Linear Control Systems (036012)
chapter 7

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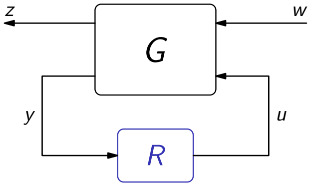
The standard problem



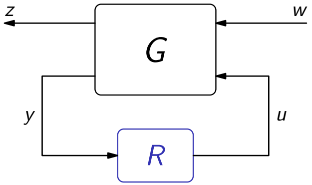
Given G (plant, weights, fixed parts of controller, etc), design R that

- internally stabilizes the system
- minimizes a size of the mapping $T_{zw} := \mathcal{F}_l(G, R) : w \mapsto z$

Depending on the measure of this "size"

- H_2 standard problem, if $\|T_{zw}\|_2$ is minimized — LQG, Kalman filtering
- H_∞ standard problem, if $\|T_{zw}\|_\infty$ is minimized — mixed sensitivity
- or mixed H_2/H_∞, L_1, et cetera

The standard problem



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Preliminary: GoF as the standard problem

We already know (Lect. 9, but for negative feedback) that

$$\begin{bmatrix} T_c & T_i \\ S_o & T_d \end{bmatrix} := \begin{bmatrix} R \\ I \end{bmatrix} (I + PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} = \mathcal{F}_l\left(\left[\begin{array}{cc|c} 0 & 0 & I \\ I & P & -P \\ \hline I & P & -P \end{array}\right], R \right).$$

It is internally stabilizable, because

$$\left[\begin{array}{cc|c} 0 & 0 & I \\ I & P & -P \\ \hline I & P & -P \end{array}\right] = \left[\begin{array}{cc|c} 0 & I & M \\ I & 0 & -N \\ \hline I & 0 & -N \end{array}\right] \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & I & M \end{array}\right]^{-1} = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & -I \\ \hline 0 & 0 & \tilde{M} \end{array}\right]^{-1} \left[\begin{array}{cc|c} 0 & 0 & I \\ 0 & 0 & 0 \\ \hline \tilde{M} & \tilde{N} & -\tilde{N} \end{array}\right]$$

are its coprime factorizations over RH_∞. All stabilizing controllers

$$R = (\tilde{Y} + MQ)(\tilde{X} - NQ)^{-1} = (X - Q\tilde{N})^{-1}(Y + Q\tilde{M})$$

(with $\det(\tilde{X}(\infty) - N(\infty)Q(\infty)) \neq 0$ or $\det(X(\infty) - Q(\infty)\tilde{N}(\infty)) \neq 0$), so

$$T_{zw} = \left(\begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix} + \begin{bmatrix} M \\ -N \end{bmatrix} Q \right) \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}.$$

Outline

Increasing modulus margin

Weighted sensitivity

Mixed sensitivity

Optimization-based design

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Increasing modulus margin

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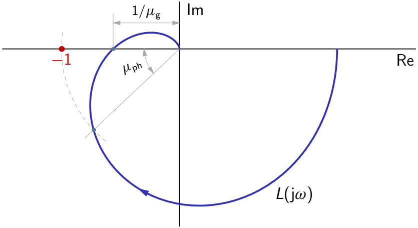
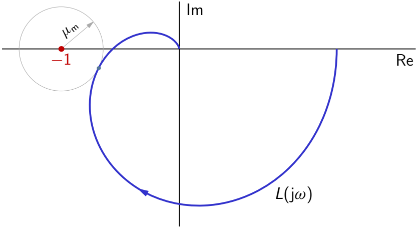
Stability margins



gain (μ_g) and phase (μ_{ph}) margins



modulus margin μ_m

If $\mu_m \leq 1$, then

$$\mu_g \geq \frac{1}{1-\mu_m} \quad \text{and} \quad \mu_{ph} \geq 2 \arcsin \frac{\mu_m}{2}.$$

Modulus margin and H_∞

If the closed-loop system is stable,

$$\mu_\mathrm{m} = \inf_{\omega\in\mathbb{R}} |1 + L(\mathrm{j}\omega)| \implies \frac{1}{\mu_\mathrm{m}} = \sup_{\omega\in\mathbb{R}} |S(\mathrm{j}\omega)| = \|S\|_\infty .$$

Hence,

$$\text{minimizing } \|S\|_\infty \iff \text{maximizing } \mu_\mathrm{m}$$

Reminder:

$$\|S\|_\infty := \sup_{s\in\mathbb{R}_0} |S(s)| = \sup_{\omega\in\mathbb{R}} |S(\mathrm{j}\omega)|$$

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The standard problem for modulus margin maximization

Generalized plant:

$$\begin{bmatrix} T_\mathrm{c} & T_\mathrm{i} \\ S_\mathrm{o} & T_\mathrm{d} \end{bmatrix} = \mathcal{F}_\mathrm{l}\left(\left[\begin{array}{cc:c} 0 & 0 & I \\ I & P & -P \\ \hdashline I & P & -P \end{array}\right], R\right),$$

so that

$$G(s) = \begin{bmatrix} I & -P(s) \\ I & -P(s) \end{bmatrix}.$$

All stable sensitivity functions:

$$T_{zw} = \left(\begin{bmatrix} \tilde{Y} \\ \tilde{X} \end{bmatrix} + \begin{bmatrix} M \\ -N \end{bmatrix} Q\right) \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix},$$

so that

$$T_{zw} = S = (\tilde{X} - NQ)\tilde{M}.$$

Example 1

Let

$$P(s) = \frac{s - z_1}{s + 1}, \quad z_1 \in \mathbb{R}.$$

Coprime factors, Bézout coefficients: $M = X = \tilde{M} = \tilde{X} = 1$, $N = \tilde{N} = P$.
All stable sensitivity functions:

$$S(s) = 1 - \frac{s - z_1}{s + 1} Q(s).$$

Minimum-phase plant: If $z_1 < 0$, then the optimal $Q(s) = (s+1)/(s-z_1)$ is not admissible, since $\tilde{X}(\infty) - N(\infty)Q(\infty) = 0$. But

$$Q(s) = \frac{(1-\epsilon)s + 1}{s - z_1} \implies S(s) = \frac{\epsilon s}{s + 1}$$

and $\|S\|_\infty$ can be made arbitrarily small.

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Example 1 (contd)

Nonminimum-phase plant: If $z_1 \geq 0$, then

$$S(s) = 1 - \frac{s - z_1}{s + 1} Q(s) \quad \Longrightarrow \quad S(z_1) = 1$$

independently of Q. Hence,

– $\|S\|_\infty \geq 1$.

In fact, the trivial choice

$$Q(s) = 0 \implies R(s) = 0$$

attains the bound, rendering $S(s) = 1$ (not quite meaningful though).

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Example 2

Let

$$P(s) = \frac{s - z_1}{s^2 - 1} = \frac{s - z_1}{(s-1)(s+1)}, \quad z_1 \in \mathbb{R}.$$

Choose

$$M(s) = \tilde{M}(s) = \frac{s-1}{s+a} \quad \text{and} \quad N(s) = \tilde{N}(s) = \frac{s - z_1}{(s+a)(s+1)}$$

for any $a > 0$. The corresponding Bézout coefficients are

$$X(s) = \tilde{X}(s) = \frac{s + (az_1 + 2z_1 + a)/(z_1 - 1)}{s+1}$$

and

$$Y(s) = \tilde{Y}(s) = -\frac{2(a+1)}{z_1 - 1}.$$

Hence,

$$R(s) = \frac{-2(a+1)(s+a) + (z_1 - 1)(s-1)Q(s)}{(s+a)((z_1-1)s + az_1 + 2z_1 + a) - (z_1 - 1)(s - z_1)Q(s)}.$$

Example 2 (contd)

We end up with

$$S(s) = \left(\frac{s + (az_1 + 2z_1 + a)/(z_1 - 1)}{s + 1} - \frac{s - z_1}{(s+1)(s+a)}\, Q(s) \right) \frac{s-1}{s+a}.$$

The choice $a = 1$ renders $\tilde{M}(s)$ co-inner (and inner), as then

$$\tilde{M}(s)\tilde{M}^{\sim}(s) = \frac{s-1}{s+1}\frac{-s-1}{-s+1} = 1.$$

Hence, $\|S\|_\infty = \|S_{\text{eq}}\|_\infty$, where

$$S_{\text{eq}}(s) := \frac{s + (3z_1 + 1)/(z_1 - 1)}{s + 1} - \frac{s - z_1}{(s+1)^2}\, Q(s).$$

If $z_1 < 0$, then $S_{\text{eq}}(\infty) = 1$ for all $Q \in RH_\infty$ and $\|S\|_\infty \geq 1$. Attainable by

$$Q(s) = \frac{\tilde{X}(s) - \tilde{X}(\infty)}{N(s)} = \frac{2(z_1 + 1)}{z_1 - 1}\frac{s+1}{s - z_1}$$

or, for $z_1 \in [-1, 0)$, by $Q = 0$.

Example 2 (contd)

We end up with

$$S(s) = \left(\frac{s + (az_1 + 2z_1 + a)/(z_1 - 1)}{s+1} - \frac{s - z_1}{(s+1)(s+a)}\, Q(s) \right) \frac{s-1}{s+a}.$$

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If $z_1 \geq 0$, then $S_{\mathrm{eq}}(z_1) = (z_1+1)/(z_1-1)$ and

$$\|S\|_\infty = \|S_{\mathrm{eq}}\|_\infty \geq \frac{z_1+1}{|z_1-1|}.$$

Attainable by non-proper

$$Q(s) = \frac{\tilde{X}(s) - \tilde{X}(z_1)}{N(s)} = -\frac{2}{z_1-1}(s+1).$$

or, almost, by

$$Q(s) = -\frac{2}{z_1-1}\frac{s+1}{\epsilon s+1}.$$

Controller:

$$R(s) = -\frac{2(s+1)((1+2\epsilon)s+1)}{\epsilon(z_1-1)s^2+(z_1+1+(3z_1+1)\epsilon)s+z_1+1} \xrightarrow{\epsilon\to 0} -\frac{2(s+1)}{z_1+1}$$

$R(0) = -2/(z_1+1)$, $R(\infty) = -2(\epsilon^{-1}+2)/(z_1-1)$, unstable for $z_1 < 1$.

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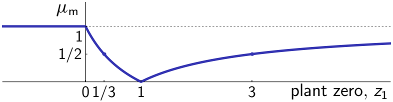
Example 2: supremal modulus margin

We end up with

$$\sup_{\text{stabilizing } R} \mu_{\mathrm{m}} = \begin{cases} 1 & \text{if } z_1 < 0 \\ \frac{|z_1 - 1|}{z_1 + 1} & \text{if } z_1 \geq 0 \end{cases} =$$



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Example 3

Let

$$P(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{s-1} + \begin{bmatrix} \alpha & -2z_1/(z_1-1) \\ -\alpha\beta & \beta \end{bmatrix} \frac{1}{s+1}$$

has an unstable pole at $s = 1$ with

$$\mathrm{pdir_i}(P, 1) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad \mathrm{pdir_o}(P, 1) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

and a (nonminimum-phase) zero at $s = z_1$ with

$$\mathrm{zdir_i}(P, z_1) = \mathrm{span}\left(\begin{bmatrix} 1 \\ \alpha \end{bmatrix}\right) \quad \text{and} \quad \mathrm{zdir_o}(P, z_1) = \mathrm{span}\left(\begin{bmatrix} \beta \\ 1 \end{bmatrix}\right).$$

Example 3 (contd)

Choose

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} = \left[\begin{array}{cc:cc} \frac{s-1}{s+1} & 0 & -\frac{\alpha(s-1)}{(s+1)^2} & \frac{(z_1+1)s-3z_1+1}{(z_1-1)(s+1)^2} \\ 0 & 1 & \frac{\alpha\beta}{s+1} & -\frac{\beta}{s+1} \\ \hdashline 0 & 0 & 1 & 0 \\ 2 & \frac{2}{\beta} & 0 & 1+\frac{z_1+1}{z_1-1}\frac{2}{s+1} \end{array}\right]$$

and

$$\begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & M \end{bmatrix} = \left[\begin{array}{cc:cc} 1+\frac{z_1}{z_1-1}\frac{4}{s+1} & 2\frac{(z_1+1)s-3z_1+1}{\beta(z_1-1)(s+1)^2} & \frac{\alpha}{s+1} & -\frac{(z_1+1)s-3z_1+1}{(z_1-1)(s+1)^2} \\ -\frac{2\beta}{s+1} & \frac{s^2+3}{(s+1)^2} & -\frac{\alpha\beta}{s+1} & -\frac{\beta(s-1)}{(s+1)^2} \\ \hdashline 0 & 0 & 1 & 0 \\ -2 & -\frac{2}{\beta}\frac{s-1}{s+1} & 0 & \frac{s-1}{s+1} \end{array}\right]$$

with co-inner $\tilde{M}(s)$.

Example 3 (contd)

As $\|S\|_\infty = \|S_{\text{eq}}\|_\infty$ for

$$S_{\text{eq}}(s) := \tilde{X}(s) - N(s)Q(s)$$

we have two constraints:

$$S_{\text{eq}}(\infty) = \tilde{X}(\infty) = I$$

and

$$\eta' S_{\text{eq}}(z_1) = \eta' \tilde{X}(z_1) = \eta' \tilde{M}^{-1}(z_1), \quad \text{for any } \eta \in \text{zdir}_\text{o}(P, z_1)$$

(the latter follows by $\tilde{X} = \tilde{M}^{-1} - P\tilde{Y}$).

Example 3: supremal modulus margin

Thus,

$$\mu_{\mathrm{m}} \le \sqrt{\left(\frac{z_1+1}{z_1-1}\right)^2 \cos^2\theta_{\mathrm{o}} + \sin^2\theta_{\mathrm{o}}} \in \left(1, \frac{z_1+1}{|z_1-1|}\right).$$

where

$$\theta_{\mathrm{o}} = \arccos\frac{\beta}{\sqrt{1+\beta^2}} \in (0,\pi)$$

is the angle between $\mathrm{pdir}_{\mathrm{o}}(P,1)$ and $\mathrm{zdir}_{\mathrm{o}}(P,z_1)$. Thus,

- if $\mathrm{pdir}_{\mathrm{o}}(P,1) = \mathrm{zdir}_{\mathrm{o}}(P,z_1)$, recovers SISO for $z_1 > 0$ NMP
- if $\mathrm{pdir}_{\mathrm{o}}(P,1) \perp \mathrm{zdir}_{\mathrm{o}}(P,z_1)$, recovers SISO for $z_1 < 0$ MP
- in between $\implies$ blending

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More comprehensive requirements:

$$|S(\mathrm{j}\omega)| \le \begin{cases} \epsilon_\sigma & \text{if } \omega \le \omega_0 \\ 1/\mu_\mathrm{m} & \text{otherwise} \end{cases}$$

for some $\epsilon_\sigma < 1$, $\omega_0 > 0$ (bandwidth), and $\mu_\mathrm{m} < 1$.

Can be cast as the H_∞ problem:

$$\|W_\sigma S\|_\infty \le 1$$

for a stable weighting function W_σ such that

$$|W_\sigma(\mathrm{j}\omega)| = \begin{cases} 1/\epsilon_\sigma & \text{if } \omega \le \omega_0 \\ \mu_\mathrm{m} & \text{otherwise} \end{cases} =$$

(norms are dumb, weighted norms may be intelligent).

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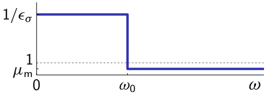

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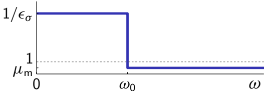

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Weighted sensitivity problem

Problems like

$$\|W_\sigma S\|_\infty \le 1 \quad \text{or} \quad \gamma_{\text{opt}} = \min_{\text{stabilizing } R} \|W_\sigma S\|_\infty$$

for a given W_σ are known as the weighted sensitivity problem.

Also a special case of the standard problem, with

$$G = \begin{bmatrix} W_\sigma & -W_\sigma P \\ I & -P \end{bmatrix}.$$

It is stabilizable whenever $W_\sigma \in H_\infty$ and all closed-loop stable systems

$$T_{zw} = W_\sigma S = W_\sigma(\tilde{X} - NQ)\tilde{M}.$$

It again makes sense to choose a co-inner $\tilde{M}(s) = \prod_{j=1}^{n_{\text{rhpp}}} \frac{s - p_j}{s + p_j}$ for $\operatorname{Re} p_j > 0$, in which case $\|T_{zw}\|_\infty = \|S_{\text{eq}}\|_\infty$, where $S_{\text{eq}} := W_\sigma(\tilde{X} - NQ)$.

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$$T_{zw} = W_\sigma S = W_\sigma(\tilde{X} - NQ)\tilde{M}.$$

It again makes sense to choose a co-inner $\tilde{M}(s) = \prod_{j=1}^{n_{\text{rhpp}}} \frac{s-p_j}{s+p_j}$ for $\operatorname{Re} p_j > 0$, in which case $\|T_{zw}\|_\infty = \|S_{\text{eq}}\|_\infty$, where $S_{\text{eq}} := W_\sigma(\tilde{X} - NQ)$.

Weighted sensitivity performance: outline

At each zero of $P(s)$ in $\bar{\mathbb{C}}_0$,

$$S_{\text{eq}}(z_i) = W_\sigma(z_i)\tilde{X}(z_i) = W_\sigma(z_i)/\tilde{M}(z_i).$$

Hence,

$$\|W_\sigma S\|_\infty \le 1 \implies |W_\sigma(z_i)/\tilde{M}(z_i)| \le 1$$

or, equivalently,

$$|W_\sigma(z_i)| \le |\tilde{M}(z_i)| = \prod_{j=1}^{n_{\text{rhpp}}} \left| \frac{z_i - p_j}{z_i + p_j} \right| \le 1$$

But

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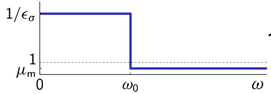
– how to calculate $|W_\sigma(z_i)|$ from $|W_\sigma(\mathrm{j}\omega)| =$  ?

Poisson integral

We know that given $G \in H_\infty$, there is an $L_\infty(\mathrm{j}\mathbb{R})$ boundary function

$$\tilde{G}(\mathrm{j}\omega) = \lim_{\sigma \downarrow 0} G(\sigma + \mathrm{j}\omega)$$

for almost all ω. Interestingly, $\tilde{G}$ completely determines G:

$$G(s) = \frac{1}{\pi} \int_{\mathbb{R}} \tilde{G}(\mathrm{j}\omega) \frac{\operatorname{Re} s}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2} \mathrm{d}\omega$$

for $s \in \mathbb{C}_0$, which is known as the Poisson integral.

But

– we need the magnitude $|W_\sigma(z_i)|$ from the magnitude $|W_\sigma(\mathrm{j}\omega)|$.

Is it possible? "No" in general (e.g. 1 and e^{-s} have the same magnitude on $\mathrm{j}\mathbb{R}$). But for a special class, "yes".

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Outer (minimum-phase) functions

Lemma

If $\phi(\omega) : \mathbb{R} \to \mathbb{R}$ *is such that*

$$\int_{\mathbb{R}} \frac{|\phi(\omega)|}{1+\omega^2}\,\mathrm{d}\omega < \infty,$$

then the (outer) function

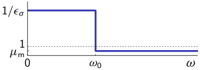
$$f(s) = \exp\biggl(\frac{1}{\pi}\int_{\mathbb{R}} \phi(\omega)\biggl(\frac{\operatorname{Re} s}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2} - \mathrm{j}\Bigl(\frac{\operatorname{Im} s - \omega}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2} + \frac{\omega}{1+\omega^2}\Bigr)\biggr)\,\mathrm{d}\omega\biggr)$$

belongs to H_∞ *and* $\lim_{\sigma\downarrow 0} \ln|f(\sigma + \mathrm{j}\omega)| = \phi(\omega)$ *for almost every* ω*.*

Implication

Given $|W_\sigma(\mathrm{j}\omega)|$, there is a unique minimum-phase (outer) $W_\sigma \in H_\infty$ such that

$$|W_\sigma(s)| = \exp\left(\frac{1}{\pi}\int_{\mathbb{R}} \ln|W_\sigma(\mathrm{j}\omega)| \frac{\operatorname{Re} s}{(\operatorname{Re} s)^2 + (\operatorname{Im} s - \omega)^2}\,\mathrm{d}\omega\right)$$
$$= \exp\left(\frac{1}{\pi}\int_{\mathbb{R}} \ln|W_\sigma(\mathrm{j}\omega)|\,\mathrm{d}\arctan\frac{\omega + \operatorname{Im} s}{\operatorname{Re} s}\right).$$

For $|W_\sigma(\mathrm{j}\omega)| =$ 

,

$$|W_\sigma(z_1)| = \exp\left(-\frac{2\ln\epsilon_\sigma}{\pi}\int_0^{\omega_0}\mathrm{d}\arctan\frac{\omega}{z_1} + \frac{2\ln\mu_{\mathrm{m}}}{\pi}\int_{\omega_0}^{\infty}\mathrm{d}\arctan\frac{\omega}{z_1}\right)$$

$$= \frac{(\mu_{\mathrm{m}})^{1-\beta_z}}{(\epsilon_\sigma)^{\beta_z}}, \quad \text{where } \beta_z := \frac{2}{\pi}\arctan\frac{\omega_0}{z_1} \in (0,1)$$

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\end{aligned}
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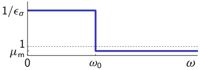
For $|W_\sigma(\mathrm{j}\omega)| =$ 

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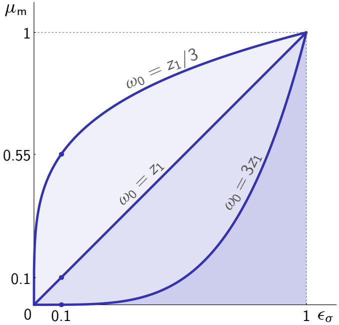
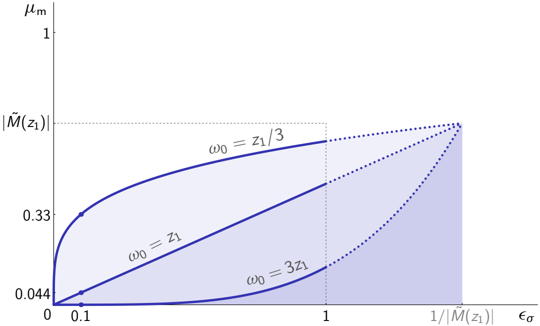
ϵ_σ–μ_m tradeoff



stable P



unstable P, with $|\tilde{M}(z_1)| = 2/3$

Supports conventional wisdom that

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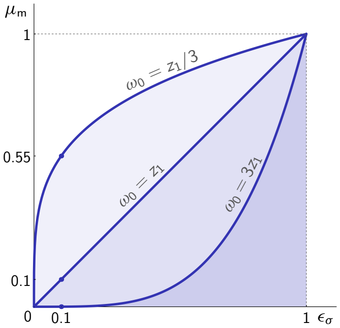

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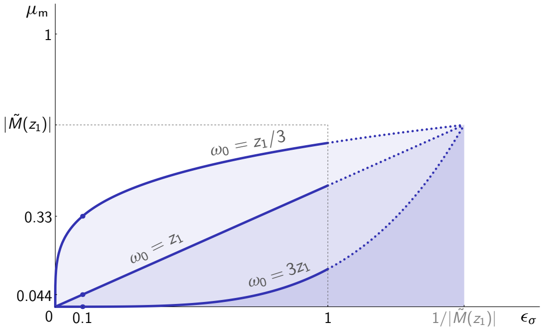

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Rational controllers

Require rational $W_\sigma(s)$. Hence, approximations (more conservatism). For example, we may use normalized Butterworth polynomials:

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For example, $B_1(s) = s+1$ and $B_2(s) = s^2 + \sqrt{2}s + 1$. Then,

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$$k = \frac{1}{\epsilon_\sigma}, \quad \omega_1 = \left(\frac{\alpha^2 - (\mu_{\mathrm{m}}\epsilon_\sigma)^2}{1-\alpha^2}\right)^{1/2n}\omega_0, \quad \omega_2 = \left(\frac{\alpha^2/(\mu_{\mathrm{m}}\epsilon_\sigma)^2 - 1}{1-\alpha^2}\right)^{1/2n}\omega_0$$

For example, if $n \in \{1, 2, 5\}$ and $\alpha = \sqrt{0.9}$,

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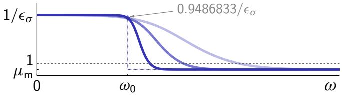
$|W_{\sigma,n}(\mathrm{j}\omega)| =$ 

Example 1 (contd)

Let

$$P(s) = \frac{s - z_1}{s + 1}, \quad z_1 > 1$$

with $\epsilon_\sigma = 0.1$ and $\mu_\mathrm{m} = 0.5$. Then Butterworth with $\alpha = \sqrt{0.9}$ yields

$$T_{zw}(s) = \frac{0.5s^2 + 5.47\omega_0 s + 30\omega_0^2}{s^2 + 2.45\omega_0 s + 3\omega_0^2}\left(1 - \frac{s - z_1}{s + 1}\,Q(s)\right).$$

Condition $|W_\sigma(z_i)| \le |\tilde{M}(z_i)|$ yields[1] $\omega_0 \le 0.0912 z_1$. In this case,

$$T_{zw}(s) = \frac{0.5s^2 + 0.499 z_1 s + 0.249 z_1^2}{s^2 + 0.223 z_1 s + 0.0249 z_1^2}\left(1 - \frac{s - z_1}{s + 1}\,Q(s)\right)$$

and optimal

$$Q(s) = \frac{W_{\sigma,2}(s) - W_{\sigma,2}(z_1)}{W_{\sigma,2}(s)P(s)} = \frac{(s + 1)(s + 0.448 z_1)}{s^2 + 0.998 z_1 s + 0.498 z_1^2}.$$

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$$R(s) = \frac{0.5(s+1)(s+0.448z_1)}{s^2+0.223z_1 s+0.0249z_1^2}$$

always cancels stable poles of the plant. Control sensitivity

$$T_c(s) = S(s)R(s) = \frac{(s+1)(s+0.448z_1)}{s^2+0.998z_1 s+0.498z_1^2}$$

and its $|T_c(0)| = 0.9/z_1$ and

$$\|T_c\|_\infty = \sqrt{0.5+0.405/z_1^2+\sqrt{0.2907+1.1717/z_1^4}} > 1.0194$$

grow as z_1 decreases $\Longrightarrow$ higher price for keeping $|S(j\omega)| < 0.1$ if the $\mathbb{C}_0$ zero approaches the origin.

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Outline

Increasing modulus margin

Weighted sensitivity

Mixed sensitivity

Optimization-based design

Motivation

Weighted sensitivity

$$\|W_\sigma S\|_\infty \le 1$$

might lead to very large T_c and

- has no mechanism for *explicitly* affecting control effort.

Remedy:

- penalize T_c explicitly, like $\|W_\varkappa T_c\|_\infty \le 1$ for some $W_\varkappa$.

$W_\varkappa$ is expected to penalize

- high control gain at high frequencies
- encourage required high-frequency roll-off

Possible choice

$$|W_\varkappa(j\omega)| = \frac{\max\{1, (\omega/\omega_1)^\nu\}}{\varkappa} =$$

for some ω_1 and roll-off $\nu \in \mathbb{N}$.

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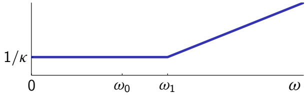

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Incorporating both S and T_c

Multidisk formulation:

$$(\|W_\sigma S\|_\infty \le 1) \wedge (\|W_\varkappa T_c\|_\infty \le 1).$$

In the SISO case is equivalent to $\begin{bmatrix} W_\sigma(j\omega)S(j\omega) \\ W_\varkappa(j\omega)T_c(j\omega) \end{bmatrix} \in \mathcal{B}_\infty$ for all ω (hard).

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Mixed sensitivity problem

Find if

$$\min_{\text{stabilizing } R} \left\| \begin{bmatrix} W_\sigma S \\ W_\varkappa T_c \end{bmatrix} \right\|_\infty \le 1.$$

Also a special case of the standard problem, with

$$G = \left[\begin{array}{c:c} W_\sigma & -W_\sigma P \\ 0 & W_\varkappa \\ \hdashline I & -P \end{array}\right].$$

It is stabilizable whenever $W_\sigma, W_\varkappa \in H_\infty$ and all closed-loop stable systems

$$T_{zw} = \begin{bmatrix} W_\sigma & 0 \\ 0 & W_\varkappa \end{bmatrix} \left(\begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} + \begin{bmatrix} -N \\ M \end{bmatrix} Q \right) \tilde{M}.$$

Unstable W_σ: G above is internally stabilizable also if $W_\sigma(s)$ contains unstable poles of $P(s)$, like $W_\sigma = W_\tau P$ for some $W_\tau \in H_\infty$. This would correspond to

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(another version of mixed sensitivity).

Example 1: problem

Plant:

$$P(s) = \frac{1}{s^2 + 0.1s + 1}$$

Specs:

- $\epsilon_\sigma = 0.1$ with ω_0 as large as possible,
- $\mu_{\mathrm{m}} \geq 0.5$,
- $\varkappa = 10$ and roll-off 1 for $\omega > \omega_1 = 2.4\,[\mathrm{rad/sec}]$.

Example 1: weighting functions

Sensitivity (with $\alpha = \sqrt{0.9}$):

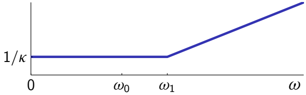
$$W_{\sigma,2}(s) = \frac{0.5s^2 + 5.47\omega_0 s + 30\omega_0^2}{s^2 + 2.45\omega_0 s + 3\omega_0^2}$$

Control sensitivity (with $\omega_2 = 1000\omega_1$):

$$W_{\varkappa,1}(s) = \frac{100(s + 2.4)}{s + 2400}$$

where

$$|W_{\varkappa}(\mathrm{j}\omega)| =$$



is replaced with

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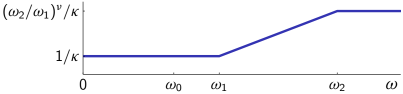
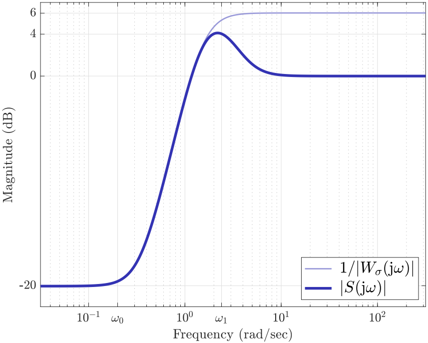
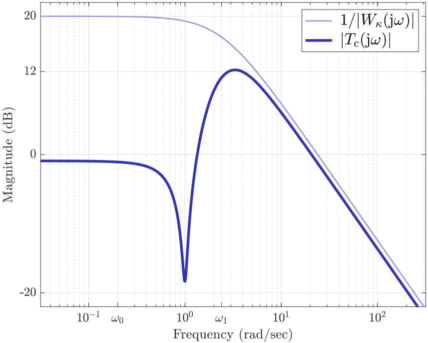


for $\omega_2 \gg \omega_1$ to have $W_{\varkappa} \in H_\infty$ and Butterworth approximations are used.

Example 1: results

Maximal $\omega_0 = 0.2$ [rad/sec], with





Example 1: controller

Off the shelf:

$$R(s) = \frac{15367(s + 2400)(s + 0.9364)(s^2 + 0.1s + 1)}{(s + 1.774 \cdot 10^6)(s^2 + 0.4896s + 0.1198)(s^2 + 6.372s + 17.97)}$$

First:

- far left pole—numerical artifact
- far left zero—pole of $W_{\varkappa,1}(s)$ general property

can be removed. Thus, after some massage

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Important:

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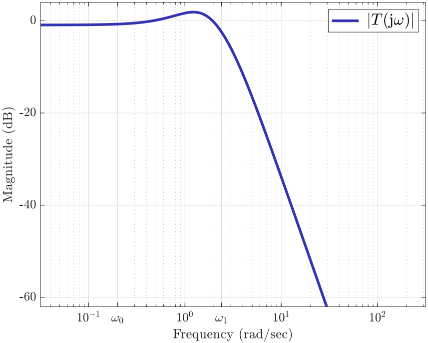
Closed-loop transfer functions, not explicitly involved in the cost:



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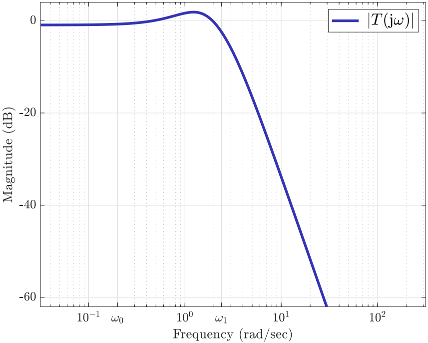

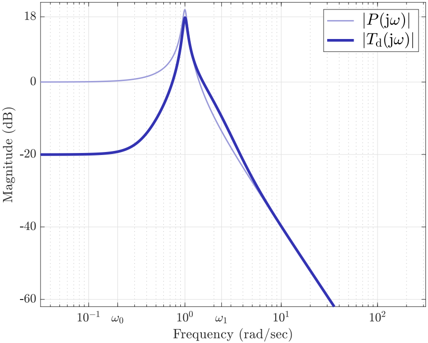

Outline

Increasing modulus margin

Weighted sensitivity

Mixed sensitivity

Optimization-based design

General remarks

- "Dumb" (non-selective) norms become "intelligent" (selective) via the use of weights

- Optimal performance level is success indicator, not of interest per se

- All requirements can never be squeezed into one cost function

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General remarks (contd)

Pros

- Separates
  - hardly formalizable, but technically simple, specs (weights) selection from
  - technically nontrivial design for given specifications
- Conclusive (e.g. optimal norm $> 1 \iff$ specs cannot be met)
- Extends to MIMO mutatis mutandis

  e.g. MIMO μ_m is still $1/\|S_o\|_\infty$, even though no graphical interpretation is available

Cons

- Optimization shows no mercy, very good on finding loopholes
  - unaccounted dynamics might be poor
  - controller order might be large
  - controller properties could be problematic (e.g. unstable R)

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