Linear Control Systems (036012)
chapter 6

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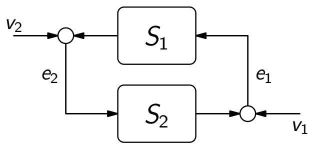
Outline

Definition and criterion



is internally stable if all four systems $v_i \mapsto e_j$ are stable (in H_∞). Because

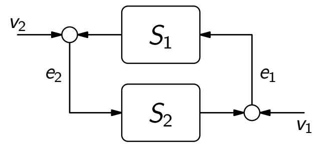
$$\begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

the system is internally stable iff

$$\begin{aligned}\begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix}^{-1} &= \left(\begin{bmatrix} I & -S_2 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} S_1 \begin{bmatrix} I & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} I & S_2 \\ 0 & I \end{bmatrix} + \begin{bmatrix} S_2 \\ I \end{bmatrix} S_1 (I - S_2 S_1)^{-1} \begin{bmatrix} I & S_2 \end{bmatrix} \\ &= \begin{bmatrix} (I - S_2 S_1)^{-1} & S_2 (I - S_1 S_2)^{-1} \\ S_1 (I - S_2 S_1)^{-1} & (I - S_1 S_2)^{-1} \end{bmatrix}\end{aligned}$$

is stable.

Example 1



If

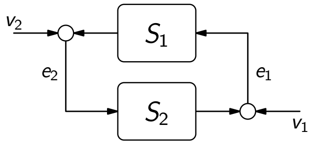
$$S_1(s) = \frac{1}{s} \quad \text{and} \quad S_2(s) = -\frac{s}{s+1},$$

then

$$\begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} (s+1)/(s+2) & -s/(s+2) \\ (s+1)/(s(s+2)) & (s+1)/(s+2) \end{bmatrix}$$

is unstable.

Example 2



If

$$S_1(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S_2(s) = -\begin{bmatrix} s/(s+1) & 0 \\ 0 & 1/s \end{bmatrix}.$$

then

$$\begin{bmatrix} I & -S_2(s) \\ -S_1(s) & I \end{bmatrix}^{-1} = \left[\begin{array}{cc:cc} \frac{s+1}{2s+1} & -\frac{s}{(s+1)(2s+1)} & -\frac{s}{2s+1} & \frac{1}{(s+1)(2s+1)} \\ 0 & \frac{s}{s+1} & 0 & -\frac{1}{s+1} \\ \hdashline \frac{s+1}{2s+1} & \frac{1}{2s+1} & \color{red}{\frac{s+1}{2s+1}} & \color{red}{-\frac{1}{s(2s+1)}} \\ 0 & \frac{s}{s+1} & \color{red}{0} & \color{red}{\frac{s}{s+1}} \end{array}\right]$$

is unstable.

Outline

Internal stability

Generic stability results

All stabilizing controllers: stable plant

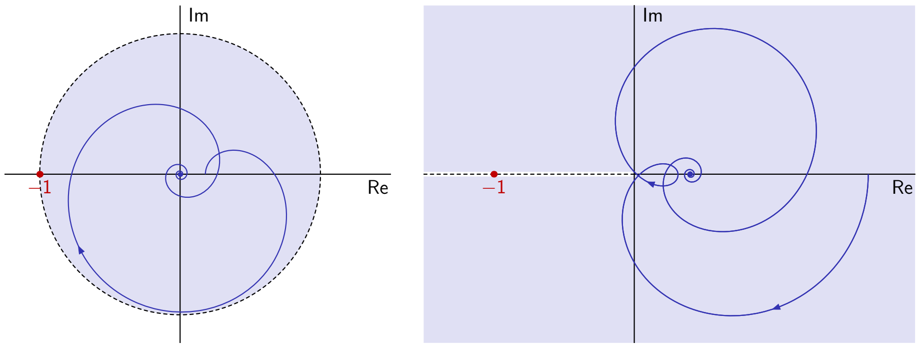
All stabilizing controllers: possibly unstable plant

Extensions

Open-loop stabilization

Stabilization in the LFT setting

Two cases



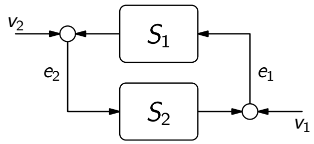
gain phase

If loop is stable, the

– closed-loop system is stable if either of these situations takes place,

no extra details about the loop are required.

The Small Gain Theorem



Theorem

If $S_i \in H_\infty$ with $\|S_i\|_\infty = \gamma_i \geq 0$ for $i = 1, 2$, then the closed-loop system is internally stable whenever $\gamma_1\gamma_2 < 1$.

Proof: If $M_1 \in \mathbb{C}^{p\times m}$, $M_2 \in \mathbb{C}^{m\times p}$, then $\|(I - M_2M_1)^{-1}\| = \frac{1}{\underline{\sigma}(I-M_2M_1)}$ and

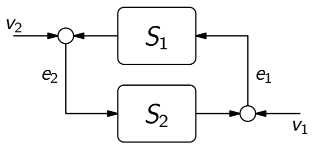
$$\underline{\sigma}\,(I - M_2M_1) = \min_{\|u\|=1} \|(I - M_2M_1)u\| \geq \min_{\|u\|=1} (\|u\| - \|M_2M_1u\|)$$
$$= 1 - \|M_2M_1\| \geq 1 - \|M_1\|\|M_2\|$$

Hence,

$$\|(I - M_2M_1)^{-1}\| \leq \frac{1}{1 - \|M_1\|\|M_2\|}$$

whenever $\|M_1\|\|M_2\| < 1$.

The Small Gain Theorem (contd)



Proof (contd): Thus, if $\gamma_1\gamma_2 < 1$, then

$$\sup_{s\in\mathbb{C}_0} \|(I - S_2(s)S_1(s))^{-1}\| = \frac{1}{\inf_{s\in\mathbb{C}_0} \underline{\sigma}\,(I - S_2(s)S_1(s))} \le \frac{1}{1-\gamma_1\gamma_2}.$$

Thus, $(I - S_2(s)S_1(s))^{-1}$ is

– bounded in $\mathbb{C}_0$,

– holomorphic in $\mathbb{C}_0$, because so are both $S_1(s)$ and $S_2(s)$.

Therefore, $(I - S_2S_1)^{-1} \in H_\infty$. This, together with the facts that $S_1 \in H_\infty$ and $S_2 \in H_\infty$, yields the internal stability of the closed-loop system. $\square$

Positive real transfer functions

Given a $m \times m$ LTI G, its transfer function is positive real (PR) if

1. $G(s)$ is holomorphic in $\mathbb{C}_0$,
2. $G(s) + [G(s)]' \ge 0$ for all $s \in \mathbb{C}_0$

Some PR functions:

s because $s + \bar{s} = 2\,\mathrm{Re}\,s > 0$ in $\mathbb{C}_0$

$\dfrac{1}{s}$ because $\dfrac{1}{s} + \dfrac{1}{\bar{s}} = 2\dfrac{\mathrm{Re}\,s}{|s|^2} > 0$ in $\mathbb{C}_0$

$\tanh s$ because $\tanh(s) + \tanh(\bar{s}) = 2\dfrac{1 - e^{-4\,\mathrm{Re}\,s}}{|1 + e^{-2s}|^2} > 0$ in $\mathbb{C}_0$

A non-PR function:

$\dfrac{1}{s^2}$ because $\dfrac{1}{s^2} + \dfrac{1}{\bar{s}^2} = 2\dfrac{(\mathrm{Re}\,s)^2 - (\mathrm{Im}\,s)^2}{|s|^4} \ge 0$ only if $\mathrm{Re}\,s \ge |\mathrm{Im}\,s|$

Strongly positive real transfer functions

Given a $m \times m$ LTI G, its transfer function is strongly positive real (SPR) if

1. $G(s)$ is holomorphic in $\mathbb{C}_0$,
2. $\exists\epsilon > 0$ such that $G(s) + [G(s)]' \ge \epsilon I$ for all $s \in \mathbb{C}_0$

Some PR functions:

$s+1$ because $s + 1 + \overline{s+1} = 2(\mathrm{Re}\,s + 1) > 2$ in $\mathbb{C}_0$

$\dfrac{s+1}{s}$ because $\dfrac{s+1}{s} + \dfrac{\overline{s+1}}{\bar{s}} = 2 + 2\dfrac{\mathrm{Re}\,s}{|s|^2} > 2$ in $\mathbb{C}_0$

A non-SPR function:

$\dfrac{1}{s}$ as $\dfrac{1}{s} + \dfrac{1}{\bar{s}} = 2\dfrac{\mathrm{Re}\,s}{|s|^2}$ can be arbitrarily small.

Properties of PR and SPR transfer functions

If $G(s)$ is PR, then

– it may only have simple pure imaginary poles, say $j\omega$, whose residues $G_i := \lim_{s\to j\omega_i}(s - j\omega_i)G(s)$ satisfy $G_i = G_i' \ge 0$,

– it has no zeros in $\mathbb{C}_0$,

– $G(j\omega) + [G(j\omega)]' \ge 0$, for all $\omega \in \mathbb{R} \setminus \{\, j\omega\text{-axis singularities of } G(s)\,\}$
MIMO counterpart of $\arg G(j\omega) \in [-\pi/2, \pi/2]$

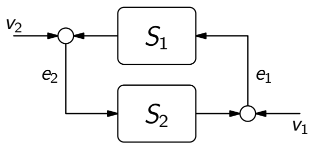
If $G(s)$ is SPR, then

– it is PR

– it may have no pure imaginary singularities

– $\exists\epsilon$ such that $G(j\omega) + [G(j\omega)]' \ge \epsilon$, for all $\omega \in \mathbb{R}$
MIMO counterpart of $\arg G(j\omega) \in (-\pi/2, \pi/2)$

The Passivity Theorem



Theorem

If $S_1(s)$ is PR, $-S_2(s)$ is SPR, and $S_2 \in H_\infty$, then the closed-loop system is internally stable.

Remark

Holds also if $-S_1$ is SPR and S_2 is PR.

Remark

Plants with PR transfer functions are stabilizable by static positive definite controllers, regardless their gain (high-gain feedback affordable).

The Passivity Theorem (contd)

Proof (outline): If $G(s)$ is PR, then

- $(I+G)^{-1} \in H_\infty$ and $\|(I+G)^{-1}\|_\infty \le 1$
- $(I-G)(I+G)^{-1} \in H_\infty$ and $\|(I-G)(I+G)^{-1}\|_\infty \le 1$

If $G(s)$ is SPR and $G \in H_\infty$, then

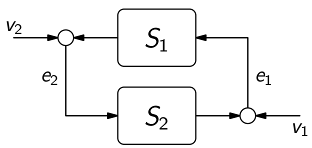
- $\|(I-G)(I+G)^{-1}\|_\infty < 1$

Now

$$\begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} I & (I+S_2)(I-S_2)^{-1} \\ -(I-S_1)(I+S_1)^{-1} & I \end{bmatrix}^{-1} \times \begin{bmatrix} (I-S_2)^{-1} & 0 \\ 0 & -(I+S_1)^{-1} \end{bmatrix}.$$

and stability follows by the Small Gain Theorem. $\square$

Beyond LTI



Small Gain: If, for all $T > 0$,

1. S_i are stable,
2. $\exists \gamma_i \ge 0, \beta_i$ such that $\|S_i e_i\|_T \le \gamma_i \|e_i\|_T + \beta_i$,

then closed-loop system is stable if $\gamma_1 \gamma_2 < 1$.

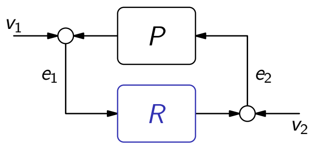
Passivity: If, for all $T > 0$,

1. $\exists \epsilon_1, \beta_1$ such that $\langle S_1 e_1, e_1 \rangle_T \ge \epsilon_1 \|S_1 e_1\|_T^2 + \beta_1$,
2. $\exists \epsilon_2, \beta_2$ such that $\langle e_2, -S_2 e_2 \rangle_T \ge \epsilon_2 \|e_2\|_T^2 + \beta_2$,
3. $\exists \gamma_2 \ge 0, \beta_3$ such that $\|S_2 e_2\|_T \le \gamma_2 \|e_2\|_T^2 + \beta_3$

then closed-loop system is stable if $\epsilon_1 + \epsilon_2 > 0$.

Outline

The problem



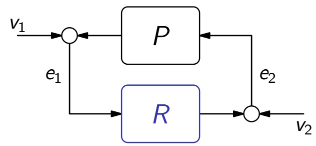
Given P (plant), design R (controller) internally stabilizing the system. In other words, we aim at rendering

$$T_{\text{aux}} := \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} R(I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix}$$
$$= \begin{bmatrix} (I - PR)^{-1} & P(I - RP)^{-1} \\ R(I - PR)^{-1} & (I - RP)^{-1} \end{bmatrix}$$

stable (i.e. $T_{\text{aux}} \in RH_\infty$). Meanwhile, assume that

– P is itself stable (i.e. $P \in RH_\infty$)

All stabilizing controllers



Theorem

If $P \in RH_\infty$, then R stabilizes the system iff there is $Q \in RH_\infty$ such that

$$R = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, Q\right) = Q(I + PQ)^{-1}$$

and $I + P(\infty)Q(\infty)$ is nonsingular.

Proof: Because P is stable, $T_{\text{aux}} \in RH_\infty$ iff

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} T_{\text{aux}} \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -P \\ R(I - PR)^{-1} & I \end{bmatrix} \in RH_\infty$$

Hence, R is stabilizing iff it stabilizes $T_c = R(I - PR)^{-1}$.

All stabilizing controllers (contd)

Proof (contd): The control sensitivity

$$T_c = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & P \end{bmatrix}, R\right)$$

$$\Updownarrow$$

$$R = \mathcal{F}_u\left(\begin{bmatrix} 0 & I \\ I & P \end{bmatrix}^{-1}, T_c\right) = \mathcal{F}_u\left(\begin{bmatrix} -P & I \\ I & 0 \end{bmatrix}, T_c\right) = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, T_c\right)$$

Thus

"if" $R = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, Q\right) \implies T_c = Q \in RH_\infty$

"only if" $T_c \in RH_\infty \implies R = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, Q\right)$ for $Q = T_c$

□

All stable closed-loop systems

With $Q = R(I - PR)^{-1}$,

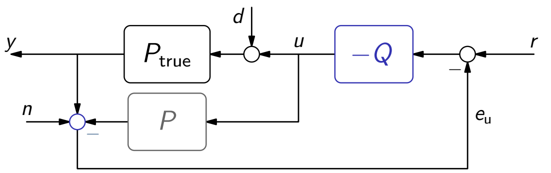
$$T_{\text{aux}} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} R(I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix}$$
$$= \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} Q \begin{bmatrix} I & P \end{bmatrix} = \begin{bmatrix} I + PQ & P + PQP \\ Q & I + QP \end{bmatrix}$$

is an affine function of the free parameter Q. It also means that

$$\begin{bmatrix} T_c & T_i \\ S_o & T_d \end{bmatrix} := \begin{bmatrix} R \\ I \end{bmatrix} (I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} = \begin{bmatrix} Q & QP \\ I + PQ & P + PQP \end{bmatrix}$$

whenever the closed-loop system is stable.

Interpretation



(mind negative feedback).

If no uncertainty ($P_{\text{true}} = P$, $d = 0$, $n = 0$), then

- $e_{\mathrm{u}} = 0$ and we have open-loop control.

If there is uncertainty ($P_{\text{true}} \neq P$, or $d \neq 0$, or $n \neq 0$), then

- $e_{\mathrm{u}} = (P_{\text{true}} - P)u + P_{\text{true}}d + n \neq 0$ (uncertainty indicator)
- feedback is based on mismatches

Outline

Internal stability

Generic stability results

All stabilizing controllers: stable plant

All stabilizing controllers: possibly unstable plant

Extensions

Open-loop stabilization

Stabilization in the LFT setting

Reminder: doubly coprime factorization

Given an LTI finite-dimensional P, its transfer function can be factorized as

$$P(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

with coprime $N, M, \tilde{N}, \tilde{M} \in RH_\infty$. There are $X, Y, \tilde{X}, \tilde{Y} \in RH_\infty$ such that

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} \tilde{M}(s) & -\tilde{N}(s) \\ Y(s) & X(s) \end{bmatrix} \begin{bmatrix} \tilde{X}(s) & N(s) \\ -\tilde{Y}(s) & M(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Also,

$$\begin{bmatrix} \tilde{X}(s) & N(s) \\ -\tilde{Y}(s) & M(s) \end{bmatrix} \begin{bmatrix} \tilde{M}(s) & -\tilde{N}(s) \\ Y(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

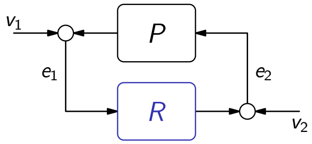
Useful relations

1: $$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} M^{-1}$$

2: $$\begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} = \tilde{M}^{-1} \begin{bmatrix} I & 0 \end{bmatrix}$$

3: $$\begin{aligned} \begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} &= \begin{bmatrix} \tilde{M} & 0 \\ Y & M^{-1} \end{bmatrix} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} \\ &= \begin{bmatrix} \tilde{M}\tilde{X} & -\tilde{M}N \\ Y\tilde{X} & I - YN \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} M^{-1}\tilde{Y} \begin{bmatrix} I & 0 \end{bmatrix} \end{aligned}$$

Internal stability



requires

$$T_{\text{aux}} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} R(I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} \in RH_\infty.$$

This is equivalent to

$$\tilde{T}_{\text{aux}} := \begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} T_{\text{aux}} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} \in RH_\infty$$

(mind that $\begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & M \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ is bi-stable too).

Structure of $\tilde{T}_{\text{aux}}$

By the relations above,

$$\begin{aligned} \tilde{T}_{\text{aux}} &= \begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} T_{\text{aux}} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} \\ &= \begin{bmatrix} \tilde{M}\tilde{X} & -\tilde{M}N \\ Y\tilde{X} & I - YN \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underbrace{\left(M^{-1}\tilde{Y} + M^{-1}R(I - PR)^{-1}\tilde{M}^{-1}\right)}_{Q} \begin{bmatrix} I & 0 \end{bmatrix} \end{aligned}$$

and

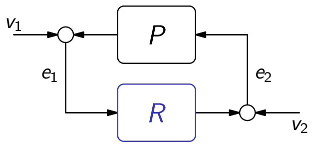
$$\tilde{T}_{\text{aux}} \in RH_\infty \iff Q = \mathcal{F}_{\text{u}}\left(\begin{bmatrix} P & \tilde{M}^{-1} \\ M^{-1} & M^{-1}\tilde{Y} \end{bmatrix}, R\right) \in RH_\infty.$$

Now,

$$\begin{bmatrix} P & \tilde{M}^{-1} \\ M^{-1} & M^{-1}\tilde{Y} \end{bmatrix} = \begin{bmatrix} -\tilde{Y}\tilde{X}^{-1} & M + \tilde{Y}\tilde{X}^{-1}N \\ \tilde{X}^{-1} & -\tilde{X}^{-1}N \end{bmatrix}^{-1}$$

is invertible and so are its $(2,1)$ and $(1,2)$ blocks.

All stabilizing controllers



Theorem (Youla–Kučera parametrization)

R stabilizes the system iff there is $Q \in RH_\infty$ such that

$$\begin{aligned} R &= \mathcal{F}_l\left(\begin{bmatrix} -\tilde{Y}\tilde{X}^{-1} & M + \tilde{Y}\tilde{X}^{-1}N \\ \tilde{X}^{-1} & -\tilde{X}^{-1}N \end{bmatrix}, Q\right) = (-\tilde{Y} + MQ)(\tilde{X} + NQ)^{-1} \\ &= \mathcal{F}_l\left(\begin{bmatrix} -X^{-1}Y & X^{-1} \\ \tilde{M} + \tilde{N}X^{-1}Y & -\tilde{N}X^{-1} \end{bmatrix}, Q\right) = (X + Q\tilde{N})^{-1}(-Y + Q\tilde{M}) \end{aligned}$$

and $\tilde{X}(\infty) + N(\infty)Q(\infty)$ or $X(\infty) + Q(\infty)\tilde{N}(\infty)$ is nonsingular.

Coprime factor interpretations

The factors in $R = (-\tilde{Y} + MQ)(\tilde{X} + NQ)^{-1}$ are right coprime, because

$$\tilde{M}(\tilde{X} + NQ) + (-\tilde{N})(-\tilde{Y} + MQ) = I.$$

The factors in $R = (X + Q\tilde{N})^{-1}(-Y + Q\tilde{M})$ are left coprime, because

$$(X + Q\tilde{N})M + (-Y + Q\tilde{M})(-N) = I.$$

Thus,

- Bézout coefficients of coprime factors of the plant are coprime factors of every stabilizing controller

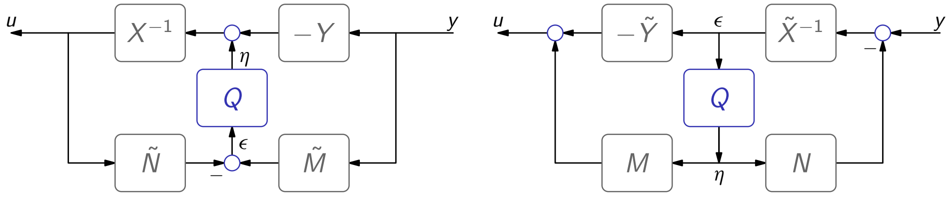
and

- Bézout coefficients of coprime factors of every stabilizing controller are coprime factors of the plant.

In other words, coprime factorizations are about stabilizing controllers and

- finding coprime factors $\iff$ finding a stabilizing controller.

Uncertainty interpretation



With $\tilde{M}y = \tilde{N}u$ and $\epsilon = \tilde{M}y - \tilde{N}u$,

– Q is activated on uncertainty (for P_{true}, d, n)

With $y = NM^{-1}u$, the measurement add-on is $PM\eta - N\eta$, so

– Q is activated on uncertainty (for P_{true} only)

All stable closed-loop systems

With

$$\tilde{T}_{\text{aux}} = \begin{bmatrix} \tilde{M}\tilde{X} & -\tilde{M}N \\ Y\tilde{X} & I - YN \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} I & 0 \end{bmatrix},$$

we have that

$$\begin{aligned} T_{\text{aux}} &= \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & M \end{bmatrix} \tilde{T}_{\text{aux}} \begin{bmatrix} \tilde{M} & \tilde{N} \\ -Y & X \end{bmatrix} \\ &= \begin{bmatrix} I - NY & NX \\ -MY & MX \end{bmatrix} + \begin{bmatrix} N \\ M \end{bmatrix} Q \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \\ &= \begin{bmatrix} I - NY + NQ\tilde{M} & NX + NQ\tilde{N} \\ -MY + MQ\tilde{M} & MX + MQ\tilde{N} \end{bmatrix} \end{aligned}$$

is also affine in Q. The signal

$$e_2 = M\big((-Y + Q\tilde{M})v_1 + (X + Q\tilde{N})v_2\big) \in \mathfrak{D}_P = ML_2,$$

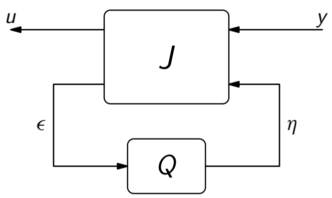
which can be viewed as the ultimate goal of every stabilizing controller.

YK in state space

Let

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be stabilizable and detectable and K and L render $A + BK$ and $A + LC$ Hurwitz. All stabilizing $C : y \mapsto u$ are given by



with

$$J(s) = \left[\begin{array}{c|cc} A + BK + LC + LDK & -L & B + LD \\ \hline K & 0 & I \\ -(C + DK) & I & -D \end{array}\right].$$

YK in state space: derivation

Key observation:

$$\begin{bmatrix} u \\ \epsilon \end{bmatrix} = \overbrace{\begin{bmatrix} -\tilde{Y}\tilde{X}^{-1} & M + \tilde{Y}\tilde{X}^{-1}N \\ \tilde{X}^{-1} & -\tilde{X}^{-1}N \end{bmatrix}}^{J} \begin{bmatrix} y \\ \eta \end{bmatrix} \iff \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M & -\tilde{Y} \\ N & \tilde{X} \end{bmatrix} \begin{bmatrix} \eta \\ \epsilon \end{bmatrix}$$

We know that

$$\begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right],$$

or

$$\begin{bmatrix} M & -\tilde{Y} \\ N & \tilde{X} \end{bmatrix} : \begin{cases} \dot{\hat{x}}(t) = (A + BK)\hat{x}(t) + B\eta(t) - L\epsilon(t) \\ u(t) = K\hat{x}(t) + \eta(t) \\ y(t) = (C + DK)\hat{x}(t) + D\eta(t) + \epsilon(t) \end{cases}$$

and all we need is to

– swap y and ϵ back.

YK in state space: derivation (contd)

The resulting

$$J:\begin{cases}\dot{\hat{x}}(t)=(A+BK+LC+LDK)\hat{x}(t)-Ly(t)+(B+LD)\eta(t)\\ u(t)=K\hat{x}(t)+\eta(t)\\ \epsilon(t)=-(C+DK)\hat{x}(t)+y(t)-D\eta(t)\end{cases}$$

is equivalent to

$$J:\begin{cases}\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)-L(\overbrace{y(t)-C\hat{x}(t)-Du(t)}^{\epsilon(t)})\\ u(t)=K\hat{x}(t)+\eta(t)\\ \epsilon(t)=y(t)-C\hat{x}(t)-Du(t)\end{cases}$$

is an observer-based controller, complemented by $\eta=Q\epsilon$.

Outline

Internal stability

Generic stability results

All stabilizing controllers: stable plant

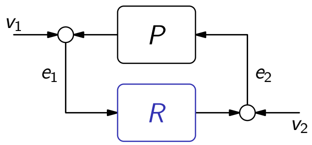
All stabilizing controllers: possibly unstable plant

Extensions

Open-loop stabilization

Stabilization in the LFT setting

All stabilizing controllers from a given one



Theorem

Let R_0 stabilizes P and $N_{R_0}, M_{R_0}, \tilde{N}_{R_0}, \tilde{M}_{R_0}\in RH_\infty$ be its coprime factors. R stabilizes the system iff there is $Q\in RH_\infty$ such that

$$R=\mathcal{F}_l\left(\begin{bmatrix}R_0 & I\\ I & -P(I-R_0P)^{-1}\end{bmatrix},\tilde{M}_{R_0}^{-1}QM_{R_0}^{-1}\right)$$

and such that $I+P(\infty)(\tilde{M}_{R_0}^{-1}(\infty)Q(\infty)M_{R_0}^{-1}(\infty)-R_0(\infty))$ is nonsingular.

Remark: If R stabilizes P_0, then all plants stabilized by this controller are

$$P=\mathcal{F}_u\left(\begin{bmatrix}-R(I-P_0R)^{-1} & I\\ I & P_0\end{bmatrix},\tilde{M}_0^{-1}QM_0^{-1}\right)$$

for $Q\in RH_\infty$.

Stabilizing controllers from those for the unstable part

If

$$P=P_{\rm fd}+\Pi\quad\text{for some }\Pi\in H_\infty$$

then its doubly coprime factorization

$$\begin{bmatrix}\tilde{M} & -\tilde{N}\\ Y & X\end{bmatrix}\begin{bmatrix}\tilde{X} & N\\ -\tilde{Y} & M\end{bmatrix}=\begin{bmatrix}\tilde{M}_{\rm fd} & -\tilde{N}_{\rm fd}-\tilde{M}_{\rm fd}\Pi\\ Y_{\rm fd} & X_{\rm fd}-Y_{\rm fd}\Pi\end{bmatrix}\begin{bmatrix}\tilde{X}_{\rm fd}-\Pi\tilde{Y}_{\rm fd} & N_{\rm fd}+\Pi M_{\rm fd}\\ -\tilde{Y}_{\rm fd} & M_{\rm fd}\end{bmatrix}=I$$

If $R_{\rm fd}=(-\tilde{Y}_{\rm fd}+M_{\rm fd}Q)(\tilde{X}_{\rm fd}+N_{\rm fd}Q)^{-1}$ stabilizes $P_{\rm fd}$, then all stabilizing R for P are

$$\begin{aligned}R&=(-\tilde{Y}+MQ)(\tilde{X}+NQ)^{-1}\\ &=(-\tilde{Y}_{\rm fd}+M_{\rm fd}Q)(\tilde{X}_{\rm fd}+N_{\rm fd}Q+\Pi(-\tilde{Y}_{\rm fd}+M_{\rm fd}))^{-1}\\ &=(-\tilde{Y}_{\rm fd}+M_{\rm fd}Q)(\tilde{X}_{\rm fd}+N_{\rm fd}Q)^{-1}(I+\Pi(-\tilde{Y}_{\rm fd}+M_{\rm fd})(\tilde{X}_{\rm fd}+N_{\rm fd}Q)^{-1})^{-1}\\ &=R_{\rm fd}(I+\Pi R_{\rm fd})^{-1},\end{aligned}$$

which is Π in negative feedback with $R_{\rm fd}$.

Dead-time compensation as YK

Let

$$P(s) = P_0(s)\mathrm{e}^{-\tau s} = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \mathrm{e}^{-\tau s}.$$

- If P_0 is stable, take $P_{\mathrm{fd}}(s) = P_0(s)$, for which

$$\Pi(s) = P_0(s)\mathrm{e}^{-\tau s} - P_{\mathrm{fd}}(s) = -P_0(s)(1 - \mathrm{e}^{-\tau s})$$

  and we end up with the Smith controller.

- If P_0 is unstable, take $P_{\mathrm{fd}}(s) = C\mathrm{e}^{-A\tau}(sI - A)^{-1}B$, for which

$$\begin{aligned}\Pi(s) &= P_0(s)\mathrm{e}^{-\tau s} - P_{\mathrm{fd}}(s) = C(\mathrm{e}^{-\tau s}I - \mathrm{e}^{-A\tau})(sI - A)^{-1}B \\ &= -C\int_0^\tau \mathrm{e}^{A(t-\tau)}\mathrm{e}^{-st}\mathrm{d}tB\end{aligned}$$

  and we end up with the modified Smith controller.

Outline

Open-loop stabilization problems

Given G_{11}, G_{12}, and G_{21},
- find $R \in RH_\infty$ such that $G_{\mathrm{e}} := G_{11} + G_{12}RG_{21} \in RH_\infty$.

Known as two-sided and could be messy.

Simplified one-sided versions:
- "tracking" setup with $G_{21} = I$ (and $G_{11} = G_1$ and $G_{12} = G_2$), so

$$G_{\mathrm{e}} := G_1 + G_2R$$

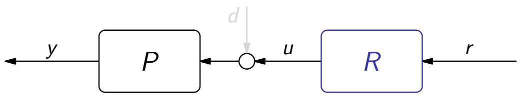
- "estimation" setup with $G_{12} = I$ (and $G_{11} = G_1$ and $G_{21} = G_2$), so

$$G_{\mathrm{e}} := G_1 + RG_2$$

Motivation

Example

Let



for $P(s) = 1/(s+1)$ and $R \in RH_\infty$. If $r = \mathbb{1}$, then

$$\lim_{t\to\infty} e(t) = 0 \iff R(0) = 1.$$

Let $W(s) = 1/s$, then

$$W(s)(1 - P(s)R(s)) = \frac{1}{s} - \frac{R(s)}{s(s+1)} = \frac{1 - R(0)}{s} - \overbrace{\frac{R(s) - R(0)(s+1)}{s(s+1)}}^{\in RH_\infty \text{ for all } R \in RH_\infty}$$

and

$$W(1 - PR) \in RH_\infty \iff R(0) = 1.$$

Estimation stabilization: insight

Intuitively, there is $R \in RH_\infty$ stabilizing $G_1 + RG_2$ if

- unstable poles of $G_1(s)$ are also poles of $G_2(s)$ and
- $R(s)$ reshapes directions of poles to cancel them with those of $G_1(s)$

But directional properties complicate matters.

Example

Let

$$G_1(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix} \quad \text{and} \quad G_2(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/(s^2+1) \end{bmatrix}$$

In this case

$$G_e(s) = \begin{bmatrix} (1+R_{11}(s))/s & R_{12}(s)/(s^2+1) \\ R_{21}(s)/s & 1/s + R_{22}(s)/(s^2+1) \end{bmatrix}$$

and stability conditions are $R_{11}(0) = -1$, $R_{21}(0) = 0$, $R_{12}(\pm \mathrm{j}) = 0$, but no stabilizing $R_{22}(s)$ exists.

Estimation stabilization: stabilizability

Lemma

Let $G_2 = N_2 M_2^{-1} = \tilde{M}_2^{-1}\tilde{N}_2$ be coprime factorizations over RH_∞. The following conditions are equivalent:

1. *there is $R \in RH_\infty$ such that $G_e = G_1 + RG_2 \in RH_\infty$,*
2. *$G_1 M_2 \in RH_\infty$ (equivalently, $\mathfrak{D}_{G_1} \subset \mathfrak{D}_{G_2}$),*
3. *there is a left coprime factorization of the form*

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_1 \\ 0 & \tilde{M}_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix} = \begin{bmatrix} \tilde{N}_1 - \tilde{M}_1 \tilde{M}_2^{-1}\tilde{N}_2 \\ \tilde{M}_2^{-1}\tilde{N}_2 \end{bmatrix}$$

for some $\tilde{N}_1, \tilde{M}_1 \in RH_\infty$.

Estimation stabilization: all stabilizing R

Theorem

If a factorization of the form

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_1 \\ 0 & \tilde{M}_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix}$$

exists, then $R \in RH_\infty$ stabilizes G_e iff there is $Q \in RH_\infty$ such that

$$R = \tilde{M}_1 + Q\tilde{M}_2$$

and then

$$G_e = \tilde{N}_1 + Q\tilde{N}_2$$

is the set of all attainable stable error systems.

Estimation stabilization: state-space condition

Lemma

If the realization

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right]$$

is stabilizable, then G admits a left coprime factorization of form

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_1 \\ 0 & \tilde{M}_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix}$$

iff (C_2, A) is detectable, in which case

$$\begin{bmatrix} \tilde{M}_1(s) & \tilde{N}_1(s) \\ \tilde{M}_2(s) & \tilde{N}_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + L_2C_2 & L_2 & B + L_2D_2 \\ \hline C_1 & 0 & D_1 \\ C_2 & I & D_2 \end{array}\right]$$

for any L_2 such that $A + L_2C_2$ is Hurwitz.

Estimation stabilization: state-space condition (contd)

Example

Return to tracking, which can be written as

$$G_e(s) = \frac{1}{s} - \frac{R(s)}{s(s+1)} = \begin{bmatrix} 1 & R(s) \end{bmatrix} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 1 \\ \hline 1 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right].$$

With $L_2 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, for which $\operatorname{spec}(A + L_2C_2) = \{-1, -1\}$,

$$\begin{bmatrix} \tilde{M}_1(s) & \tilde{N}_1(s) \\ \tilde{M}_2(s) & \tilde{N}_2(s) \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} 1/(s+1) & (s+2)/(s+1)^2 \\ s/(s+1) & -1/(s+1)^2 \end{bmatrix},$$

so

$$R(s) = \frac{1 + sQ(s)}{s+1} \quad \text{and} \quad G_e(s) = \frac{s + 2 - Q(s)}{(s+1)^2}$$

and $G_e \in RH_\infty$, with $1 - P(s)R(s) = s(s + 2 - Q(s))/(s+1)^2$.

Integral action in R

Assume an integral action is required in the controller (for all good reasons we know). We may think of two approaches:

1. augmented design — immediate
   1.1 augment an integrator to the plant
   1.2 design a stabilizer for the augmented plant
   1.3 move the integrator to the stabilizer
2. constrained YK parametrization — need to figure out how
   2.1 start with all stabilizing controllers, say $R = (X + Q\tilde{N})^{-1}(-Y + Q\tilde{M})$
   2.2 find under what conditions on Q it contains an integral action

For the sake of simplicity, we consider the SISO case only.

Integral action in R: YK condition

So let

$$R(s) = \frac{-Y(s) + Q(s)\tilde{M}(s)}{X(s) + Q(s)\tilde{N}(s)}$$

We know that unstable poles of $R(s)$ are unstable zeros of its denominator. Hence,

$$R(s) = \frac{\tilde{R}(s)}{s} \iff X(0) + Q(0)\tilde{N}(0) = 0 \iff (X + Q\tilde{N})W \in RH_\infty$$

where $W(s) = 1/s$. But this is

– the estimation stability setup with $G_1 = XW$ and $G_2 = \tilde{N}W$.

If $\tilde{N}(0) = 0$, then $X(0) \neq 0$ and

– $G_2 = \tilde{N}W \in RH_\infty$, while $G_1 = XW \notin RH_\infty \overset{\text{by Lemma}}{\Longrightarrow}$ not stabilizable

(naturally). We thus assume that $\tilde{N}(0) \neq 0$.

Integral action in R: YK condition (contd)

If $\tilde{N}(0) \neq 0$, then $G_2(s) = \tilde{N}(s)/s$ has its only unstable pole at $s = 0$ and a possible choice $\tilde{M}_2 = s/(s+1)$, for which

$$\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} 1 & -X(0)/\tilde{N}(0) \\ 0 & s/(s+1) \end{bmatrix}^{-1} \begin{bmatrix} (\tilde{N}(0)X(s) - \tilde{N}(s)X(0))/(\tilde{N}(0)s) \\ \tilde{N}(s)/(s+1) \end{bmatrix}$$

is the required *lcf*. Hence, all $Q \in RH_\infty$ for which R has an integral action are given as

$$Q(s) = -\frac{X(0)}{\tilde{N}(0)} + Q_0(s)\frac{s}{s+1}$$

All such Q's ensure that

$$Q(0) = -\frac{X(0)}{\tilde{N}(0)}$$

irrespective of Q_0.

Example

Consider

$$G(s) = \frac{1}{s}$$

Its coprime factors and their Bézout coefficients are

$$\tilde{N}(s) = \frac{1}{s+1}, \quad \tilde{M}(s) = \frac{s}{s+1}, \quad X(s) = 1, \quad \text{and} \quad Y(s) = 1$$

All stabilizing:

$$R(s) = \frac{-s-1+sQ(s)}{s+1+Q(s)}$$

All Q's guaranteeing an integral action in $R(s)$:

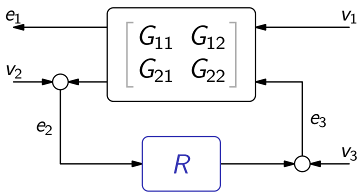
$$Q(s) = -1 + \frac{sQ_0(s)}{s+1}$$

All stabilizing controllers with an integral action:

$$R(s) = \frac{-2s^2-3s-1+s^2Q_0(s)}{s(s+1+Q_0(s))}$$

Outline

Internal stability



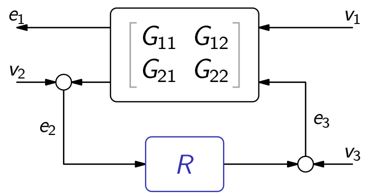
is said to be internally stable if all nine systems $v_i \mapsto e_j$ are stable (in H_∞).

$$\begin{bmatrix} I & 0 & -G_{12} \\ 0 & I & -G_{22} \\ 0 & -R & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

If $T_{\text{aux}} : (v_1, v_2, v_3) \mapsto (e_1, e_2, e_3)$, then

$$T_{\text{aux}} = \begin{bmatrix} G_{11} & 0 & G_{12} \\ G_{21} & I & G_{22} \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} G_{12} \\ G_{22} \\ I \end{bmatrix} R(I - G_{22}R)^{-1} \begin{bmatrix} G_{21} & I & G_{22} \end{bmatrix}.$$

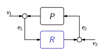
Stabilizability conditions



Theorem

There is an internally stabilizing R iff there are coprime factorizations of the form

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_{12} \\ 0 & \tilde{M}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ M_{21} & M_{22} \end{bmatrix}^{-1}$$

with right coprime $\tilde{N}_{22}$, $\tilde{M}_{22}$ and left coprime N_{22}, M_{22}. If factorizations as above exist, then R internally stabilizes the system iff it internally stabilizes  *under $P = G_{22}$.*

Stabilizability conditions (contd)

Proof (rough outline): First, note that

$$T_{\rm aux}\begin{bmatrix} I & 0 & 0 \\ 0 & \tilde X_{22} & -N_{22} \\ 0 & \tilde Y_{22} & M_{22} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12}\tilde Y_{22} & G_{12}M_{22} \\ G_{21} & \tilde M_{22}^{-1} & 0 \\ 0 & \tilde Y_{22} & M_{22} \end{bmatrix} + \begin{bmatrix} G_{12} \\ G_{22} \\ I \end{bmatrix} T_{\rm c} \begin{bmatrix} G_{21} & M_{22}^{-1} & 0 \end{bmatrix}$$

Hence, necessity of $G_{12}M_{22} \in RH_\infty$ (cf. open-loop stabilizability). Second,

$$\begin{bmatrix} I & \tilde M_{12} & -\tilde N_{12} \\ 0 & \tilde M_{22} & -\tilde N_{22} \\ 0 & Y_{22} & X_{22} \end{bmatrix} T_{\rm aux} = \begin{bmatrix} G_{11}+\tilde M_{12}G_{21} & \tilde M_{12} & 0 \\ \tilde M_{22}G_{21} & \tilde M_{22} & 0 \\ Y_{22}G_{21} & Y_{22} & M_{22}^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_{22}^{-1} \end{bmatrix} T_{\rm c} \begin{bmatrix} G_{21} & I & G_{22} \end{bmatrix},$$

Hence, necessity of $G_{11}+\tilde M_{12}G_{21} \in RH_\infty$ and $\tilde M_{22}G_{21} \in RH_\infty$. Then a lot of hard labor and the result follows . . . □

Stabilizability conditions: interpretation

Conditions

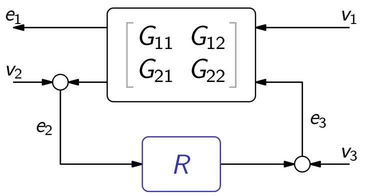
$$\begin{aligned}
\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} &= \begin{bmatrix} \tilde N_{11} - \tilde M_{12}\tilde M_{22}^{-1}\tilde N_{21} & \tilde N_{12} - \tilde M_{12}\tilde M_{22}^{-1}\tilde N_{22} \\ \tilde M_{22}^{-1}\tilde N_{21} & \tilde M_{22}^{-1}\tilde N_{22} \end{bmatrix} \\
&= \begin{bmatrix} \tilde N_{11} & \tilde N_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde M_{12} \\ -I \end{bmatrix} \tilde M_{22}^{-1} \begin{bmatrix} \tilde N_{21} & \tilde N_{22} \end{bmatrix} \\
&= \begin{bmatrix} N_{11} - N_{12}M_{22}^{-1}M_{21} & N_{12}M_{22}^{-1} \\ N_{21} - N_{22}M_{22}^{-1}M_{21} & N_{22}M_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} N_{11} & 0 \\ N_{21} & 0 \end{bmatrix} - \begin{bmatrix} N_{12} \\ N_{22} \end{bmatrix} M_{22}^{-1} \begin{bmatrix} M_{21} & -I \end{bmatrix}
\end{aligned}$$

imply that

- all unstable modes of G must be present in G_{22},

around which the feedback loop is closed.

All stable closed-loop maps



From

$$T_{\rm aux} = \begin{bmatrix} G_{11} & 0 & G_{12} \\ G_{21} & I & G_{22} \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} G_{12} \\ G_{22} \\ I \end{bmatrix} R(I - G_{22}R)^{-1} \begin{bmatrix} G_{21} & I & G_{22} \end{bmatrix}$$

and $R(I - G_{22}R)^{-1} = -\tilde Y_{22}\tilde M_{22} + M_{22}Q\tilde M_{22}$, we have that

$$\begin{aligned}
T_{e_1v_1} &= G_{11} - G_{12}\tilde Y_{22}\tilde M_{22}G_{21} + G_{12}M_{22}Q\tilde M_{22}G_{21} \\
&= \tilde N_{11} - (\tilde M_{12} + N_{12}Y_{22})\tilde M_{22}^{-1}\tilde N_{21} + N_{12}Q\tilde N_{21} \\
&= \tilde N_{11} + \tilde X_{12}\tilde N_{21} + N_{12}Q\tilde N_{21}
\end{aligned}$$

(with the help of $M_{22}^{-1}\tilde Y_{22} = Y_{22}\tilde M_{22}^{-1}$ and $\tilde X_{12}\tilde M_{22} = -\tilde M_{12} - N_{12}Y_{22}$).

Stabilizability conditions in state space

Proposition

Let

$$\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

It is stabilizable iff (A, B_2) *is stabilizable and* (C_2, A) *is detectable. If these conditions hold, then all stabilizing controllers are given by*

$$R(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} A + B_2K + LC_2 + LD_{22}K & -L & B_2 + LD_{22} \\ \hline K & 0 & I \\ -C_2 - D_{22}K & I & -D_{22} \end{array}\right], Q(s)\right),$$

where K *and* L *are such that* $A + B_2K$ *and* $A + LC_2$ *are Hurwitz and* $Q \in RH_\infty$ *and such that* $\det(I - Q(\infty)D_{22}) \neq 0$*, but otherwise arbitrary.*