Linear Control Systems (036012)
chapter 9

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Outline

Outline

Nobody's perfect

In other words, any

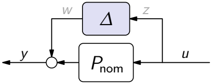
– mathematical model is merely a (more / less accurate) approximation

of the real world.

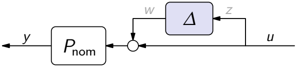
Unstructured uncertainty models

Under an arbitrary stable and norm-bounded Δ:

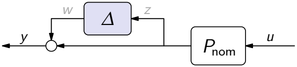
Additive $P = P_{\text{nom}} + \Delta$:



Input multiplicative $P = P_{\text{nom}}(I + \Delta)$:



Output multiplicative $P = (I + \Delta)P_{\text{nom}}$:



Inverse additive $P = (I + P_{\text{nom}}\Delta)^{-1}P_{\text{nom}}$:

Inverse input multiplicative $P = P_{\text{nom}}(I + \Delta)^{-1}$:

Inverse output multiplicative $P = (I + \Delta)^{-1}P_{\text{nom}}$:

Unstructured uncertainty models

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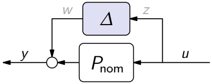

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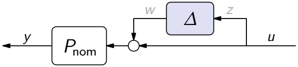
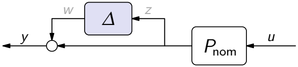


Output multiplicative $P = (I + \Delta)P_{\text{nom}}$:



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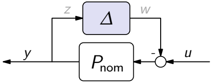

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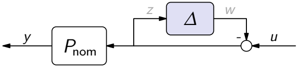
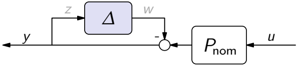
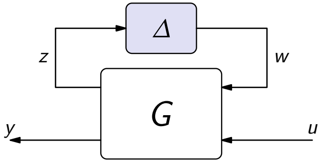


Inverse output multiplicative $P = (I + \Delta)^{-1}P_{\text{nom}}$:



General form



Plant:

$$P = \mathcal{F}_{\mathrm{u}}(G, \Delta).$$

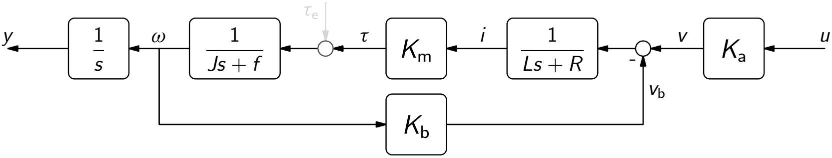
For the particular cases above:

$$G_{\mathrm{a}} = \begin{bmatrix} 0 & I \\ I & P_{\mathrm{nom}} \end{bmatrix}, \qquad G_{\mathrm{a,inv}} = \begin{bmatrix} -P_{\mathrm{nom}} & P_{\mathrm{nom}} \\ -P_{\mathrm{nom}} & P_{\mathrm{nom}} \end{bmatrix},$$

$$G_{\mathrm{im}} = \begin{bmatrix} 0 & I \\ P_{\mathrm{nom}} & P_{\mathrm{nom}} \end{bmatrix}, \qquad G_{\mathrm{im,inv}} = \begin{bmatrix} -I & I \\ -P_{\mathrm{nom}} & P_{\mathrm{nom}} \end{bmatrix},$$

$$G_{\mathrm{om}} = \begin{bmatrix} 0 & P_{\mathrm{nom}} \\ I & P_{\mathrm{nom}} \end{bmatrix}, \qquad G_{\mathrm{om,inv}} = \begin{bmatrix} -I & P_{\mathrm{nom}} \\ -I & P_{\mathrm{nom}} \end{bmatrix}.$$

Example: DC motor



Model:

$$P_L(s) := \frac{K_m K_a}{s((Ls + R)(Js + f) + K_m K_b)}.$$

Nominal (simplified) model with $L = 0$:

$$P_{nom}(s) = \frac{K_m K_a}{s(RJs + (Rf + K_b K_m))}.$$

Nominal values:

K_a	K_m [N m/A]	R [Ω]	L [H]	J [kg m^2]	f [N m s/rad]
12	0.126	2.08	0.000264	0.008	0.005

but actual $R \in [1.87, 2.29]$, $J \in [0.0072, 0.0088]$, and $f \in [0.0045, 0.0055]$.

Example: DC motor (contd)

Our goal is to

– calculate bounds on $\Delta(\mathrm{j}\omega)$ at each frequency

for various choices of the uncertainty configuration.

– $P = P_{\mathrm{nom}} + \Delta \implies |\Delta(\mathrm{j}\omega)| \ge |P(\mathrm{j}\omega) - P_{\mathrm{nom}}(\mathrm{j}\omega)|$

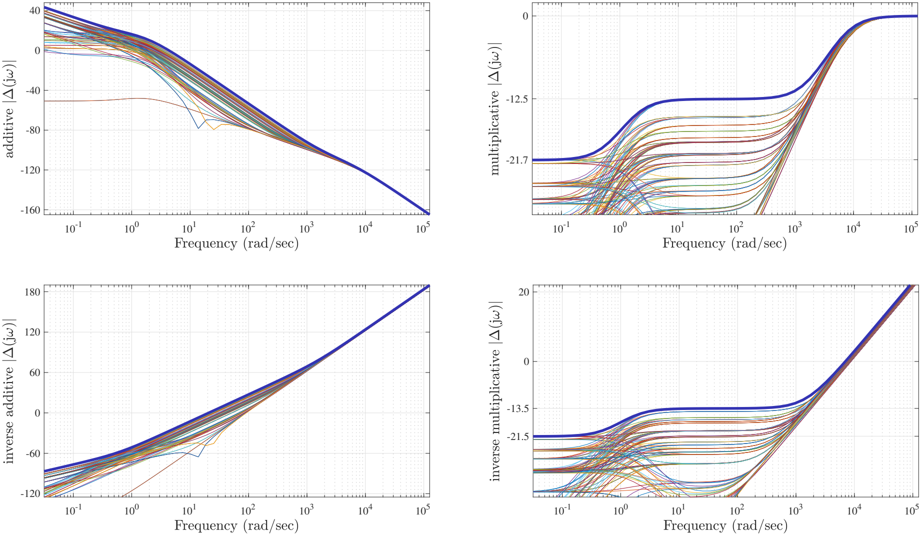
– $P = P_{\mathrm{nom}}(1 + \Delta) \implies |\Delta(\mathrm{j}\omega)| \ge \left| \dfrac{P(\mathrm{j}\omega)}{P_{\mathrm{nom}}(\mathrm{j}\omega)} - 1 \right|$

– $P = \dfrac{P_{\mathrm{nom}}}{1 + P_{\mathrm{nom}}\Delta} \implies |\Delta(\mathrm{j}\omega)| \ge \left| \dfrac{1}{P(\mathrm{j}\omega)} - \dfrac{1}{P_{\mathrm{nom}}(\mathrm{j}\omega)} \right|$

– $P = \dfrac{P_{\mathrm{nom}}}{1 + \Delta} \implies |\Delta(\mathrm{j}\omega)| \ge \left| \dfrac{P_{\mathrm{nom}}(\mathrm{j}\omega)}{P(\mathrm{j}\omega)} - 1 \right|$

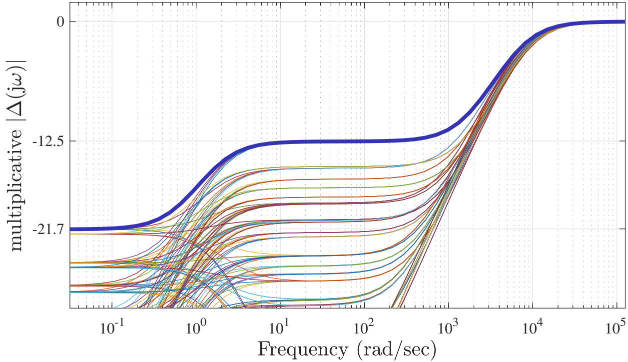
for every possible P from a given class.

Example: DC motor (contd)



and only the multiplicative Δ is stable.

Example: DC motor (contd)



This $\Delta \in H_\infty$, with $\|\Delta\|_\infty \leq 1$. More accurate is to

– introduce frequency-dependent weight, say $W(j\omega)$

such that $\Delta = W\Delta_0$ for some stable Δ_0 such that $\|\Delta_0\|_\infty \leq 1$ (in our case $|W(j\omega)| < 0.1$ at low frequencies and $|W(j\omega)| \uparrow 1$ at high). The weight

– W can always be absorbed into G.

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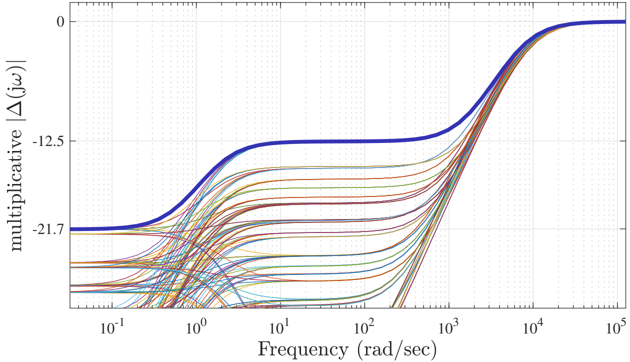

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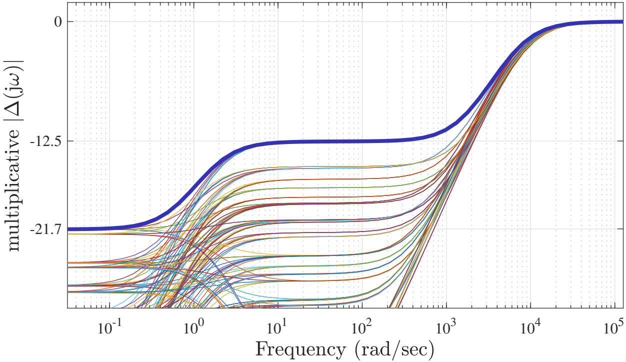

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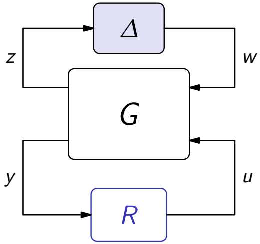
Robust stability problem in H_∞

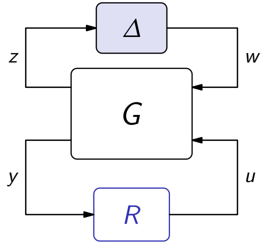


Goal:

- guarantee that the system is stable for all $\Delta \in \mathcal{B}_{H_\infty}$

(may be $\mathcal{B}_{L_2 \to L_2}$ in the time-varying nonlinear case).

Robust stability theorem



Theorem

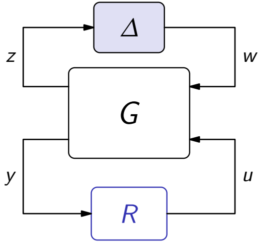
The system is robustly stable iff $\|\mathcal{F}_l(G, R)\|_\infty < 1$.

Proof (outline): Sufficiency follows by the SGT. Necessity by constructing a destabilizing $\Delta \in \mathcal{B}_{H_\infty}$ for ω at which $\|\mathcal{F}_l(G(j\omega), R(j\omega))\| = 1$. $\square$

Thus,

– robust stability $\iff$ the standard H_∞ problem for P_{nom}

Robust stability theorem



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Thus,

– robust stability $\iff$ the standard H_∞ problem for P_{nom}

Robust stability theorem: special cases

Additive uncertainty:

$$G = \begin{bmatrix} 0 & I \\ I & P_{\text{nom}} \end{bmatrix} \implies \mathcal{F}_{\text{l}}(G, R) = R(I - P_{\text{nom}} R)^{-1} = T_{\text{c}}$$

Input multiplicative uncertainty:

$$G = \begin{bmatrix} 0 & I \\ P_{\text{nom}} & P_{\text{nom}} \end{bmatrix} \implies \mathcal{F}_{\text{l}}(G, R) = R(I - P_{\text{nom}} R)^{-1} P_{\text{nom}} = T_{\text{i}}$$

Inverse additive uncertainty:

$$G = \begin{bmatrix} -P_{\text{nom}} & P_{\text{nom}} \\ -P_{\text{nom}} & P_{\text{nom}} \end{bmatrix} \implies \mathcal{F}_{\text{l}}(G, R) = -(I - P_{\text{nom}} R)^{-1} P_{\text{nom}} = -T_{\text{d}}$$

Inverse output multiplicative uncertainty:

$$G = \begin{bmatrix} -I & P_{\text{nom}} \\ -I & P_{\text{nom}} \end{bmatrix} \implies \mathcal{F}_{\text{l}}(G, R) = -(I - P_{\text{nom}} R)^{-1} = -S_{\text{o}}$$

Beyond SG

Structured uncertainty:

$$\Delta = \operatorname{diag}\{\Delta_i\}, \quad \text{with } \begin{cases} \text{some } \Delta_i \in \mathcal{B}_{H_\infty^{p_i \times m_i}} \\ \text{some } \Delta_i = \delta_i I_{m_i} \text{ for } \delta_i \in \mathcal{B}_{H_\infty^{1\times 1}} \end{cases}$$

Theory is based on structured singular values, aka μ.

Robust performance:

with the goal to

- guarantee bounds on $\|T_{z_p w_p}\|$ for all $\Delta \in \mathcal{B}_{H_\infty}$.

In some cases also reduces to μ.

Beyond SG

Structured uncertainty:

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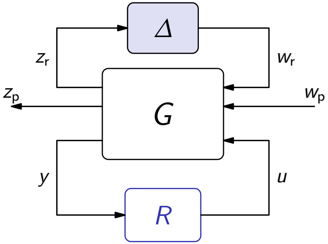

with the goal to

- guarantee bounds on $\|T_{z_\text{p} w_\text{p}}\|$ for all $\varDelta \in \mathcal{B}_{H_\infty}$.

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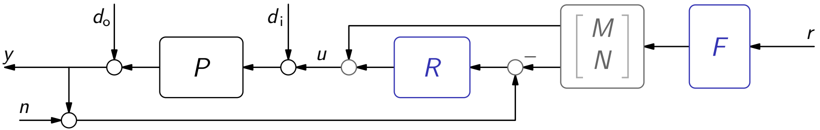
Modeling uncertainty and robust stability

Loop shaping

MIMO extensions

Case studies

2DOF control architecture



Nominal closed-loop system, if $NM^{-1} = P$:

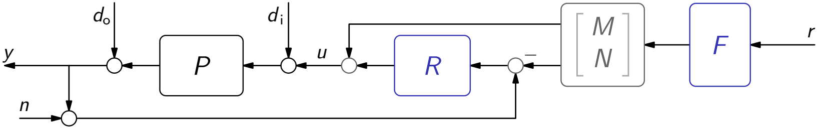
$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} N \\ M \end{bmatrix} F r + \begin{bmatrix} S_o & T_d \\ T_c & T_i \end{bmatrix} \begin{bmatrix} d_o \\ d_i \end{bmatrix} + \begin{bmatrix} T_o \\ T_c \end{bmatrix} n,$$

where

$$\begin{bmatrix} S_o & T_d \\ T_c & T_i \end{bmatrix} := \begin{bmatrix} I \\ R \end{bmatrix} (I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix}$$

and $T_o := (I - PR)^{-1} PR = S_o - I$.

2DOF control architecture (contd)



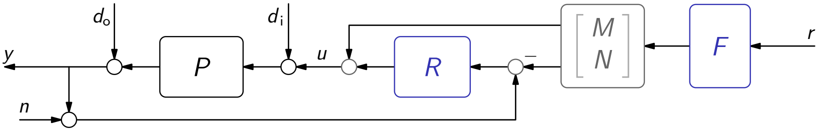
Closed-loop system if $NM^{-1} \neq P$:

$$\begin{aligned} \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} N \\ M \end{bmatrix} Fr + \begin{bmatrix} S_o & T_d \\ T_c & T_i \end{bmatrix} \left(\begin{bmatrix} -N \\ M \end{bmatrix} Fr + \begin{bmatrix} d_o \\ d_i \end{bmatrix} \right) + \begin{bmatrix} T_o \\ T_c \end{bmatrix} n \\ &= \left(\begin{bmatrix} N \\ M \end{bmatrix} + \begin{bmatrix} S_o \\ T_c \end{bmatrix} (PM - N) \right) Fr + \begin{bmatrix} S_o & T_d \\ T_c & T_i \end{bmatrix} \begin{bmatrix} d_o \\ d_i \end{bmatrix} + \begin{bmatrix} T_o \\ T_c \end{bmatrix} n \end{aligned}$$

because

$$\begin{bmatrix} S_o & T_d \\ T_c & T_i \end{bmatrix} \begin{bmatrix} -N \\ M \end{bmatrix} F = \begin{bmatrix} I \\ R \end{bmatrix} (I - PR)^{-1} (PM - N) F.$$

Typical requirements: SISO



with

$$y = NFr + S((PM - N)Fr + d_o + Pd_i) + Tn,$$

so we need to have

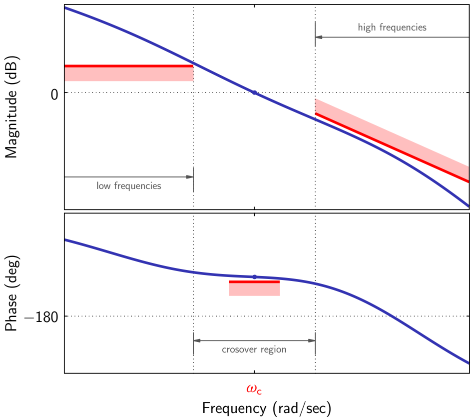
- $|S(j\omega)| \ll 1$ at dominant ω's of
  - the spectrum of r, where modeling uncertainty is non-negligible
  - the spectrum of d_o
  - the spectrum of d_i, where plant gain is non-negligible
- $|T(j\omega)| \ll 1$ at dominant ω's of
  - the spectrum of n

Typically,

- $|S(j\omega)| \ll 1$ at low frequencies and $|T(j\omega)| \ll 1$ at high frequencies.

SISO loop shaping

The loop transfer function is $L(s) := P(s)R(s)$. Then



- $|S(j\omega)| \ll 1 \iff |L(j\omega)| \gg 1$
- $|T(j\omega)| \ll 1 \iff |L(j\omega)| \ll 1$
- closed-loop system is stable iff $L(j\omega)$ agrees with the Nyquist criterion
- closed-loop system is robust if $L(j\omega)$ is "far" from the critical point (measured by stability margins, like μ_{ph}, μ_g, μ_m)

Magnitude shaping is relatively easy, phase shaping is tricky...

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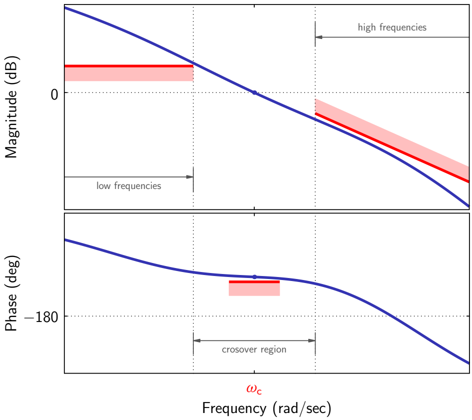

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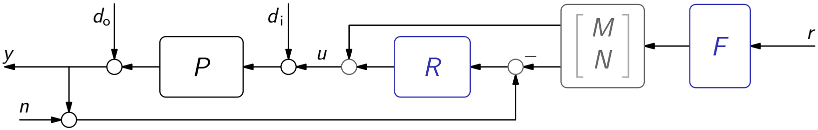
Typical requirements: MIMO



with

$$y = NFr + S_o((PM - N)Fr + d_o + Pd_i) + T_o n,$$

so we need to have

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  - the spectrum of r, where modeling uncertainty is non-negligible
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Typically,

- $\|S_o(j\omega)\| \ll 1$ at low frequencies and $\|T_o(j\omega)\| \ll 1$ at high freq-s.

MIMO magnitude shaping

In terms of the output loop transfer function $L_o(s) := P(s)R(s)$,

- $\|S_o(j\omega)\| \ll 1 \;\Longleftarrow\; \underline{\sigma}\left(L_o(j\omega)\right) \gg 1$
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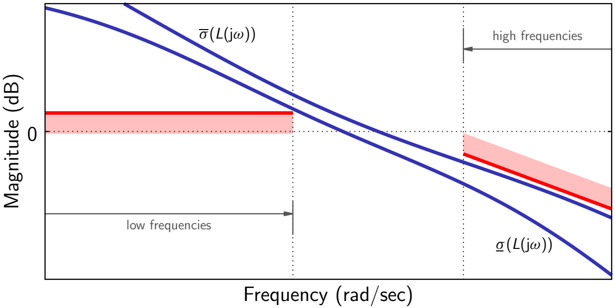
so we have:



But

- how to shape phase?

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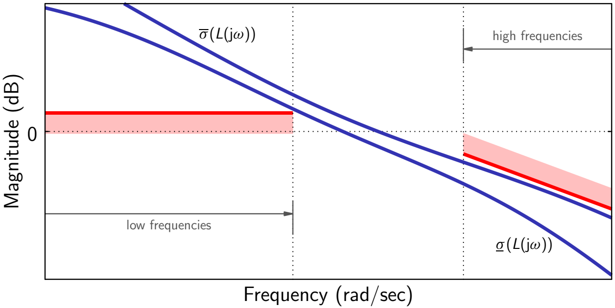

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MIMO phase shaping

Idea of McFarlane & Glover (in essence):

- cast phase shaping as an H_∞ robust stabilization problem
  ("far from the critical point" may indeed be interpreted as robustness requirement)

The latter

- can be solved analytically, via H_∞ optimization
- applies to MIMO systems

Design steps

1. Shape loop magnitude via weighs as $P_{\mathrm{msh}}(s) := W_{\mathrm{o}}(s)P(s)W_{\mathrm{i}}(s)$
   - typically P, PI, or PID, may include LPF
   - can cancel poles or zeros of $P(s)$ if required
   - in MIMO case can decouple
2. Design $R_{\mathrm{rob}}(s)$ for $P_{\mathrm{msh}}(s)$ via robust stabilization
   - robustness level, say $\epsilon_{\max} = 1/\gamma_{\min}$, serves as the success indicator
3. If successfull, pick $R(s) = W_{\mathrm{i}}(s)R_{\mathrm{rob}}(s)W_{\mathrm{o}}(s)$
   - in the SISO case, $L(s) = P(s)R(s) = P_{\mathrm{msh}}(s)R_{\mathrm{rob}}(s)$ is the designed loop
   - in the MIMO case, $L_{\mathrm{o}}(s) = P(s)R(s) = W_{\mathrm{o}}^{-1}(s)P_{\mathrm{msh}}(s)R_{\mathrm{rob}}(s)W_{\mathrm{o}}(s)$, so that certain care (balance) should be taken in the choice of W_{o}

Properties:

- closed-loop stability is guaranteed
- controller order = plant order + 2 × (weights order)
- integral actions / internal model can be easily enforced in $R(s)$ (neither $W_{\mathrm{i}}(s)$ nor $W_{\mathrm{o}}(s)$ is constrained to be stable)

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Choice of robust stability problem

Keep in mind:

- not necessarily reflects physics of the problem
- rather, should possess favorable properties from the design viewpoint

Rationale:

- the H_∞-norm of every GoF system reflects some robustness
- each one of GoF systems encourages cancellations

So, let's balance the design, via solving

$$\underset{R_{\text{rob}}}{\text{minimize}} \left\| \begin{bmatrix} R_{\text{rob}} \\ I \end{bmatrix} (I - P_{\text{msh}} R_{\text{rob}})^{-1} \begin{bmatrix} I & P_{\text{msh}} \end{bmatrix} \right\|_\infty = \underset{R_{\text{rob}}}{\text{minimize}} \left\| \begin{bmatrix} T_{\text{c}} & T_{\text{i}} \\ S_{\text{o}} & T_{\text{d}} \end{bmatrix} \right\|_\infty$$

which may be dubbed balanced sensitivity problem. Quantity

$$\epsilon_{\max} := \frac{1}{\gamma_{\min}} \in (0, 1), \quad \text{where } \gamma_{\min} := \min_{R_{\text{rob}}} \left\| \begin{bmatrix} T_{\text{c}} & T_{\text{i}} \\ S_{\text{o}} & T_{\text{d}} \end{bmatrix} \right\|_\infty$$

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Keep in mind:

- not necessarily reflects physics of the problem
- rather, should possess favorable properties from the design viewpoint

Rationale:

- the H_∞-norm of every GoF system reflects some robustness
- each one of GoF systems encourages cancellations

So, let's balance the design, via solving

$$\underset{R_{\mathrm{rob}}}{\text{minimize}} \left\| \begin{bmatrix} R_{\mathrm{rob}} \\ I \end{bmatrix} (I - P_{\mathrm{msh}} R_{\mathrm{rob}})^{-1} \begin{bmatrix} I & P_{\mathrm{msh}} \end{bmatrix} \right\|_\infty = \underset{R_{\mathrm{rob}}}{\text{minimize}} \left\| \begin{bmatrix} T_{\mathrm{c}} & T_{\mathrm{i}} \\ S_{\mathrm{o}} & T_{\mathrm{d}} \end{bmatrix} \right\|_\infty$$

which may be dubbed balanced sensitivity problem. Quantity

$$\epsilon_{\max} := \frac{1}{\gamma_{\min}} \in (0,1), \quad \text{where } \gamma_{\min} := \min_{R_{\mathrm{rob}}} \left\| \begin{bmatrix} T_{\mathrm{c}} & T_{\mathrm{i}} \\ S_{\mathrm{o}} & T_{\mathrm{d}} \end{bmatrix} \right\|_\infty$$

is the success indicator.

Choice of robust stability problem

Keep in mind:

- not necessarily reflects physics of the problem
- rather, should possess favorable properties from the design viewpoint

Rationale:

- the H_∞-norm of every GoF system reflects some robustness
- each one of GoF systems encourages cancellations

So, let's balance the design, via solving

$$\underset{R_{\text{rob}}}{\text{minimize}} \left\| \begin{bmatrix} R_{\text{rob}} \\ I \end{bmatrix} (I - P_{\text{msh}} R_{\text{rob}})^{-1} \begin{bmatrix} I & P_{\text{msh}} \end{bmatrix} \right\|_\infty = \underset{R_{\text{rob}}}{\text{minimize}} \left\| \begin{bmatrix} T_{\text{c}} & T_{\text{i}} \\ S_{\text{o}} & T_{\text{d}} \end{bmatrix} \right\|_\infty$$

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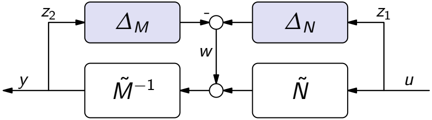
$$\epsilon_{\max} := \frac{1}{\gamma_{\min}} \in (0,1), \quad \text{where } \gamma_{\min} := \min_{R_{\text{rob}}} \left\| \begin{bmatrix} T_{\text{c}} & T_{\text{i}} \\ S_{\text{o}} & T_{\text{d}} \end{bmatrix} \right\|_\infty$$

is the success indicator.

Balanced sensitivity problem: associated uncertainty

Consider robust stability problem for



under normalized *lcf* uncertainty, i.e. $P = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N)$ with

$$\tilde{N}\tilde{N}^{\sim} + \tilde{M}\tilde{M}^{\sim} = I$$

(i.e. $\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$ is co-inner). It can be shown that

– closed-loop system robustly stable for all $\| \begin{bmatrix} \Delta_M & \Delta_N \end{bmatrix} \|_\infty < \alpha$ iff

$$\left\| \begin{bmatrix} T_\text{c} & T_\text{i} \\ S_\text{o} & T_\text{d} \end{bmatrix} \right\|_\infty \leq \frac{1}{\alpha}.$$

Hence,

– solving balanced sensitivity $\iff$ maximizing robustness radius.

Balanced sensitivity problem: solution

Bring in a stabilizable and detectable realization

$$P_{\mathrm{msh}}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and let $X \geq 0$ and $Y \geq 0$ be the stabilizing solutions to the H_2 AREs

$$A'X + XA - XBB'X + C'C = 0 \quad \text{and} \quad AY + YA' - YC'CY + BB' = 0$$

such that $A - BB'X$ and $A - YC'C$ are Hurwitz (exist and are unique).

Remark It can be shown that $Y = (I - W_X X)^{-1} W_X$, where $W_X \geq 0$ solves the Lyapunov equation

$$(A - BB'X)W_X + W_X(A - BB'X)' + BB' = 0$$

(i.e. is the controllability Gramian of $(A - BB'X, B)$).

Balanced sensitivity problem: solution (contd)

Theorem

The minimal attainable performance

$$\gamma_{min} = \sqrt{1 + \rho(YX)} > 1.$$

Given then any $\gamma > \gamma_{min}$, all γ-suboptimal controllers are given by

$$R_{rob}(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} A - BB'X - Z_\gamma^{-1}YC'C & Z_\gamma^{-1}YC' & Z_\gamma^{-1}B \\ \hline -B'X & 0 & I \\ -C & I & 0 \end{array}\right], Q(s)\right)$$

for any $Q \in RH_\infty$ such that

$$\|Q\|_\infty < \sqrt{\gamma^2 - 1},$$

where $Z_\gamma := (1 - \gamma^{-2})I - \gamma^{-2}YX$.

MATLAB code 1

```
% H-inf loop-shaping design (assuming negative feedback)
%
 Pmsh = minreal(Wo*P*Wi);
 [A,B,C,D] = ssdata(Pmsh);
 X = icare(A,B,C'*C);
 Y = icare(A',C',B*B');
 epsmax2 = 1/(1+max(eig(Y*X)));
%
 suboptf = 1.05^2;                        % any number >1
 gammam2inv = epsmax2/suboptf;
 Z = (1-gammam2inv)*eye(size(A))-gammam2inv*Y*X;
 K = -B'*X;
 L = -inv(Z)*Y*C';
 Rrob = ss(A+B*K+L*C,L,K,zeros(size(D')));
%
 Rgam = minreal(Wi*Rrob*Wo);
```

Balanced sensitivity problem: what if $\gamma \downarrow \gamma_{\text{opt}}$

If $\gamma^2 \downarrow 1 + \rho(YX) = \rho(I + YX)$, then

$$Z_\gamma = I - \gamma^{-2}(I + YX)$$

turns singular. To avoid inverting singular matrices, the central controller

$$R_{\text{rob}}(s) = \left[\begin{array}{c|c} A - BB'X - Z_\gamma^{-1}YC'C & -Z_\gamma^{-1}YC' \\ \hline B'X & 0 \end{array}\right]$$

can be implemented in the descriptor (algebraic differential equation) form

$$R_{\text{rob}}(s) = -B'X(sZ_\gamma - Z_\gamma A + Z_\gamma BB'X + YC'C)^{-1}YC'$$

(is well defined, because $\operatorname{nrank}(sZ_{\gamma_{\text{opt}}} - Z_{\gamma_{\text{opt}}}A + Z_{\gamma_{\text{opt}}}BB'X + YC'C) = n$). As a result,

- the order of the optimal controller equals $\operatorname{rank}(Z_{\gamma_{\text{opt}}}) < n$.

MATLAB code 2

```
% H-inf loop-shaping design (assuming negative feedback)
%
 Pmsh = minreal(Wo*P*Wi);
 [A,B,C,D] = ssdata(Pmsh);
 X = icare(A,B,C'*C);
 Y = icare(A',C',B*B');
 epsmax2 = 1/(1+max(eig(Y*X)));
%
 suboptf = 1.05^2;                           % any number >=1
 gammam2inv = epsmax2/subopt;
 Z = (1-gammam2inv)*eye(size(A))-gammam2inv*Y*X;
 K = -B'*X;
 L = -Y*C';
 Rrob = dss(Z*A+Z*B*K+L*C,L,K,zeros(size(D')),Z);
%
 Rgam = minreal(Wi*Rrob*Wo);
```

Outline

Modeling uncertainty and robust stability

Loop shaping

MIMO extensions

Case studies

Example 1

Let

$$P(s) = \frac{1}{s^2} \quad \text{and} \quad W(s) = \tilde{\omega}_c^2,$$

so that the magnitude-shaped loop,

$$P_{msh}(s) = \frac{\tilde{\omega}_c^2}{s^2} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & \tilde{\omega}_c \\ \hline \tilde{\omega}_c & 0 & 0 \end{array}\right],$$

has its crossover frequency $\omega_c = \tilde{\omega}_c$.

Example 1: optimal cost

AREs

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - X \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\omega}_c^2 \end{bmatrix} X + \begin{bmatrix} \tilde{\omega}_c^2 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y + Y \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - Y \begin{bmatrix} \tilde{\omega}_c^2 & 0 \\ 0 & 0 \end{bmatrix} Y + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\omega}_c^2 \end{bmatrix} = 0$$

have stabilizing solutions

$$X = \begin{bmatrix} \sqrt{2}\,\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}/\tilde{\omega}_c \end{bmatrix} > 0 \quad \text{and} \quad Y = \begin{bmatrix} \sqrt{2}/\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}\,\tilde{\omega}_c \end{bmatrix} > 0.$$

Optimal performance:

$$\gamma_{\min}^2 = 1 + \rho\left(\begin{bmatrix} \sqrt{2}/\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}\,\tilde{\omega}_c \end{bmatrix} \begin{bmatrix} \sqrt{2}\,\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}/\tilde{\omega}_c \end{bmatrix} \right)$$

$$= 1 + \rho\left(\begin{bmatrix} 3 & 2\sqrt{2}/\tilde{\omega}_c \\ 2\sqrt{2}\,\tilde{\omega}_c & 3 \end{bmatrix} \right) = 4 + 2\sqrt{2} \approx 6.8284$$

is independent of $\tilde{\omega}_c$ (because loop phase is independent of $\tilde{\omega}_c$ either).

Example 1: optimal cost

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$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} X + X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - X \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\omega}_c^2 \end{bmatrix} X + \begin{bmatrix} \tilde{\omega}_c^2 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y + Y \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - Y \begin{bmatrix} \tilde{\omega}_c^2 & 0 \\ 0 & 0 \end{bmatrix} Y + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\omega}_c^2 \end{bmatrix} = 0$$

have stabilizing solutions

$$X = \begin{bmatrix} \sqrt{2}\,\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}/\tilde{\omega}_c \end{bmatrix} > 0 \quad \text{and} \quad Y = \begin{bmatrix} \sqrt{2}/\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}\,\tilde{\omega}_c \end{bmatrix} > 0.$$

Optimal performance:

$$\begin{aligned} \gamma_{\min}^2 &= 1 + \rho\left(\begin{bmatrix} \sqrt{2}/\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}\,\tilde{\omega}_c \end{bmatrix}\begin{bmatrix} \sqrt{2}\,\tilde{\omega}_c & 1 \\ 1 & \sqrt{2}/\tilde{\omega}_c \end{bmatrix}\right) \\ &= 1 + \rho\left(\begin{bmatrix} 3 & 2\sqrt{2}/\tilde{\omega}_c \\ 2\sqrt{2}\,\tilde{\omega}_c & 3 \end{bmatrix}\right) = 4 + 2\sqrt{2} \approx 6.8284 \end{aligned}$$

is independent of $\tilde{\omega}_c$ (because loop phase is independent of $\tilde{\omega}_c$ either).

Example 1: optimal controller

With

$$Z_{\gamma_{\min}} = (\sqrt{2}-1)\begin{bmatrix} 1 & -1/\tilde{\omega}_c \\ -\tilde{\omega}_c & 1 \end{bmatrix} = (\sqrt{2}-1)\begin{bmatrix} 1 \\ -\tilde{\omega}_c \end{bmatrix}\begin{bmatrix} 1 & -1/\tilde{\omega}_c \end{bmatrix}$$

(indeed singular),

$$R_{\text{rob}}(s) = -\frac{(1+\sqrt{2})s + \tilde{\omega}_c}{s + \tilde{\omega}_c(1+\sqrt{2})}$$

which is

- the first-order lead, having the maximal phase lead 45° at $\omega = \tilde{\omega}_c$.

Hence,

$$R(s) = W(s)R_{\text{rob}}(s) = -\tilde{\omega}_c^2\,\frac{(1+\sqrt{2})s + \tilde{\omega}_c}{s + \tilde{\omega}_c(1+\sqrt{2})}$$

(mind that positive feedback is assumed).

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(mind that positive feedback is assumed).

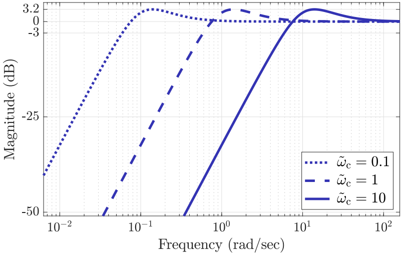
Example 1: resulted loop

For every $\tilde{\omega}_c$,

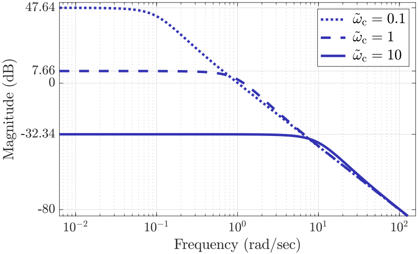
- $\epsilon_{max} = 1/\sqrt{4 + 2\sqrt{2}} \approx 0.3827$
- $\omega_c = \tilde{\omega}_c$
- $\mu_{ph} = 45^\circ$
- $\mu_g = \infty$
- $\mu_m \approx 0.6921$

Example 1: resulted GoF

S



T_d



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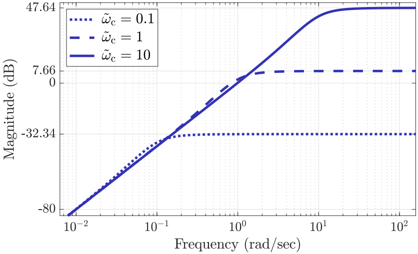

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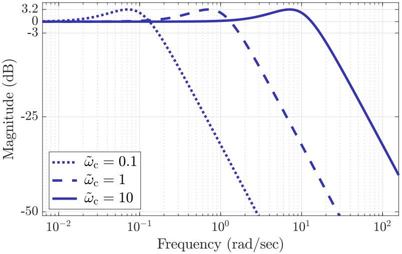

Example 2

Let

$$P(s) = \frac{1}{s^3} \quad \text{and} \quad W(s) = \tilde{\omega}_c^3,$$

so that the magnitude-shaped loop,

$$P_{\text{msh}}(s) = \frac{\tilde{\omega}_c^3}{s^3} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{\tilde{\omega}_c^3} \\ \hline \sqrt{\tilde{\omega}_c^3} & 0 & 0 & 0 \end{array}\right],$$

has its crossover frequency $\omega_c = \tilde{\omega}_c$.

Optimal cost (about 44% of what we had in the double integrator case):

$$\epsilon_{\max} = \sqrt{\frac{1}{2} - \frac{\sqrt{2}}{3}} \approx 0.1691,$$

is independent of $\tilde{\omega}_c$ too (because loop phase is independent of $\tilde{\omega}_c$ either).

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is independent of $\tilde{\omega}_c$ too (because loop phase is independent of $\tilde{\omega}_c$ either).

Example 2: optimal controller

is

$$R_{\mathrm{rob}}(s) = -\frac{(1+\sqrt{2})^2 s^2 + (2+\sqrt{2})\tilde{\omega}_{\mathrm{c}} s + \tilde{\omega}_{\mathrm{c}}^2}{s^2 + (2+\sqrt{2})\tilde{\omega}_{\mathrm{c}} s + (1+\sqrt{2})^2 \tilde{\omega}_{\mathrm{c}}^2},$$

which is

- the second-order complex lead, having the maximal phase lead $\approx 109^\circ$ at $\omega = \tilde{\omega}_{\mathrm{c}}$ and the damping $\zeta = 1/\sqrt{2}$.

In fact,

$$R_{\mathrm{rob}}(s) = -\beta \frac{B_2(s/(\tilde{\omega}_{\mathrm{c}}\beta))}{B_2(s\beta/\tilde{\omega}_{\mathrm{c}})}$$

for the Butterworth polynomial $B_2(s)$ and $\beta = 3 - 2\sqrt{2} \approx 0.1716$.

Hence,

$$R(s) = -\tilde{\omega}_{\mathrm{c}}^2 \frac{(1+\sqrt{2})^2 s^2 + (2+\sqrt{2})\tilde{\omega}_{\mathrm{c}} s + \tilde{\omega}_{\mathrm{c}}^2}{s^2 + (2+\sqrt{2})\tilde{\omega}_{\mathrm{c}} s + (1+\sqrt{2})^2 \tilde{\omega}_{\mathrm{c}}^2}$$

(mind that positive feedback is assumed).

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(mind that positive feedback is assumed).

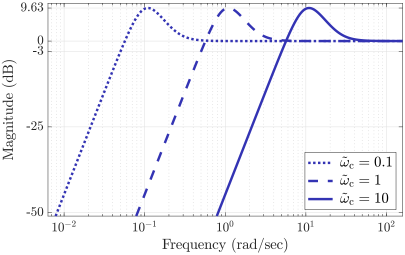
Example 2: resulted loop

For every $\tilde{\omega}_c$,

- $\epsilon_{max} = \sqrt{1/2 - \sqrt{2}/3} \approx 0.1691$
- $\omega_c = \tilde{\omega}_c$
- $\mu_{ph} \approx 19^\circ$
- $\mu_g \approx 2.29$
- $\mu_m \approx 0.3298$

Example 2: resulted GoF

S



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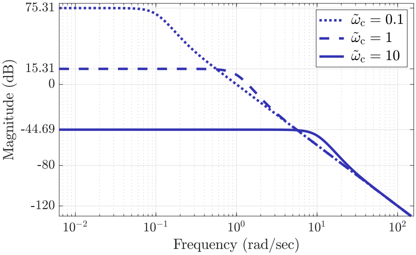
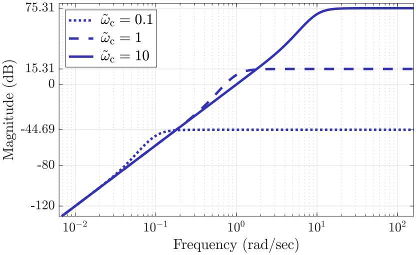
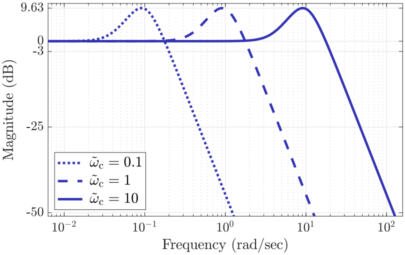


T_c



T



Example 3

Let

$$P(s) = \frac{1}{s(s+1)} \quad \text{and} \quad W(s) = \frac{\tilde{\omega}_c^2\sqrt{1+\tilde{\omega}_c^2}}{s},$$

so that

$$P_{\text{msh}}(s) = \frac{\tilde{\omega}_c^2\sqrt{1+\tilde{\omega}_c^2}}{s^2(s+1)}.$$

This weight intends to provide

- an integral action in the controller
- high-frequency roll-off of 1
- crossover frequency $\omega_c = \tilde{\omega}_c$

This loop approaches the

- double integrator if $\tilde{\omega}_c \ll 1$ — a phase lag of $\approx -180^\circ$ at $\tilde{\omega}_c$
- triple integrator if $\tilde{\omega}_c \gg 1$ — a phase lag of $\approx -270^\circ$ at $\tilde{\omega}_c$

Example 3: optimal controllers

Three cases:

$\tilde{\omega}_c = 0.1$

$$R(s) = -\frac{0.0259(s+0.0407)(s+1)}{s(s+0.2819)(s+0.9603)} \approx -\frac{0.027(s+0.0407)}{s(s+0.2819)}$$

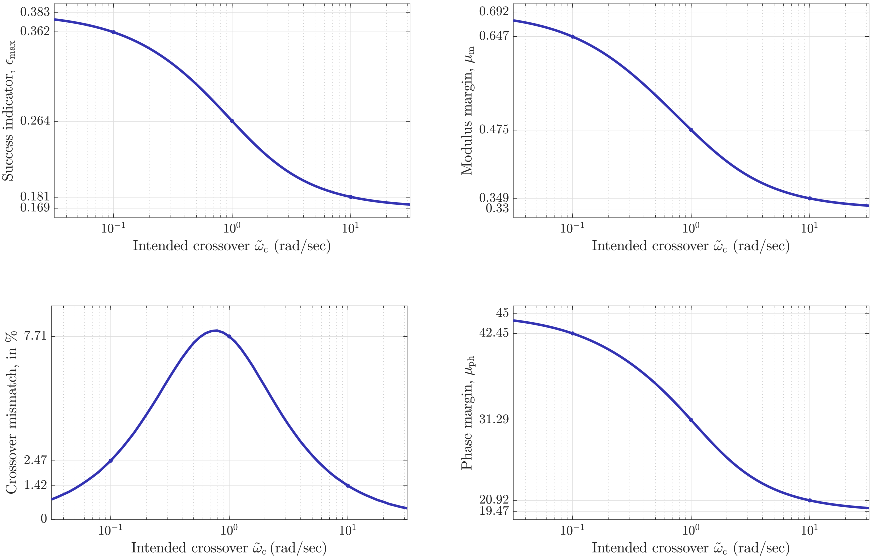
$\tilde{\omega}_c = 1$

$$R(s) = -\frac{5.1584(s+0.4733)(s+0.9477)}{s(s^2+3.614s+5.968)}$$

$\tilde{\omega}_c = 10$

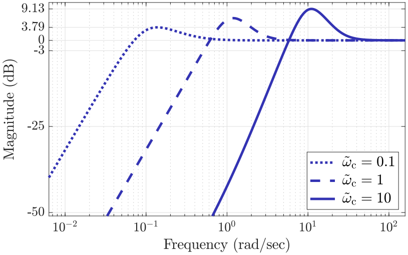
$$R(s) = -\frac{5448.9(s^2+6.512s+19.16)}{s(s^2+33.62s+563.1)}$$

Example 3: resulted loop

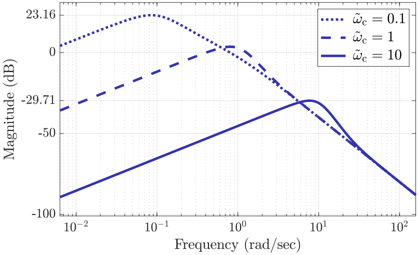


Example 3: resulted GoF

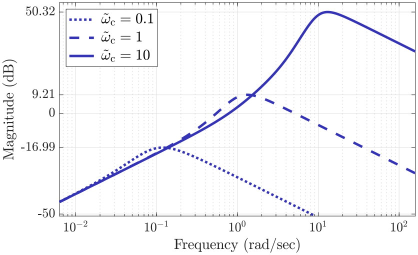
S



T_d



T_c



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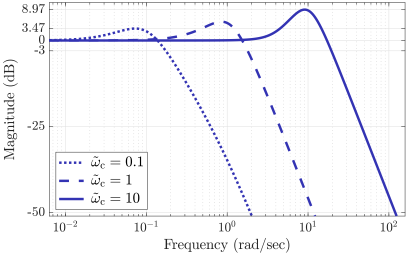
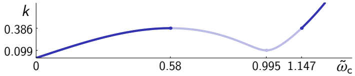


Example 4

Let

$$P(s) = \frac{1}{s^2 + 0.1s + 1} \quad \text{and} \quad W(s) = \frac{k}{s} \quad \text{for}$$



so that

$$P_{msh}(s) = \frac{k}{s(s^2 + 0.1s + 1)}.$$

This weight intends to provide

– an integral action in the controller
– high-frequency roll-off of 1
– crossover frequency $\omega_c = \tilde{\omega}_c$

This loop approaches the

– single integrator if $\tilde{\omega}_c \ll 1$ — a phase lag of $\approx -90^\circ$ at $\tilde{\omega}_c$
– triple integrator if $\tilde{\omega}_c \gg 1$ — a phase lag of $\approx -270^\circ$ at $\tilde{\omega}_c$

(the phase drops rapidly around the resonance).

Example 4: optimal controllers

Three cases:

$\tilde{\omega}_c = 0.13$

$$R(s) = -\frac{0.1396\overbrace{(s^2 - 0.03291s + 0.9464)}^{\approx s^2+0.1s+1}}{s(s^2 + 0.6051s + 1.129)}$$

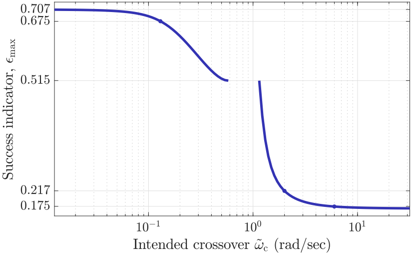
$\tilde{\omega}_c = 2$

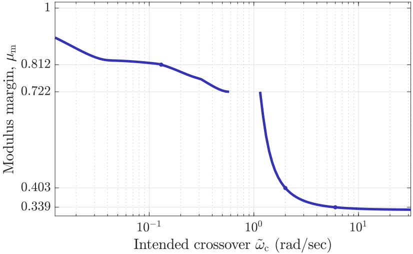
$$R(s) = -\frac{27.083(s^2 + 0.8226s + 0.8323)}{s(s^2 + 5.725s + 16.88)}$$

$\tilde{\omega}_c = 6$

$$R(s) = -\frac{1184.8(s^2 + 3.453s + 6.361)}{s(s^2 + 20.1s + 202.4)}$$

Example 4: resulted loop

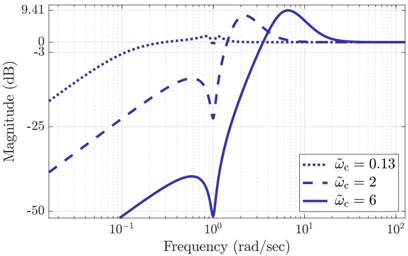




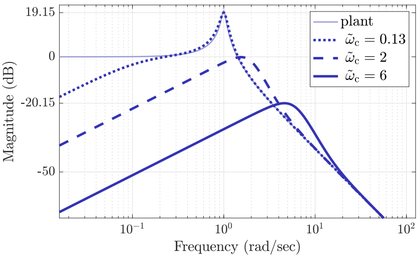
Robustness deteriorates rapidly after $\tilde{\omega}_c$ passed the resonance.

Example 4: resulted GoF

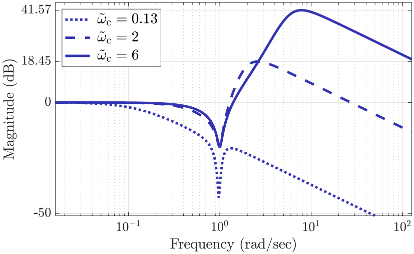
S



T_d



T_c



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