.e. if $\|x\| = 0$ for certain $x \neq 0$, then the function is called semi-norm.

Size matters

We frequently need to decide on whether a signal is ‘large’ or ‘small’, think of

- measurements — how accurate a measurement is?
- precipitation level — was it a wet winter?
- blood sugar level — is it normal?
- ...

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Commonly used norms

L_1 norm

$$\|x\|_1 := \int_{\mathbb{R}} |x(t)| \mathrm{d}t$$

If $\|x\|_1 < \infty$, then we say that $x \in L_1$ and call it *absolutely integrable*.

L_2 norm

$$\|x\|_2 := \left(\int_{\mathbb{R}} |x(t)|^2 \mathrm{d}t \right)^{1/2}$$

If $\|x\|_2 < \infty$, then we say that $x \in L_2$ and call it *square integrable*.

L_∞ norm

$$\|x\|_\infty := \sup_{t \in \mathbb{R}} |x(t)|$$

If $\|x\|_\infty < \infty$, then we say that $x \in L_\infty$ and call it *bounded*.

Norms: (lack of) equivalence

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$$x \in L_1 \ (\|x\|_1 = \tfrac{1}{-\lambda}), \quad x \in L_2 \ (\|x\|_2 = \tfrac{1}{\sqrt{-2\lambda}}), \quad x \in L_\infty \ (\|x\|_\infty = 1)$$

- if $x = \mathrm{sinc}$

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- if $x = \operatorname{sinc}$, then

$$x \notin L_1, \quad \text{but} \quad x \in L_2 \ (\|x\|_2 = \sqrt{\pi}) \quad \text{and} \quad x \in L_\infty \ (\|x\|_\infty = 1)$$

- if $x = \mathbb{1}$, then

$$x \notin L_1 \quad \text{and} \quad x \notin L_2, \quad \text{but} \quad x \in L_\infty \ (\|x\|_\infty = 1)$$

- if $x = d_\epsilon$, then

$$x \in L_1 \ (\|x\|_1 = 1), \quad x \in L_2 \ (\|x\|_2 = \tfrac{1}{\sqrt{\epsilon}}), \quad x \in L_\infty \ (\|x\|_\infty = \tfrac{1}{\epsilon})$$

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Other measures of sizes

Energy

$$E_x := \int_{\mathbb{R}} |x(t)|^2 \mathrm{d}t = \|x\|_2^2$$

Power (energy per unit time)

$$P_x := \lim_{M\to\infty} \frac{1}{M} \int_{-M/2}^{M/2} |x(t)|^2 \mathrm{d}t$$

Properties:

- $E_x < \infty$ (finite-energy signals) $\implies P_x = 0$ $\sqrt{P_x}$ is a semi-norm
- x is bounded and have finite support $\implies E_x$ is finite and $P_x = 0$
- $E_{ax} = a^2 E_x$ and $P_{ax} = a^2 P_x$ for every $a \in \mathbb{R}$

Periodic signals

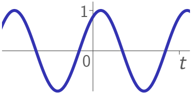
We say that x is T-periodic if

- $\exists T > 0$ such that $x(t) = x(t+T)$ for all t

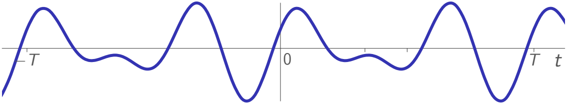
(otherwise, aperiodic). If x is T-periodic, then it's also kT-periodic $\forall k \in \mathbb{N}$. We normally refer to the smallest such T as the period.

Examples:

- if $x(t) = \sin(\omega t + \phi)$, then x is $\frac{2\pi}{\omega}$-periodic

$x =$ 

- if $x(t) = a_1 \sin(2\omega_0 t + \phi_1) + a_2 \sin(3\omega_0 t + \phi_2)$, then x is $\frac{2\pi}{\omega_0}$-periodic

$x =$ 

Periodic signals: integral over a period

If x is T-periodic, then

$$\begin{aligned}\int_a^{a+T} x(t)\mathrm{d}t &= \int_a^0 x(t)\mathrm{d}t + \int_0^T x(t)\mathrm{d}t + \int_T^{a+T} x(t)\mathrm{d}t\Big|_{t=s+T} \\ &= -\int_0^a x(t)\mathrm{d}t + \int_0^T x(t)\mathrm{d}t + \int_0^a x(s+T)\mathrm{d}s \\ &= -\int_0^a x(t)\mathrm{d}t + \int_0^T x(t)\mathrm{d}t + \int_0^a x(s)\mathrm{d}s \\ &= \int_0^T x(t)\mathrm{d}t\end{aligned}$$

for every $a \in \mathbb{R}$.

Power of periodic signals

If x is T-periodic, then

$$
\begin{aligned}
P_x &= \lim_{k\to\infty} \frac{1}{kT} \int_{-kT/2}^{kT/2} |x(t)|^2 \mathrm{d}t = \lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \int_{iT-kT/2}^{iT-kT/2+T} |x(t)|^2 \mathrm{d}t \\
&= \lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \int_0^T |x(t)|^2 \mathrm{d}t = \lim_{k\to\infty} \frac{1}{kT} k \int_0^T |x(t)|^2 \mathrm{d}t \\
&= \frac{1}{T} \int_0^T |x(t)|^2 \mathrm{d}t \le \max_{0\le t\le T} |x(t)|^2
\end{aligned}
$$

If $x(t) = \sin(\omega t + \phi)$, then $T = 2\pi/\omega$ and

$$
\begin{aligned}
P_x &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sin^2(\omega t + \phi) \mathrm{d}t = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \frac{1 - \cos(2\omega t + 2\phi)}{2} \mathrm{d}t \\
&= \frac{\omega}{4\pi} \int_0^{2\pi/\omega} \mathrm{d}t = \frac{1}{2}
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since $\sin^2\theta = (1 - \cos(2\theta))/2$ and the integral of cos over a period is zero

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Outline

Continuous-time signals

Discrete-time signals

From continuous to discrete and back again

Basic definitions

Discrete-time signals are functions with domains in (a subset of) $\mathbb{Z}$, like

$$\mathbb{Z}, \quad \mathbb{Z}_+ := \{t \in \mathbb{Z} \mid t \geq 0\}, \quad \mathbb{N} := \{t \in \mathbb{Z} \mid t \geq 1\},$$
$$\mathbb{Z}_{a..b} := \{t \in \mathbb{Z} \mid a \leq t \geq b\},$$

and where the independent variable is understood as the discrete time. A subset of the domain in which a signal is nonzero is called its support, e.g.

$$\operatorname{supp}(x) := \{t \in \mathbb{Z} \mid x[t] \neq 0\}$$

A signal x is said to be

- **scalar-valued** if the codomain is a scalar, like $\mathbb{R}$ or $\mathbb{C}$ (we use $\mathbb{F}$ if either)
- **vector-valued** if the codomain is a vector, like $\mathbb{R}^n$ or $\mathbb{C}^m$
- **decaying** if $\lim_{t\to\infty} x[t] = 0$
- **converging** if $\lim_{t\to\infty} x[t] = x_{ss}$ for some constant x_{ss} from its codomain
- **periodic** if $\exists T \in \mathbb{N}$ such that $x[t] = x[t+T]$ for all t

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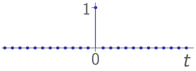
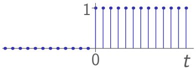
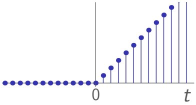
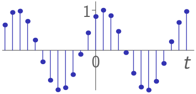
Elementary signals

pulse: $\delta[t] = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} =$ 

step: $\mathbb{1}[t] = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} =$ 

ramp: $\text{ramp}[t] = t\mathbb{1}[t] = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} =$ 

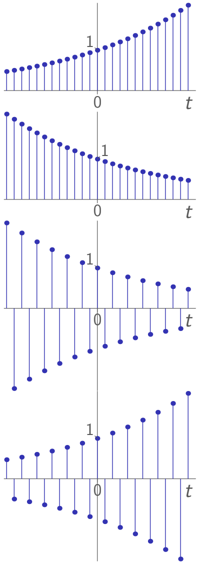
sinusoidal[2]: $\sin[\theta t + \phi] =$ 

frequency $\theta \geq 0$, phase ϕ

[2]Periodic only if $2\pi/\theta \in \mathbb{Q}$ (rational), the period equals $2\pi/|\theta|$ only if $2\pi/|\theta| \in \mathbb{N}$.

Elementary signals (contd)

exponential: $\exp_\lambda[t] = \lambda^t =$



- if $\lambda > 1$
- if $0 < \lambda < 1$
- if $-1 < \lambda < 0$
- if $\lambda < -1$

Operations on discrete-time signals

Exactly as those on continuous-time signals[3], mutatis mutandis.

Convolution: given $x : \mathbb{Z} \to \mathbb{F}$ and $y : \mathbb{Z} \to \mathbb{F}$, signal $x * y : \mathbb{Z} \to \mathbb{F}$ satisfies

$$(x * y)[t] = \sum_{s \in \mathbb{Z}} x[s]y[t-s] = \sum_{s \in \mathbb{Z}} x[t-s]y[s], \quad \forall t \in \mathbb{Z}$$

Properties:

- $x * y = y * x$
- $(ax) * y = a(x * y)$
- $x * (y * z) = (x * y) * z$
- $(x + y) * z = x * z + y * z$
- $\delta * x = x$ — similar to the Dirac delta

Convolution with step:

$$(\mathbb{1} * x)[t] = \sum_{s \in \mathbb{Z}} \mathbb{1}[t-s]x[s] = \sum_{s=-\infty}^{t} x[s]$$

[3] The only exception is the time scaling, which is not well defined in the discrete time.

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Commonly used norms for discrete signals

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$$\|x\|_1 := \sum_{t \in \mathbb{Z}} |x[t]|$$

If $\|x\|_1 < \infty$, then we say that $x \in \ell_1$ and call it *absolutely summable*.

ℓ_2 norm

$$\|x\|_2 := \left(\sum_{t \in \mathbb{Z}} |x[t]|^2 \right)^{1/2}$$

If $\|x\|_2 < \infty$, then we say that $x \in \ell_2$ and call it *square summable*.

ℓ_∞ norm

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If $\|x\|_\infty < \infty$, then we say that $x \in \ell_\infty$ and call it *bounded*.

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– if $x = \exp_\lambda$ for $|\lambda| < 1$, then

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– if $x[t] = 1/(1+|t|)$, then

$$x \notin \ell_1, \quad \text{but} \quad x \in \ell_2 \; (\|x\|_2 = \sqrt{\tfrac{\pi^2}{3} - 1}) \quad \text{and} \quad x \in \ell_\infty \; (\|x\|_\infty = 1)$$

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- if $x = \mathbb{1}$, then

$$x \notin \ell_1 \quad \text{and} \quad x \notin \ell_2, \quad \text{but} \quad x \in \ell_\infty \ (\|x\|_\infty = 1)$$

In the discrete case $x \in \ell_1 \implies x \in \ell_2 \implies x \in \ell_\infty$.

Other measures of sizes

Energy

$$E_x := \sum_{t\in\mathbb{Z}} |x[t]|^2 = \|x\|_2^2$$

Power (energy per step)

$$P_x := \lim_{M\to\infty} \frac{1}{2M} \sum_{t=-M}^{M} |x[t]|^2$$

Properties:

- $E_x < \infty$ (finite-energy signals) $\implies$ $P_x = 0$ $\sqrt{P_x}$ is a semi-norm
- x is bounded and have finite support $\implies$ E_x is finite and $P_x = 0$
- $E_{ax} = a^2 E_x$ and $P_{ax} = a^2 P_x$ for every $a \in \mathbb{R}$

Outline

Continuous-time signals

Discrete-time signals

From continuous to discrete and back again

A/D conversion

A conversion of a continuous-time (analog) signal, say x, to a discrete-time (digital) signal, say $\bar{x}$, is known as sampling. If for all $i \in \mathbb{Z}$

$$\bar{x}[i] = x(s_i), \quad s_i < s_{i+1}$$

then the term ideal sampling is used.

Terminology:

- time instances s_i are called sampling instances
- if $s_i = ih$ for some $h > 0$, we say that the sampling is periodic and call h the sampling period

This ideal sampling operation is

- well defined only if x is continuous at each sampling instance s_i

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Some other sampling algorithms

- averaging sampling

$$\bar{x}[i] = \frac{1}{s_i - s_{i-1}} \int_{s_{i-1}}^{s_i} x(t)\mathrm{d}t \quad \text{or} \quad \bar{x}[i] = \frac{1}{\epsilon} \int_{s_i-\epsilon}^{s_i} x(t)\mathrm{d}t$$

- BoI sampling

  each representative rate is calculated as

  - average of randomly taken samples of banks rates in the last 2 hours prior publishing (either 3:15pm or 12:15pm), excluding those deviating from the sample average by more than two standard deviations
  - the same as in the previous day on Saturdays, Sundays, and holidays
  - exercising discretion in exceptional cases

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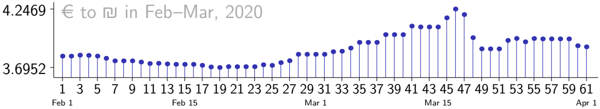

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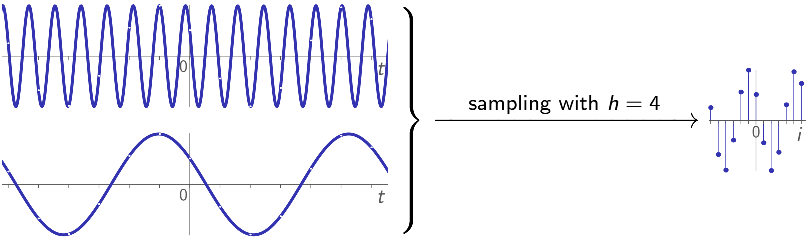
A/D conversion: information loses

Sampling is frequently (but not always) a

- lossy process

in which some information about the analog signal x is lost. For example,



and there is no way to recover the source (unless additional information is available).

D/A conversion

A conversion of a discrete-time (digital) signal, say $\bar{x}$, to a continuous-time (analog) signal, say x, is known as hold (interpolation). Common choices:
zero-order hold (ZOH) acts as

$$x(t) = \bar{x}[i], \quad \forall t \in (s_i, s_{i+1})$$

first-order hold (FOH or linear interpolator) acts as

$$x(t) = \bar{x}[i] + \frac{t - s_i}{s_{i+1} - s_i}(\bar{x}[i+1] - \bar{x}[i]), \quad \forall t \in (s_i, s_{i+1})$$

for given sampling instances s_i. For example,

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