

Linear Control Systems (036012)

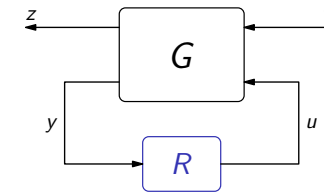
chapter 7

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT



The standard problem



Given G (plant, weights, fixed parts of controller, etc), design R that

- internally stabilizes the system
- minimizes a size of the mapping $T_{zw} := \mathcal{F}_1(G, R) : w \mapsto z$

Depending on the measure of this “size”

- H_2 standard problem, if $\|T_{zw}\|_2$ is minimized LQG, Kalman filtering
- H_∞ standard problem, if $\|T_{zw}\|_\infty$ is minimized mixed sensitivity
- or mixed H_2/H_∞ , L_1 , et cetera

Preliminary: GoF as the standard problem

We already know (Lect. 9, but for negative feedback) that

$$\begin{bmatrix} T_c & T_i \\ S_o & T_d \end{bmatrix} := \begin{bmatrix} R \\ I \end{bmatrix} (I + PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} = \mathcal{F}_1 \left(\begin{bmatrix} 0 & 0 & I \\ I & P & -P \\ I & P & -P \end{bmatrix}, R \right).$$

It is internally stabilizable, because

$$\begin{bmatrix} 0 & 0 & I \\ I & P & -P \\ I & P & -P \end{bmatrix} = \begin{bmatrix} 0 & I & M \\ I & 0 & -N \\ I & 0 & -N \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & M \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -I \\ 0 & 0 & \tilde{M} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ \tilde{M} & \tilde{N} & -\tilde{N} \end{bmatrix}$$

are its coprime factorizations over RH_∞ . All stabilizing controllers

$$R = (\tilde{Y} + MQ)(\tilde{X} - NQ)^{-1} = (X - Q\tilde{N})^{-1}(Y + Q\tilde{M})$$

(with $\det(\tilde{X}(\infty) - N(\infty)Q(\infty)) \neq 0$ or $\det(X(\infty) - Q(\infty)\tilde{N}(\infty)) \neq 0$), so

$$T_{zw} = \left(\begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix} + \begin{bmatrix} M \\ -N \end{bmatrix} Q \right) \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}.$$

Outline

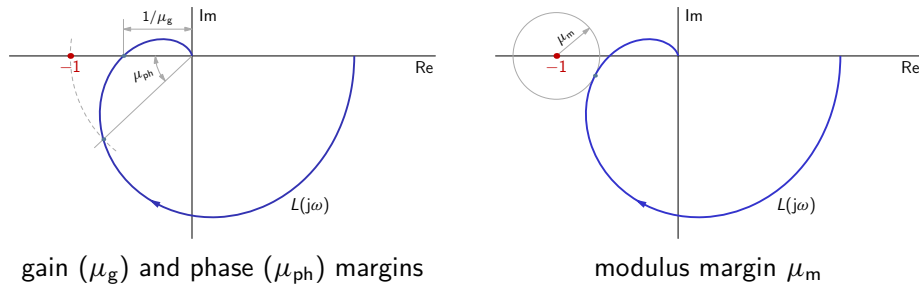
Increasing modulus margin

Weighted sensitivity

Mixed sensitivity

Optimization-based design

Stability margins



If $\mu_m \leq 1$, then

$$\mu_g \geq \frac{1}{1 - \mu_m} \quad \text{and} \quad \mu_{ph} \geq 2 \arcsin \frac{\mu_m}{2}.$$

5/40

Modulus margin and H_∞

If the closed-loop system is stable,

$$\mu_m = \inf_{\omega \in \mathbb{R}} |1 + L(j\omega)| \implies \frac{1}{\mu_m} = \sup_{\omega \in \mathbb{R}} |S(j\omega)| = \|S\|_\infty.$$

Hence,

$$\text{minimizing } \|S\|_\infty \iff \text{maximizing } \mu_m$$

Reminder:

$$\|S\|_\infty := \sup_{s \in \mathbb{R}_0} |S(s)| = \sup_{\omega \in \mathbb{R}} |S(j\omega)|$$

6/40

The standard problem for modulus margin maximization

Generalized plant:

$$\begin{bmatrix} T_c & T_i \\ S_o & T_d \end{bmatrix} = \mathcal{F}_l \left(\left[\begin{array}{cc|c} 0 & 0 & I \\ I & P & -P \\ \hline I & P & -P \end{array} \right], R \right),$$

so that

$$G(s) = \begin{bmatrix} I & -P(s) \\ I & -P(s) \end{bmatrix}.$$

All stable sensitivity functions:

$$T_{zw} = \left(\begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix} + \begin{bmatrix} M \\ -N \end{bmatrix} Q \right) \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix},$$

so that

$$T_{zw} = S = (\tilde{X} - NQ)\tilde{M}.$$

7/40

Example 1

Let

$$P(s) = \frac{s - z_1}{s + 1}, \quad z_1 \in \mathbb{R}.$$

Coprime factors, Bézout coefficients: $M = X = \tilde{M} = \tilde{X} = 1$, $N = \tilde{N} = P$.
All stable sensitivity functions:

$$S(s) = 1 - \frac{s - z_1}{s + 1} Q(s).$$

Minimum-phase plant: If $z_1 < 0$, then the optimal $Q(s) = (s + 1)/(s - z_1)$ is not admissible, since $\tilde{X}(\infty) - N(\infty)Q(\infty) = 0$. But

$$Q(s) = \frac{(1 - \epsilon)s + 1}{s - z_1} \implies S(s) = \frac{\epsilon s}{s + 1}$$

and $\|S\|_\infty$ can be made arbitrarily small.

8/40

Example 1 (contd)

Nonminimum-phase plant: If $z_1 \geq 0$, then

$$S(s) = 1 - \frac{s - z_1}{s + 1} Q(s) \implies S(z_1) = 1$$

independently of Q . Hence,

$$- \|S\|_\infty \geq 1.$$

In fact, the trivial choice

$$Q(s) = 0 \implies R(s) = 0$$

attains the bound, rendering $S(s) = 1$ (not quite meaningful though).

9/40

Example 2

Let

$$P(s) = \frac{s - z_1}{s^2 - 1} = \frac{s - z_1}{(s - 1)(s + 1)}, \quad z_1 \in \mathbb{R}.$$

Choose

$$M(s) = \tilde{M}(s) = \frac{s - 1}{s + a} \quad \text{and} \quad N(s) = \tilde{N}(s) = \frac{s - z_1}{(s + a)(s + 1)}$$

for any $a > 0$. The corresponding Bézout coefficients are

$$X(s) = \tilde{X}(s) = \frac{s + (az_1 + 2z_1 + a)/(z_1 - 1)}{s + 1}$$

and

$$Y(s) = \tilde{Y}(s) = -\frac{2(a + 1)}{z_1 - 1}.$$

Hence,

$$R(s) = \frac{-2(a + 1)(s + a) + (z_1 - 1)(s - 1)Q(s)}{(s + a)((z_1 - 1)s + az_1 + 2z_1 + a) - (z_1 - 1)(s - z_1)Q(s)}.$$

10/40

Example 2 (contd)

We end up with

$$S(s) = \left(\frac{s + (az_1 + 2z_1 + a)/(z_1 - 1)}{s + 1} - \frac{s - z_1}{(s + 1)(s + a)} Q(s) \right) \frac{s - 1}{s + a}.$$

The choice $a = 1$ renders $\tilde{M}(s)$ co-inner (and inner), as then

$$\tilde{M}(s)\tilde{M}^*(s) = \frac{s - 1 - s - 1}{s + 1 - s + 1} = 1.$$

Hence, $\|S\|_\infty = \|S_{\text{eq}}\|_\infty$, where

$$S_{\text{eq}}(s) := \frac{s + (3z_1 + 1)/(z_1 - 1)}{s + 1} - \frac{s - z_1}{(s + 1)^2} Q(s).$$

If $z_1 < 0$, then $S_{\text{eq}}(\infty) = 1$ for all $Q \in RH_\infty$ and $\|S\|_\infty \geq 1$. Attainable by

$$Q(s) = \frac{\tilde{X}(s) - \tilde{X}(\infty)}{N(s)} = \frac{2(z_1 + 1)}{z_1 - 1} \frac{s + 1}{s - z_1}$$

or, for $z_1 \in [-1, 0)$, by $Q = 0$.

11/40

Example 2 (contd)

If $z_1 \geq 0$, then $S_{\text{eq}}(z_1) = (z_1 + 1)/(z_1 - 1)$ and

$$\|S\|_\infty = \|S_{\text{eq}}\|_\infty \geq \frac{z_1 + 1}{|z_1 - 1|}.$$

Attainable by non-proper

$$Q(s) = \frac{\tilde{X}(s) - \tilde{X}(z_1)}{N(s)} = -\frac{2}{z_1 - 1} (s + 1).$$

or, almost, by

$$Q(s) = -\frac{2}{z_1 - 1} \frac{s + 1}{\epsilon s + 1}.$$

Controller:

$$R(s) = -\frac{2(s + 1)((1 + 2\epsilon)s + 1)}{\epsilon(z_1 - 1)s^2 + (z_1 + 1 + (3z_1 + 1)\epsilon)s + z_1 + 1} \xrightarrow{\epsilon \rightarrow 0} -\frac{2(s + 1)}{z_1 + 1}$$

$R(0) = -2/(z_1 + 1)$, $R(\infty) = -2(\epsilon^{-1} + 2)/(z_1 - 1)$, unstable for $z_1 < 1$.

12/40

Example 2: supremal modulus margin

We end up with

$$\sup_{\text{stabilizing } R} \mu_m = \begin{cases} 1 & \text{if } z_1 < 0 \\ \frac{|z_1-1|}{z_1+1} & \text{if } z_1 \geq 0 \end{cases} = \begin{array}{c} \mu_m \\ \text{---} \\ 1 \\ \text{---} \\ 1/2 \\ \text{---} \\ 0 \end{array} \begin{array}{c} \text{---} \\ 0 \end{array} \begin{array}{c} \text{---} \\ 1/3 \end{array} \begin{array}{c} \text{---} \\ 1 \end{array} \begin{array}{c} \text{---} \\ 3 \end{array} \begin{array}{c} \text{---} \\ \text{plant zero, } z_1 \end{array}$$

13/40

Example 3

Let

$$P(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{s-1} + \begin{bmatrix} \alpha & -2z_1/(z_1-1) \\ -\alpha\beta & \beta \end{bmatrix} \frac{1}{s+1}$$

has an unstable pole at $s = 1$ with

$$\text{pdir}_i(P, 1) = \text{span} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad \text{pdir}_o(P, 1) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

and a (nonminimum-phase) zero at $s = z_1$ with

$$\text{zdir}_i(P, z_1) = \text{span} \left(\begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right) \quad \text{and} \quad \text{zdir}_o(P, z_1) = \text{span} \left(\begin{bmatrix} \beta \\ 1 \end{bmatrix} \right).$$

14/40

Example 3 (contd)

Choose

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} = \begin{array}{cc|cc} \frac{s-1}{s+1} & 0 & -\frac{\alpha(s-1)}{(s+1)^2} & \frac{(z_1+1)s-3z_1+1}{(z_1-1)(s+1)^2} \\ 0 & 1 & \frac{\alpha\beta}{s+1} & -\frac{\beta}{s+1} \\ \hline 0 & 0 & 1 & 0 \\ 2 & \frac{2}{\beta} & 0 & 1 + \frac{z_1+1}{z_1-1} \frac{2}{s+1} \end{array}$$

and

$$\begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & M \end{bmatrix} = \begin{array}{cc|cc} 1 + \frac{z_1}{z_1-1} \frac{4}{s+1} & 2 \frac{(z_1+1)s-3z_1+1}{\beta(z_1-1)(s+1)^2} & \frac{\alpha}{s+1} & -\frac{(z_1+1)s-3z_1+1}{(z_1-1)(s+1)^2} \\ -\frac{2\beta}{s+1} & \frac{s^2+3}{(s+1)^2} & -\frac{\alpha\beta}{s+1} & -\frac{\beta(s-1)}{(s+1)^2} \\ \hline 0 & 0 & 1 & 0 \\ -2 & -\frac{2}{\beta} \frac{s-1}{s+1} & 0 & \frac{s-1}{s+1} \end{array}$$

with co-inner $\tilde{M}(s)$.

15/40

Example 3 (contd)

As $\|S\|_\infty = \|S_{\text{eq}}\|_\infty$ for

$$S_{\text{eq}}(s) := \tilde{X}(s) - N(s)Q(s)$$

we have two constraints:

$$S_{\text{eq}}(\infty) = \tilde{X}(\infty) = I$$

and

$$\eta' S_{\text{eq}}(z_1) = \eta' \tilde{X}(z_1) = \eta' \tilde{M}^{-1}(z_1), \quad \text{for any } \eta \in \text{zdir}_o(P, z_1)$$

(the latter follows by $\tilde{X} = \tilde{M}^{-1} - P\tilde{Y}$).

16/40

Example 3: supremal modulus margin

Thus,

$$\mu_m \leq \sqrt{\left(\frac{z_1 + 1}{z_1 - 1}\right)^2 \cos^2 \theta_o + \sin^2 \theta_o} \in \left(1, \frac{z_1 + 1}{|z_1 - 1|}\right).$$

where

$$\theta_o = \arccos \frac{\beta}{\sqrt{1 + \beta^2}} \in (0, \pi)$$

is the angle between $\text{pdir}_o(P, 1)$ and $\text{zdir}_o(P, z_1)$. Thus,

- if $\text{pdir}_o(P, 1) = \text{zdir}_o(P, z_1)$, recovers SISO for $z_1 > 0$
- if $\text{pdir}_o(P, 1) \perp \text{zdir}_o(P, z_1)$, recovers SISO for $z_1 < 0$
- in between \implies blending

NMP

MP

Outline

Increasing modulus margin

Weighted sensitivity

Mixed sensitivity

Optimization-based design

Beyond modulus margin

More comprehensive requirements:

$$|S(j\omega)| \leq \begin{cases} \epsilon_\sigma & \text{if } \omega \leq \omega_0 \\ 1/\mu_m & \text{otherwise} \end{cases}$$

for some $\epsilon_\sigma < 1$, $\omega_0 > 0$ (bandwidth), and $\mu_m < 1$.

Can be cast as the H_∞ problem:

$$\|W_\sigma S\|_\infty \leq 1$$

for a stable **weighting function** W_σ such that

$$|W_\sigma(j\omega)| = \begin{cases} 1/\epsilon_\sigma & \text{if } \omega \leq \omega_0 \\ \mu_m & \text{otherwise} \end{cases} = \begin{array}{c} \text{graph of } |W_\sigma(j\omega)| \\ \text{y-axis: } 1/\epsilon_\sigma, \mu_m \\ \text{x-axis: } \omega \\ \text{step function at } \omega_0 \end{array}$$

(norms are dumb, weighted norms may be intelligent).

Weighted sensitivity problem

Problems like

$$\|W_\sigma S\|_\infty \leq 1 \quad \text{or} \quad \gamma_{\text{opt}} = \min_{\text{stabilizing } R} \|W_\sigma S\|_\infty$$

for a given W_σ are known as the **weighted sensitivity** problem.

Also a special case of the standard problem, with

$$G = \begin{bmatrix} W_\sigma & -W_\sigma P \\ I & -P \end{bmatrix}.$$

It is stabilizable whenever $W_\sigma \in H_\infty$ and all closed-loop stable systems

$$T_{zw} = W_\sigma S = W_\sigma(\tilde{X} - NQ)\tilde{M}.$$

It again makes sense to choose a co-inner $\tilde{M}(s) = \prod_{j=1}^{n_{\text{rhp}}} \frac{s-p_j}{s+p_j}$ for $\text{Re } p_j > 0$, in which case $\|T_{zw}\|_\infty = \|S_{\text{eq}}\|_\infty$, where $S_{\text{eq}} := W_\sigma(\tilde{X} - NQ)$.

Weighted sensitivity performance: outline

At each zero of $P(s)$ in $\bar{\mathbb{C}}_0$,

$$S_{eq}(z_i) = W_\sigma(z_i)\tilde{X}(z_i) = W_\sigma(z_i)/\tilde{M}(z_i).$$

Hence,

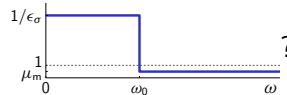
$$\|W_\sigma S\|_\infty \leq 1 \implies |W_\sigma(z_i)/\tilde{M}(z_i)| \leq 1$$

or, equivalently,

$$|W_\sigma(z_i)| \leq |\tilde{M}(z_i)| = \prod_{j=1}^{n_{rhpp}} \left| \frac{z_i - p_j}{z_i + p_j} \right| \leq 1$$

But

– how to calculate $|W_\sigma(z_i)|$ from $|W_\sigma(j\omega)| =$



21/40

Poisson integral

We know that given $G \in H_\infty$, there is an $L_\infty(j\mathbb{R})$ boundary function

$$\tilde{G}(j\omega) = \lim_{\sigma \downarrow 0} G(\sigma + j\omega)$$

for almost all ω . Interestingly, \tilde{G} completely determines G :

$$G(s) = \frac{1}{\pi} \int_{\mathbb{R}} \tilde{G}(j\omega) \frac{\operatorname{Re} s}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2} d\omega$$

for $s \in \mathbb{C}_0$, which is known as the Poisson integral.

But

– we need the magnitude $|W_\sigma(z_i)|$ from the magnitude $|W_\sigma(j\omega)|$.

Is it possible? “No” in general (e.g. 1 and e^{-s} have the same magnitude on $j\mathbb{R}$). But for a special class, “yes”.

22/40

Outer (minimum-phase) functions

Lemma

If $\phi(\omega) : \mathbb{R} \rightarrow \mathbb{R}$ is such that

$$\int_{\mathbb{R}} \frac{|\phi(\omega)|}{1 + \omega^2} d\omega < \infty,$$

then the (outer) function

$$f(s) = \exp\left(\frac{1}{\pi} \int_{\mathbb{R}} \phi(\omega) \left(\frac{\operatorname{Re} s}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2} - j \left(\frac{\operatorname{Im} s - \omega}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2} + \frac{\omega}{1 + \omega^2} \right) \right) d\omega\right)$$

belongs to H_∞ and $\lim_{\sigma \downarrow 0} \ln|f(\sigma + j\omega)| = \phi(\omega)$ for almost every ω .

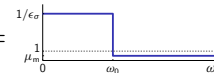
23/40

Implication

Given $|W_\sigma(j\omega)|$, there is a unique minimum-phase (outer) $W_\sigma \in H_\infty$ such that

$$\begin{aligned} |W_\sigma(s)| &= \exp\left(\frac{1}{\pi} \int_{\mathbb{R}} \ln|W_\sigma(j\omega)| \frac{\operatorname{Re} s}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2} d\omega\right) \\ &= \exp\left(\frac{1}{\pi} \int_{\mathbb{R}} \ln|W_\sigma(j\omega)| d \arctan \frac{\omega + \operatorname{Im} s}{\operatorname{Re} s}\right). \end{aligned}$$

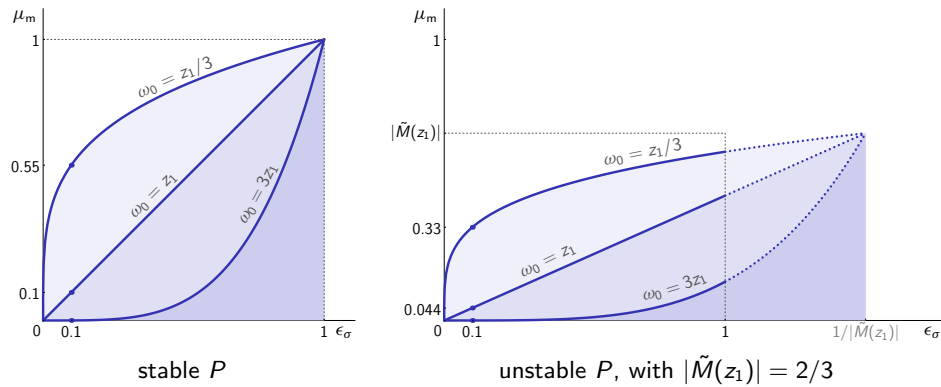
For $|W_\sigma(j\omega)| =$



$$\begin{aligned} |W_\sigma(z_1)| &= \exp\left(-\frac{2 \ln \epsilon_\sigma}{\pi} \int_0^{\omega_0} d \arctan \frac{\omega}{z_1} + \frac{2 \ln \mu_m}{\pi} \int_{\omega_0}^{\infty} d \arctan \frac{\omega}{z_1}\right) \\ &= \frac{(\mu_m)^{1-\beta_z}}{(\epsilon_\sigma)^{\beta_z}}, \quad \text{where } \beta_z := \frac{2}{\pi} \arctan \frac{\omega_0}{z_1} \in (0, 1) \end{aligned}$$

24/40

$\epsilon_\sigma - \mu_m$ tradeoff



Supports conventional wisdom that

- bandwidth cannot go beyond the location of RHP zeros.

25/40

Rational controllers

Require rational $W_\sigma(s)$. Hence, approximations (more conservatism). For example, we may use normalized Butterworth polynomials:

$$|B_n(j\omega)| = \sqrt{1 + \omega^{2n}}$$

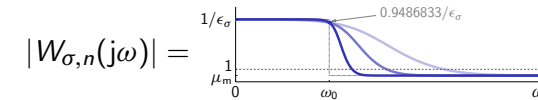
For example, $B_1(s) = s + 1$ and $B_2(s) = s^2 + \sqrt{2}s + 1$. Then,

$$W_{\sigma,n}(s) = k \frac{B_n(s/\omega_2)}{B_n(s/\omega_1)},$$

with $W_{\sigma,n}(0) = 1/\epsilon_\sigma$, $W_{\sigma,n}(\infty) = \mu_m$, and $|W_{\sigma,n}(\omega_0)| = \alpha/\epsilon_\sigma$. This yields

$$k = \frac{1}{\epsilon_\sigma}, \quad \omega_1 = \left(\frac{\alpha^2 - (\mu_m \epsilon_\sigma)^2}{1 - \alpha^2} \right)^{1/2n} \omega_0, \quad \omega_2 = \left(\frac{\alpha^2 / (\mu_m \epsilon_\sigma)^2 - 1}{1 - \alpha^2} \right)^{1/2n} \omega_0$$

For example, if $n \in \{1, 2, 5\}$ and $\alpha = \sqrt{0.9}$,



26/40

Example 1 (contd)

Let

$$P(s) = \frac{s - z_1}{s + 1}, \quad z_1 > 1$$

with $\epsilon_\sigma = 0.1$ and $\mu_m = 0.5$. Then Butterworth with $\alpha = \sqrt{0.9}$ yields

$$T_{zw}(s) = \frac{0.5s^2 + 5.47\omega_0 s + 30\omega_0^2}{s^2 + 2.45\omega_0 s + 3\omega_0^2} \left(1 - \frac{s - z_1}{s + 1} Q(s) \right).$$

Condition $|W_\sigma(z_i)| \leq |\tilde{M}(z_i)|$ yields¹ $\omega_0 \leq 0.0912z_1$. In this case,

$$T_{zw}(s) = \frac{0.5s^2 + 0.499z_1 s + 0.249z_1^2}{s^2 + 0.223z_1 s + 0.0249z_1^2} \left(1 - \frac{s - z_1}{s + 1} Q(s) \right)$$

and optimal

$$Q(s) = \frac{W_{\sigma,2}(s) - W_{\sigma,2}(z_1)}{W_{\sigma,2}(s)P(s)} = -\frac{(s + 1)(s + 0.448z_1)}{s^2 + 0.998z_1 s + 0.498z_1^2}.$$

¹Would be $\omega_0 \leq 0.38z_1$ with the original $W_\sigma(j\omega)$.

27/40

Example 1 (contd)

Optimal controller:

$$R(s) = \frac{0.5(s + 1)(s + 0.448z_1)}{s^2 + 0.223z_1 s + 0.0249z_1^2}$$

always **cancels stable poles** of the plant. Control sensitivity

$$T_c(s) = S(s)R(s) = \frac{(s + 1)(s + 0.448z_1)}{s^2 + 0.998z_1 s + 0.498z_1^2}$$

and its $|T_c(0)| = 0.9/z_1$ and

$$\|T_c\|_\infty = \sqrt{0.5 + 0.405/z_1^2} + \sqrt{0.2907 + 1.1717/z_1^4} > 1.0194$$

grow as z_1 decreases \implies higher price for keeping $|S(j\omega)| < 0.1$ if the \mathbb{C}_0 zero approaches the origin.

28/40

Outline

Increasing modulus margin

Weighted sensitivity

Mixed sensitivity

Optimization-based design

29/40

Motivation

Weighted sensitivity

$$\|W_\sigma S\|_\infty \leq 1$$

might lead to very large T_c and

- has no mechanism for *explicitly* affecting control effort.

Remedy:

- penalize T_c explicitly, like $\|W_\varkappa T_c\|_\infty \leq 1$ for some W_\varkappa .

W_\varkappa is expected to penalize

- high control gain at high frequencies
- encourage required high-frequency roll-off

Possible choice

$$|W_\varkappa(j\omega)| = \frac{\max\{1, (\omega/\omega_1)^\nu\}}{\varkappa} = \frac{1}{\varkappa} \begin{cases} 1 & \omega < \omega_1 \\ (\omega/\omega_1)^\nu & \omega > \omega_1 \end{cases}$$

for some ω_1 and roll-off $\nu \in \mathbb{N}$.

30/40

Incorporating both S and T_c

Multidisk formulation:

$$(\|W_\sigma S\|_\infty \leq 1) \wedge (\|W_\varkappa T_c\|_\infty \leq 1).$$

In the SISO case is equivalent to $\begin{bmatrix} W_\sigma(j\omega)S(j\omega) \\ W_\varkappa(j\omega)T_c(j\omega) \end{bmatrix} \in \mathcal{B}_\infty$ for all ω (**hard**).

Mixed sensitivity formulation:

$$\left\| \begin{bmatrix} W_\sigma S \\ W_\varkappa T_c \end{bmatrix} \right\|_\infty \leq 1.$$

In the SISO case is equivalent to $\begin{bmatrix} W_\sigma(j\omega)S(j\omega) \\ W_\varkappa(j\omega)T_c(j\omega) \end{bmatrix} \in \mathcal{B}_2$ for all ω (**doable**).
Justified by $\mathcal{B}_2 \subset \mathcal{B}_\infty$, but

- adds conservatism.

31/40

Mixed sensitivity problem

Find if

$$\min_{\text{stabilizing } R} \left\| \begin{bmatrix} W_\sigma S \\ W_\varkappa T_c \end{bmatrix} \right\|_\infty \leq 1.$$

Also a special case of the standard problem, with

$$G = \begin{bmatrix} W_\sigma & -W_\sigma P \\ 0 & W_\varkappa \\ I & -P \end{bmatrix}.$$

It is stabilizable whenever $W_\sigma, W_\varkappa \in H_\infty$ and all closed-loop stable systems

$$T_{zw} = \begin{bmatrix} W_\sigma & 0 \\ 0 & W_\varkappa \end{bmatrix} \left(\begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} + \begin{bmatrix} -N \\ M \end{bmatrix} Q \right) \tilde{M}.$$

Unstable W_\varkappa G above is internally stabilizable also if $W_\varkappa(s)$ contains unstable poles of $P(s)$, like $W_\varkappa = W_\tau P$ for some $W_\tau \in H_\infty$. This would correspond to

$$\min_{\text{stabilizing } R} \left\| \begin{bmatrix} W_\sigma S \\ W_\tau T \end{bmatrix} \right\|_\infty \leq 1$$

(another version of mixed sensitivity).

32/40

Example 1: problem

Plant:

$$P(s) = \frac{1}{s^2 + 0.1s + 1}$$

Specs:

- $\epsilon_\sigma = 0.1$ with ω_0 as large as possible,
- $\mu_m \geq 0.5$,
- $\varkappa = 10$ and roll-off 1 for $\omega > \omega_1 = 2.4$ [rad/sec].

33/40

Example 1: weighting functions

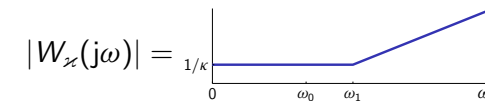
Sensitivity (with $\alpha = \sqrt{0.9}$):

$$W_{\sigma,2}(s) = \frac{0.5s^2 + 5.47\omega_0s + 30\omega_0^2}{s^2 + 2.45\omega_0s + 3\omega_0^2}$$

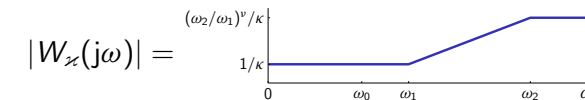
Control sensitivity (with $\omega_2 = 1000\omega_1$):

$$W_{\varkappa,1}(s) = \frac{100(s + 2.4)}{s + 2400}$$

where



is replaced with

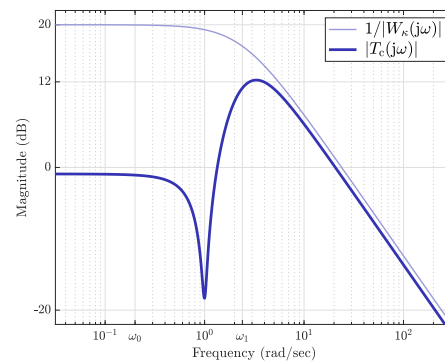
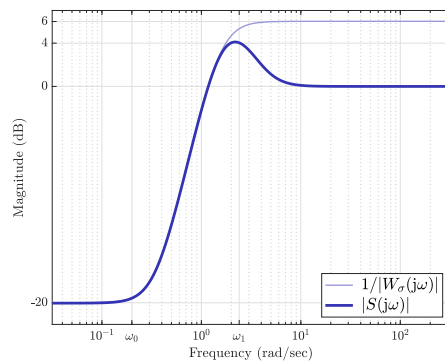


for $\omega_2 \gg \omega_1$ to have $W_\varkappa \in H_\infty$ and Butterworth approximations are used.

34/40

Example 1: results

Maximal $\omega_0 = 0.2$ [rad/sec], with



35/40

Example 1: controller

Off the shelf:

$$R(s) = \frac{15367(s + 2400)(s + 0.9364)(s^2 + 0.1s + 1)}{(s + 1.774 \cdot 10^6)(s^2 + 0.4896s + 0.1198)(s^2 + 6.372s + 17.97)}$$

First:

- far left pole—numerical artifact
- far left zero—pole of $W_{\varkappa,1}(s)$

general property

can be removed. Thus, after some massage

$$R(s) = \frac{20.787(s + 0.9364)(s^2 + 0.1s + 1)}{(s^2 + 0.4896s + 0.1198)(s^2 + 6.372s + 17.97)}$$

Important:

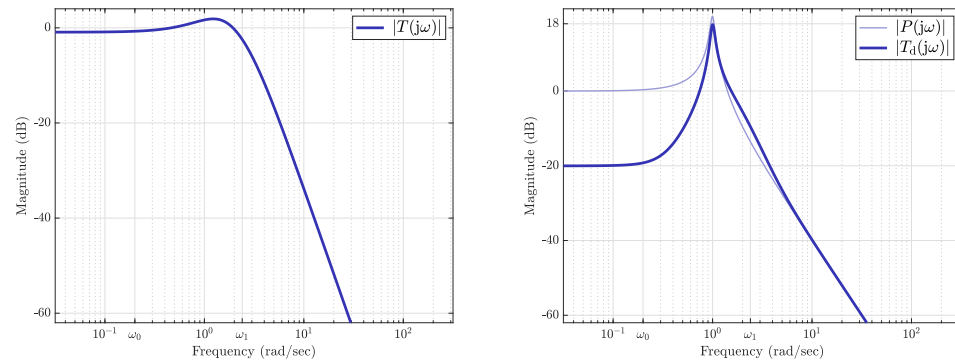
- cancels all stable plant poles

general property too

36/40

Example 1: results (contd)

Closed-loop transfer functions, not explicitly involved in the cost:



37/40

Outline

Increasing modulus margin

Weighted sensitivity

Mixed sensitivity

Optimization-based design

38/40

General remarks

- “Dumb” (non-selective) norms become “intelligent” (selective) via the use of **weights**
- Optimal performance level is **success indicator**, not of interest per se
- All requirements can never be squeezed into one cost function

39/40

General remarks (contd)

Pros

- Separates
 - hardly formalizable, but technically simple, specs (weights) selection from
 - technically nontrivial design for given specifications
- Conclusive (e.g. optimal norm $> 1 \iff$ specs cannot be met)
- Extends to MIMO mutatis mutandis
e.g. MIMO μ_m is still $1/\|S_o\|_\infty$, even though no graphical interpretation is available

Cons

- Optimization shows **no mercy**, very good on finding loopholes
 - unaccounted dynamics might be poor
 - controller order might be large
 - controller properties could be problematic (e.g. unstable R)

40/40