Linear Control Systems (036012)
chapter 6

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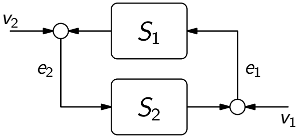
Outline

Definition



is internally stable if all four systems $v_i \mapsto e_j$ are stable (in H_∞). Because

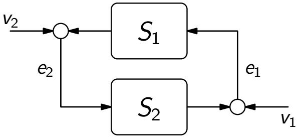
$$\begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

the system is internally stable iff

$$\begin{aligned} \begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix}^{-1} &= \left(\begin{bmatrix} I & -S_2 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} S_1 \begin{bmatrix} I & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} I & S_2 \\ 0 & I \end{bmatrix} + \begin{bmatrix} S_2 \\ I \end{bmatrix} S_1 (I - S_2 S_1)^{-1} \begin{bmatrix} I & S_2 \end{bmatrix} \\ &= \begin{bmatrix} (I - S_2 S_1)^{-1} & S_2 (I - S_1 S_2)^{-1} \\ S_1 (I - S_2 S_1)^{-1} & (I - S_1 S_2)^{-1} \end{bmatrix} \end{aligned}$$

is stable.

Definition and criterion



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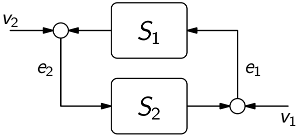
the system is internally stable iff

$$\begin{aligned} \begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix}^{-1} &= \left(\begin{bmatrix} I & -S_2 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} S_1 \begin{bmatrix} I & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} I & S_2 \\ 0 & I \end{bmatrix} + \begin{bmatrix} S_2 \\ I \end{bmatrix} S_1 (I - S_2 S_1)^{-1} \begin{bmatrix} I & S_2 \end{bmatrix} \\ &= \begin{bmatrix} (I - S_2 S_1)^{-1} & S_2 (I - S_1 S_2)^{-1} \\ S_1 (I - S_2 S_1)^{-1} & (I - S_1 S_2)^{-1} \end{bmatrix} \end{aligned}$$

is stable.

Example 1



If

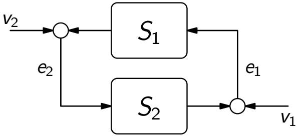
$$S_1(s) = \frac{1}{s} \quad \text{and} \quad S_2(s) = -\frac{s}{s+1},$$

then

$$\begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} (s+1)/(s+2) & -s/(s+2) \\ {\color{red}(s+1)/(s(s+2))} & (s+1)/(s+2) \end{bmatrix}$$

is unstable.

Example 2



If

$$S_1(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S_2(s) = -\begin{bmatrix} s/(s+1) & 0 \\ 0 & 1/s \end{bmatrix}.$$

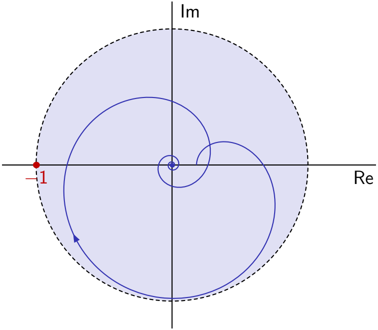
then

$$\begin{bmatrix} I & -S_2(s) \\ -S_1(s) & I \end{bmatrix}^{-1} = \left[\begin{array}{cc:cc} \frac{s+1}{2s+1} & -\frac{s}{(s+1)(2s+1)} & -\frac{s}{2s+1} & \frac{1}{(s+1)(2s+1)} \\ 0 & \frac{s}{s+1} & 0 & -\frac{1}{s+1} \\ \hdashline \frac{s+1}{2s+1} & \frac{1}{2s+1} & \frac{s+1}{2s+1} & -\frac{1}{s(2s+1)} \\ 0 & \frac{s}{s+1} & 0 & \frac{s}{s+1} \end{array}\right]$$

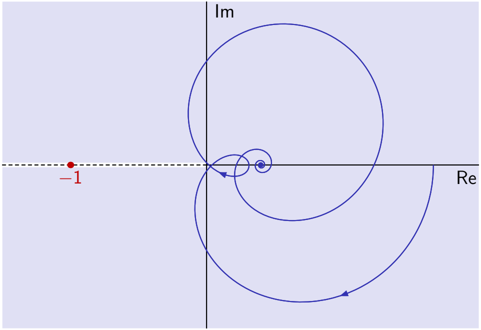
is unstable.

Outline

Two cases



gain



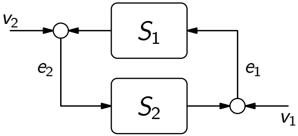
phase

If loop is stable, the

– closed-loop system is stable if either of these situations takes place,

no extra details about the loop are required.

The Small Gain Theorem



Theorem

If $S_i \in H_\infty$ with $\|S_i\|_\infty = \gamma_i \geq 0$ for $i = 1, 2$, then the closed-loop system is internally stable whenever $\gamma_1\gamma_2 < 1$.

Proof: If $M_1 \in \mathbb{C}^{p\times m}$, $M_2 \in \mathbb{C}^{m\times p}$, then $\|(I - M_2M_1)^{-1}\| = \frac{1}{\underline{\sigma}(I - M_2M_1)}$ and

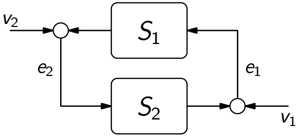
$$\begin{aligned}\underline{\sigma}\,(I - M_2M_1) &= \min_{\|u\|=1} \|(I - M_2M_1)u\| \geq \min_{\|u\|=1} (\|u\| - \|M_2M_1u\|)\\ &= 1 - \|M_2M_1\| \geq 1 - \|M_1\|\|M_2\|\end{aligned}$$

Hence,

$$\|(I - M_2M_1)^{-1}\| \leq \frac{1}{1 - \|M_1\|\|M_2\|}$$

whenever $\|M_1\|\|M_2\| < 1$.

The Small Gain Theorem (contd)



Proof (contd): Thus, if $\gamma_1\gamma_2 < 1$, then

$$\sup_{s\in\mathbb{C}_0} \|(I - S_2(s)S_1(s))^{-1}\| = \frac{1}{\inf_{s\in\mathbb{C}_0} \underline{\sigma}\left(I - S_2(s)S_1(s)\right)} \le \frac{1}{1-\gamma_1\gamma_2}.$$

Thus, $(I - S_2(s)S_1(s))^{-1}$ is

- bounded in $\mathbb{C}_0$,
- holomorphic in $\mathbb{C}_0$, because so are both $S_1(s)$ and $S_2(s)$.

Therefore, $(I - S_2S_1)^{-1} \in H_\infty$. This, together with the facts that $S_1 \in H_\infty$ and $S_2 \in H_\infty$, yields the internal stability of the closed-loop system. $\square$

Positive real transfer functions

Given a $m \times m$ LTI G, its transfer function is positive real (PR) if

1. $G(s)$ is holomorphic in $\mathbb{C}_0$,
2. $G(s) + [G(s)]' \geq 0$ for all $s \in \mathbb{C}_0$

Some PR functions:

$$s \quad \text{because } s + \bar{s} = 2\operatorname{Re} s > 0 \text{ in } \mathbb{C}_0$$

$$\frac{1}{s} \quad \text{because } \frac{1}{s} + \frac{1}{\bar{s}} = 2\frac{\operatorname{Re} s}{|s|^2} > 0 \text{ in } \mathbb{C}_0$$

$$\tanh s \quad \text{because } \tanh(s) + \tanh(\bar{s}) = 2\frac{1 - e^{-4\operatorname{Re} s}}{|1 + e^{-2s}|^2} > 0 \text{ in } \mathbb{C}_0$$

A non-PR function:

$$\frac{1}{s^2} \quad \text{because } \frac{1}{s^2} + \frac{1}{\bar{s}^2} = 2\frac{(\operatorname{Re} s)^2 - (\operatorname{Im} s)^2}{|s|^4} \geq 0 \text{ only if } \operatorname{Re} s \geq |\operatorname{Im} s|$$

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Strongly positive real transfer functions

Given a $m \times m$ LTI G, its transfer function is strongly positive real (SPR) if

1. $G(s)$ is holomorphic in $\mathbb{C}_0$,
2. $\exists \epsilon > 0$ such that $G(s) + [G(s)]' \geq \epsilon I$ for all $s \in \mathbb{C}_0$

Some PR functions:

$s+1$ because $s+1+\overline{s+1} = 2(\mathrm{Re}\, s + 1) > 2$ in $\mathbb{C}_0$

$\dfrac{s+1}{s}$ because $\dfrac{s+1}{s} + \overline{\dfrac{s+1}{s}} = 2 + 2\dfrac{\mathrm{Re}\, s}{|s|^2} > 2$ in $\mathbb{C}_0$

A non-SPR function:

$\dfrac{1}{s}$ as $\dfrac{1}{s} + \dfrac{1}{\bar{s}} = 2\dfrac{\mathrm{Re}\, s}{|s|^2}$ can be arbitrarily small.

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Properties of PR and SPR transfer functions

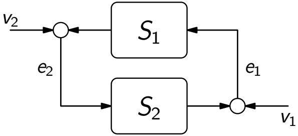
If $G(s)$ is PR, then

- it may only have simple pure imaginary poles, say jω, whose residues $G_i := \lim_{s \to \mathrm{j}\omega_i}(s - \mathrm{j}\omega_i)G(s)$ satisfy $G_i = G_i' \geq 0$,
- it has no zeros in $\mathbb{C}_0$,
- $G(\mathrm{j}\omega) + [G(\mathrm{j}\omega)]' \geq 0$, for all $\omega \in \mathbb{R} \setminus \{\, \mathrm{j}\omega\text{-axis singularities of } G(s)\,\}$
  MIMO counterpart of $\arg G(\mathrm{j}\omega) \in [-\pi/2, \pi/2]$

If $G(s)$ is SPR, then

- it is PR
- it may have no pure imaginary singularities
- $\exists\epsilon$ such that $G(\mathrm{j}\omega) + [G(\mathrm{j}\omega)]' \geq \epsilon$, for all $\omega \in \mathbb{R}$
  MIMO counterpart of $\arg G(\mathrm{j}\omega) \in (-\pi/2, \pi/2)$

The Passivity Theorem



Theorem

If $S_1(s)$ is PR, $-S_2(s)$ is SPR, and $S_2 \in H_\infty$, then the closed-loop system is internally stable.

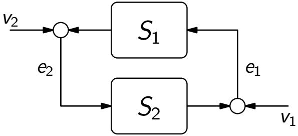
Remark

Holds also if $-S_1$ is SPR and S_2 is PR.

Remark

Plants with PR transfer functions are stabilizable by static positive definite controllers, regardless their gain (high-gain feedback affordable).

The Passivity Theorem



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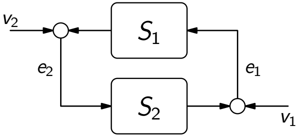

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The Passivity Theorem (contd)

Proof (outline) : If $G(s)$ is PR, then

- $(I+G)^{-1} \in H_\infty$ and $\|(I+G)^{-1}\|_\infty \leq 1$
- $(I-G)(I+G)^{-1} \in H_\infty$ and $\|(I-G)(I+G)^{-1}\|_\infty \leq 1$

If $G(s)$ is SPR and $G \in H_\infty$, then

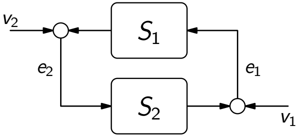
- $\|(I-G)(I+G)^{-1}\|_\infty < 1$

Now

$$\begin{bmatrix} I & -S_2 \\ -S_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} I & (I+S_2)(I-S_2)^{-1} \\ -(I-S_1)(I+S_1)^{-1} & I \end{bmatrix}^{-1} \times \begin{bmatrix} (I-S_2)^{-1} & 0 \\ 0 & -(I+S_1)^{-1} \end{bmatrix}.$$

and stability follows by the Small Gain Theorem. $\square$

Beyond LTI



Small Gain: If, for all $T > 0$,

1. S_i are stable,
2. $\exists \gamma_i \geq 0, \beta_i$ such that $\|S_i e_i\|_T \leq \gamma_i \|e_i\|_T + \beta_i$,

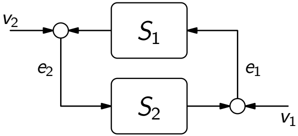
then closed-loop system is stable if $\gamma_1 \gamma_2 < 1$.

Passivity: If, for all $T > 0$,

1. $\exists \epsilon_1, \beta_1$ such that $\langle S_1 e_1, e_1 \rangle_T \geq \epsilon_1 \|S_1 e_1\|_T^2 + \beta_1$,
2. $\exists \epsilon_2, \beta_2$ such that $\langle e_2, -S_2 e_2 \rangle_T \geq \epsilon_2 \|e_2\|_T^2 + \beta_2$,
3. $\exists \gamma_2 \geq 0, \beta_3$ such that $\|S_2 e_2\|_T \leq \gamma_2 \|e_2\|_T^2 + \beta_3$

then closed-loop system is stable if $\epsilon_1 + \epsilon_2 > 0$.

Beyond LTI



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Passivity: If, for all $T > 0$,

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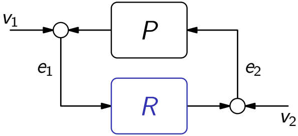
Outline

The problem



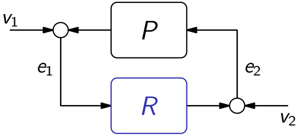
Given P (plant), design R (controller) internally stabilizing the system. In other words, we aim at rendering

$$
\begin{aligned}
T_{\text{aux}} &:= \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} R(I-PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} \\
&= \begin{bmatrix} (I-PR)^{-1} & P(I-RP)^{-1} \\ R(I-PR)^{-1} & (I-RP)^{-1} \end{bmatrix}
\end{aligned}
$$

stable (i.e. $T_{\text{aux}} \in RH_\infty$). Meanwhile, assume that

- P is itself stable (i.e. $P \in RH_\infty$)

All stabilizing controllers



Theorem

If $P \in RH_\infty$, then R stabilizes the system iff there is $Q \in RH_\infty$ such that

$$R = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, Q\right) = Q(I + PQ)^{-1}$$

and $I + P(\infty)Q(\infty)$ is nonsingular.

Proof: Because P is stable, $T_{\text{aux}} \in RH_\infty$ iff

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} T_{\text{aux}} \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -P \\ R(I - PR)^{-1} & I \end{bmatrix} \in RH_\infty$$

Hence, R is stabilizing iff it stabilizes $T_c = R(I - PR)^{-1}$.

All stabilizing controllers (contd)

Proof (contd): The control sensitivity

$$T_c = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & P \end{bmatrix}, R\right)$$

$$\Updownarrow$$

$$R = \mathcal{F}_u\left(\begin{bmatrix} 0 & I \\ I & P \end{bmatrix}^{-1}, T_c\right) = \mathcal{F}_u\left(\begin{bmatrix} -P & I \\ I & 0 \end{bmatrix}, T_c\right) = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, T_c\right)$$

Thus

"if" $R = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, Q\right) \implies T_c = Q \in RH_\infty$

"only if" $T_c \in RH_\infty \implies R = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & -P \end{bmatrix}, Q\right)$ for $Q = T_c$

$\square$

All stable closed-loop systems

With $Q = R(I - PR)^{-1}$,

$$
\begin{aligned}
T_{\text{aux}} &= \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} R(I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} \\
&= \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} Q \begin{bmatrix} I & P \end{bmatrix} = \begin{bmatrix} I + PQ & P + PQP \\ Q & I + QP \end{bmatrix}
\end{aligned}
$$

is an affine function of the free parameter Q. It also means that

$$
\begin{bmatrix} T_c & T_i \\ S_o & T_d \end{bmatrix} := \begin{bmatrix} R \\ I \end{bmatrix} (I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} = \begin{bmatrix} Q & QP \\ I + PQ & P + PQP \end{bmatrix}
$$

whenever the closed-loop system is stable.

All stable closed-loop systems

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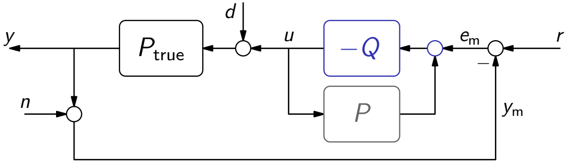
Interpretation



(mind negative feedback).

If no uncertainty ($P_{\text{true}} = P$, $d = 0$, $n = 0$), then

– $e_u = 0$ and we have open-loop control.

If there is uncertainty ($P_{\text{true}} \neq P$, or $d \neq 0$, or $n \neq 0$), then

– $e_u = (P_{\text{true}} - P)u + P_{\text{true}}d + n \neq 0$ (uncertainty indicator)

– feedback is based on mismatches

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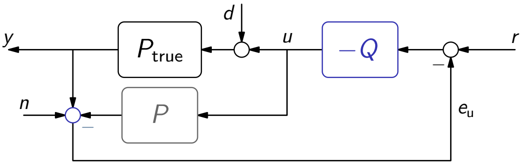

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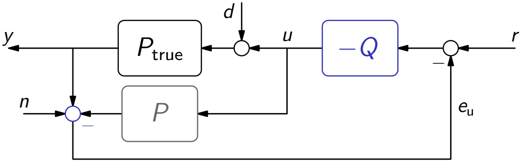

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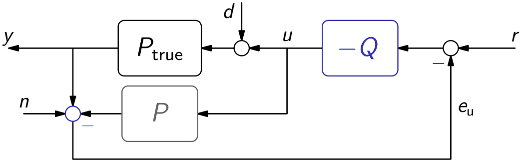
Interpretation



(mind negative feedback).

If no uncertainty ($P_{\text{true}} = P$, $d = 0$, $n = 0$), then

- $e_{\text{u}} = 0$ and we have open-loop control.

If there is uncertainty ($P_{\text{true}} \neq P$, or $d \neq 0$, or $n \neq 0$), then

- $e_{\text{u}} = (P_{\text{true}} - P)u + P_{\text{true}}d + n \neq 0$ (uncertainty indicator)
- feedback is based on mismatches

Outline

Reminder: doubly coprime factorization

Given an LTI finite-dimensional P, its transfer function can be factorized as

$$P(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

with coprime $N, M, \tilde{N}, \tilde{M} \in RH_\infty$. There are $X, Y, \tilde{X}, \tilde{Y} \in RH_\infty$ such that

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} \tilde{M}(s) & -\tilde{N}(s) \\ Y(s) & X(s) \end{bmatrix} \begin{bmatrix} \tilde{X}(s) & N(s) \\ -\tilde{Y}(s) & M(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Also,

$$\begin{bmatrix} \tilde{X}(s) & N(s) \\ -\tilde{Y}(s) & M(s) \end{bmatrix} \begin{bmatrix} \tilde{M}(s) & -\tilde{N}(s) \\ Y(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

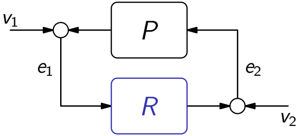
Useful relations

1: $$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} M^{-1}$$

2: $$\begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} = \tilde{M}^{-1} \begin{bmatrix} I & 0 \end{bmatrix}$$

3: $$\begin{aligned} \begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} &= \begin{bmatrix} \tilde{M} & 0 \\ Y & M^{-1} \end{bmatrix} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} \\ &= \begin{bmatrix} \tilde{M}\tilde{X} & -\tilde{M}N \\ Y\tilde{X} & I - YN \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} M^{-1} \tilde{Y} \begin{bmatrix} I & 0 \end{bmatrix} \end{aligned}$$

Internal stability



requires

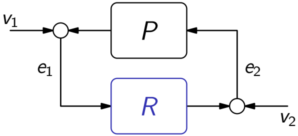
$$T_{\mathrm{aux}} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} R(I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} \in RH_\infty.$$

This is equivalent to

$$\tilde{T}_{\mathrm{aux}} := \begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} T_{\mathrm{aux}} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} \in RH_\infty$$

(mind that $\begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & M \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ is bi-stable too).

Internal stability



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Structure of $\tilde{T}_{\mathrm{aux}}$

By the relations above,

$$\begin{aligned}\tilde{T}_{\mathrm{aux}} &= \begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} T_{\mathrm{aux}} \begin{bmatrix} \tilde{X} & -N \\ \tilde{Y} & M \end{bmatrix} \\ &= \begin{bmatrix} \tilde{M}\tilde{X} & -\tilde{M}N \\ Y\tilde{X} & I - YN \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underbrace{\bigl(M^{-1}\tilde{Y} + M^{-1}R(I-PR)^{-1}\tilde{M}^{-1}\bigr)}_{Q} \begin{bmatrix} I & 0 \end{bmatrix}\end{aligned}$$

and

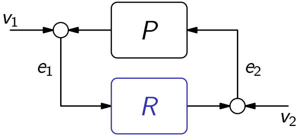
$$\tilde{T}_{\mathrm{aux}} \in RH_\infty \iff Q = \mathcal{F}_{\mathrm{u}}\left(\begin{bmatrix} P & \tilde{M}^{-1} \\ M^{-1} & M^{-1}\tilde{Y} \end{bmatrix}, R\right) \in RH_\infty .$$

Now,

$$\begin{bmatrix} P & \tilde{M}^{-1} \\ M^{-1} & M^{-1}\tilde{Y} \end{bmatrix} = \begin{bmatrix} -\tilde{Y}\tilde{X}^{-1} & M + \tilde{Y}\tilde{X}^{-1}N \\ \tilde{X}^{-1} & -\tilde{X}^{-1}N \end{bmatrix}^{-1}$$

is invertible and so are its $(2,1)$ and $(1,2)$ blocks.

All stabilizing controllers



Theorem (Youla–Kučera parametrization)

R stabilizes the system iff there is $Q \in RH_\infty$ such that

$$R = \mathcal{F}_l\left(\begin{bmatrix} -\tilde{Y}\tilde{X}^{-1} & M + \tilde{Y}\tilde{X}^{-1}N \\ \tilde{X}^{-1} & -\tilde{X}^{-1}N \end{bmatrix}, Q\right) = (-\tilde{Y} + MQ)(\tilde{X} + NQ)^{-1}$$

$$= \mathcal{F}_l\left(\begin{bmatrix} -X^{-1}Y & X^{-1} \\ \tilde{M} + \tilde{N}X^{-1}Y & -\tilde{N}X^{-1} \end{bmatrix}, Q\right) = (X + Q\tilde{N})^{-1}(-Y + Q\tilde{M})$$

and $\tilde{X}(\infty) + N(\infty)Q(\infty)$ or $X(\infty) + Q(\infty)\tilde{N}(\infty)$ is nonsingular.

Coprime factor interpretations

The factors in $R = (-\tilde{Y} + MQ)(\tilde{X} + NQ)^{-1}$ are right coprime, because

$$\tilde{M}(\tilde{X} + NQ) + (-\tilde{N})(-\tilde{Y} + MQ) = I.$$

The factors in $R = (X + Q\tilde{N})^{-1}(-Y + Q\tilde{M})$ are left coprime, because

$$(X + Q\tilde{N})M + (-Y + Q\tilde{M})(-N) = I.$$

Thus,

- Bézout coefficients of coprime factors of the plant are coprime factors of every stabilizing controller

and

- Bézout coefficients of coprime factors of every stabilizing controller are coprime factors of the plant.

In other words, coprime factorizations are about stabilizing controllers and

- finding coprime factors $\iff$ finding a stabilizing controller.

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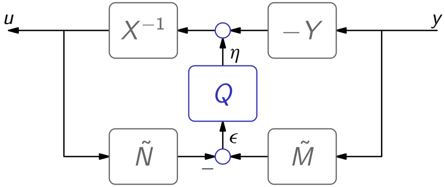
Uncertainty interpretation



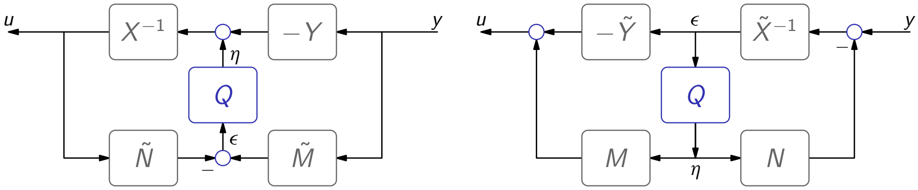
With $\tilde{M}y = \tilde{N}u$ and $\epsilon = \tilde{M}y - \tilde{N}u$,

- Q is activated on uncertainty (for P_{true}, d, n)

With $y = NM^{-1}u$, the measurement add-on is $PM\eta - N\eta$, so

- Q is activated on uncertainty (for P_{true} only)

Uncertainty interpretation



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All stable closed-loop systems

With

$$\tilde{T}_{\text{aux}} = \begin{bmatrix} \tilde{M}\tilde{X} & -\tilde{M}N \\ Y\tilde{X} & I - YN \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} I & 0 \end{bmatrix},$$

we have that

$$\begin{aligned} T_{\text{aux}} &= \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & M \end{bmatrix} \tilde{T}_{\text{aux}} \begin{bmatrix} \tilde{M} & \tilde{N} \\ -Y & X \end{bmatrix} \\ &= \begin{bmatrix} I - NY & NX \\ -MY & MX \end{bmatrix} + \begin{bmatrix} N \\ M \end{bmatrix} Q \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \\ &= \begin{bmatrix} I - NY + NQ\tilde{M} & NX + NQ\tilde{N} \\ -MY + MQ\tilde{M} & MX + MQ\tilde{N} \end{bmatrix} \end{aligned}$$

is also affine in Q. The signal

$$e_2 = M\big((-Y + Q\tilde{M})v_1 + (X + Q\tilde{N})v_2\big) \in \mathfrak{D}_P = ML_2,$$

which can be viewed as the ultimate goal of every stabilizing controller.

All stable closed-loop systems

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$$\tilde{T}_{\rm aux} = \begin{bmatrix} \tilde{M}\tilde{X} & -\tilde{M}N \\ Y\tilde{X} & I - YN \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} I & 0 \end{bmatrix},$$

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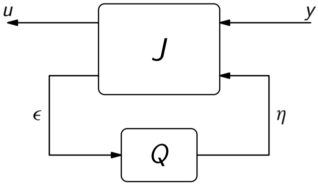
which can be viewed as the ultimate goal of every stabilizing controller.

YK in state space

Let

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be stabilizable and detectable and K and L render $A + BK$ and $A + LC$ Hurwitz. All stabilizing $C : y \mapsto u$ are given by



with

$$J(s) = \left[\begin{array}{c|cc} A + BK + LC + LDK & -L & B + LD \\ \hline K & 0 & I \\ -(C + DK) & I & -D \end{array}\right].$$

YK in state space: derivation

Key observation:

$$\begin{bmatrix} u \\ \epsilon \end{bmatrix} = \overbrace{\begin{bmatrix} -\tilde{Y}\tilde{X}^{-1} & M + \tilde{Y}\tilde{X}^{-1}N \\ \tilde{X}^{-1} & -\tilde{X}^{-1}N \end{bmatrix}}^{J} \begin{bmatrix} y \\ \eta \end{bmatrix} \iff \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M & -\tilde{Y} \\ N & \tilde{X} \end{bmatrix} \begin{bmatrix} \eta \\ \epsilon \end{bmatrix}$$

We know that

$$\begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right],$$

or

$$\begin{bmatrix} M & -\tilde{Y} \\ N & \tilde{X} \end{bmatrix} : \begin{cases} \dot{\hat{x}}(t) = (A + BK)\hat{x}(t) + B\eta(t) - L\epsilon(t) \\ u(t) = K\hat{x}(t) + \eta(t) \\ y(t) = (C + DK)\hat{x}(t) + D\eta(t) + \epsilon(t) \end{cases}$$

and all we need is to

- swap y and ϵ back.

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or

$$\begin{bmatrix} M & -\tilde{Y} \\ N & \tilde{X} \end{bmatrix} : \begin{cases} \dot{\hat{x}}(t) = (A+BK)\hat{x}(t) + B\eta(t) - L\epsilon(t) \\ u(t) = K\hat{x}(t) + \eta(t) \\ y(t) = (C+DK)\hat{x}(t) + D\eta(t) + \epsilon(t) \end{cases}$$

and all we need is to

– swap y and ϵ back.

YK in state space: derivation (contd)

The resulting

$$J : \begin{cases} \dot{\hat{x}}(t) = (A + BK + LC + LDK)\hat{x}(t) - Ly(t) + (B + LD)\eta(t) \\ u(t) = K\hat{x}(t) + \eta(t) \\ \epsilon(t) = -(C + DK)\hat{x}(t) + y(t) - D\eta(t) \end{cases}$$

is equivalent to

$$J : \begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L\overbrace{(y(t) - C\hat{x}(t) - Du(t))}^{\epsilon(t)} \\ u(t) = K\hat{x}(t) + \eta(t) \\ \epsilon(t) = y(t) - C\hat{x}(t) - Du(t) \end{cases}$$

is an observer-based controller, complemented by $\eta = Q\epsilon$.

YK in state space: derivation (contd)

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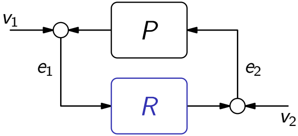
Outline

All stabilizing controllers from a given one



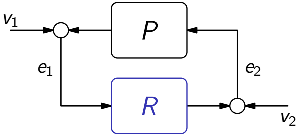
Theorem

Let R_0 stabilizes P and $N_{R_0}, M_{R_0}, \tilde{N}_{R_0}, \tilde{M}_{R_0} \in RH_\infty$ be its coprime factors. R stabilizes the system iff there is $Q \in RH_\infty$ such that

$$R = \mathcal{F}_l\left(\begin{bmatrix} R_0 & I \\ I & -P(I - R_0 P)^{-1} \end{bmatrix}, \tilde{M}_{R_0}^{-1} Q M_{R_0}^{-1} \right)$$

and such that $I + P(\infty)(\tilde{M}_{R_0}^{-1}(\infty) Q(\infty) M_{R_0}^{-1}(\infty) - R_0(\infty))$ is nonsingular.

All stabilizing controllers from a given one



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and such that $I + P(\infty)(\tilde{M}_{R_0}^{-1}(\infty) Q(\infty) M_{R_0}^{-1}(\infty) - R_0(\infty))$ is nonsingular.

Remark: If R stabilizes P_0, then all plants stabilized by this controller are

$$P = \mathcal{F}_u\left(\begin{bmatrix} -R(I - P_0 R)^{-1} & I \\ I & P_0 \end{bmatrix}, \tilde{M}_0^{-1} Q M_0^{-1}\right)$$

for $Q \in RH_\infty$.

Stabilizing controllers from those for the unstable part

If

$$P = P_{\mathrm{fd}} + \Pi \quad \text{for some } \Pi \in H_\infty$$

then its doubly coprime factorization

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ Y & X \end{bmatrix} \begin{bmatrix} \tilde{X} & N \\ -\tilde{Y} & M \end{bmatrix} = \begin{bmatrix} \tilde{M}_{\mathrm{fd}} & -\tilde{N}_{\mathrm{fd}} - \tilde{M}_{\mathrm{fd}}\Pi \\ Y_{\mathrm{fd}} & X_{\mathrm{fd}} - Y_{\mathrm{fd}}\Pi \end{bmatrix} \begin{bmatrix} \tilde{X}_{\mathrm{fd}} - \Pi\tilde{Y}_{\mathrm{fd}} & N_{\mathrm{fd}} + \Pi M_{\mathrm{fd}} \\ -\tilde{Y}_{\mathrm{fd}} & M_{\mathrm{fd}} \end{bmatrix} = I$$

If $R_{\mathrm{fd}} = (-\tilde{Y}_{\mathrm{fd}} + M_{\mathrm{fd}}Q)(\tilde{X}_{\mathrm{fd}} + N_{\mathrm{fd}}Q)^{-1}$ stabilizes P_{fd}, then all stabilizing R for P are

$$\begin{aligned} R &= (-\tilde{Y} + MQ)(\tilde{X} + NQ)^{-1} \\ &= (-\tilde{Y}_{\mathrm{fd}} + M_{\mathrm{fd}}Q)(\tilde{X}_{\mathrm{fd}} + N_{\mathrm{fd}}Q + \Pi(-\tilde{Y}_{\mathrm{fd}} + M_{\mathrm{fd}}))^{-1} \\ &= (-\tilde{Y}_{\mathrm{fd}} + M_{\mathrm{fd}}Q)(\tilde{X}_{\mathrm{fd}} + N_{\mathrm{fd}}Q)^{-1}\big(I + \Pi(-\tilde{Y}_{\mathrm{fd}} + M_{\mathrm{fd}})(\tilde{X}_{\mathrm{fd}} + N_{\mathrm{fd}}Q)^{-1}\big)^{-1} \\ &= R_{\mathrm{fd}}(I + \Pi R_{\mathrm{fd}})^{-1}, \end{aligned}$$

which is Π in negative feedback with R_{fd}.

Dead-time compensation as YK

Let

$$P(s) = P_0(s)\mathrm{e}^{-\tau s} = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \mathrm{e}^{-\tau s}.$$

– If P_0 is stable, take $P_{\mathrm{fd}}(s) = P_0(s)$, for which

$$\Pi(s) = P_0(s)\mathrm{e}^{-\tau s} - P_{\mathrm{fd}}(s) = -P_0(s)(1 - \mathrm{e}^{-\tau s})$$

and we end up with the Smith controller.

– If P_0 is unstable, take $P_{\mathrm{fd}}(s) = C\mathrm{e}^{-A\tau}(sI - A)^{-1}B$, for which

$$\begin{aligned}\Pi(s) = P_0(s)\mathrm{e}^{-\tau s} - P_{\mathrm{fd}}(s) &= C(\mathrm{e}^{-\tau s}I - \mathrm{e}^{-A\tau})(sI - A)^{-1}B \\ &= -C\int_0^{\tau} \mathrm{e}^{A(t-\tau)}\mathrm{e}^{-st}\,\mathrm{d}t B\end{aligned}$$

and we end up with the modified Smith controller.

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Outline

Open-loop stabilization problems

Given G_{11}, G_{12}, and G_{21},

- find $R \in RH_\infty$ such that $G_e := G_{11} + G_{12}RG_{21} \in RH_\infty$.

Known as two-sided and could be messy.

Simplified one-sided versions:

- "tracking" setup with $G_{21} = I$ (and $G_{11} = G_1$ and $G_{12} = G_2$), so

$$G_e := G_1 + G_2 R$$

- "estimation" setup with $G_{12} = I$ (and $G_{11} = G_1$ and $G_{21} = G_2$), so

$$G_e := G_1 + RG_2$$

Open-loop stabilization problems

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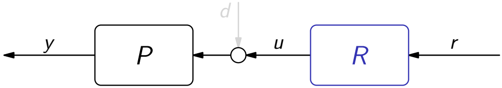
$$G_e := G_1 + R G_2$$

Motivation

Example

Let



for $P(s) = 1/(s+1)$ and $R \in RH_\infty$. If $r = \mathbb{1}$, then

$$\lim_{t\to\infty} e(t) = 0 \iff R(0) = 1.$$

Let $W(s) = 1/s$, then

$$W(s)(1 - P(s)R(s)) = \frac{1}{s} - \frac{R(s)}{s(s+1)} = \frac{1 - R(0)}{s} - \overbrace{\frac{R(s) - R(0)(s+1)}{s(s+1)}}^{\in RH_\infty \text{ for all } R \in RH_\infty}$$

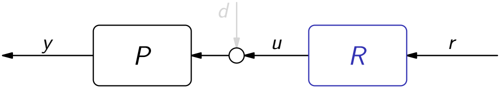
and

$$W(1 - PR) \in RH_\infty \iff R(0) = 1.$$

Motivation

Example

Let



for $P(s) = 1/(s+1)$ and $R \in RH_\infty$. If $r = \mathbb{1}$, then

$$\lim_{t\to\infty} e(t) = 0 \iff R(0) = 1.$$

Let $W(s) = 1/s$, then

$$W(s)(1 - P(s)R(s)) = \frac{1}{s} - \frac{R(s)}{s(s+1)} = \frac{1 - R(0)}{s} - \overbrace{\frac{R(s) - R(0)(s+1)}{s(s+1)}}^{\in RH_\infty \text{ for all } R \in RH_\infty}$$

and

$$W(1 - PR) \in RH_\infty \iff R(0) = 1.$$

Motivation

Example

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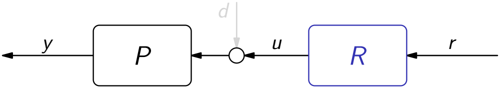

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$$W(1 - PR) \in RH_\infty \iff R(0) = 1.$$

Estimation stabilization: insight

Intuitively, there is $R \in RH_\infty$ stabilizing $G_1 + RG_2$ if

- unstable poles of $G_1(s)$ are also poles of $G_2(s)$ and
- $R(s)$ reshapes directions of poles to cancel them with those of $G_1(s)$

But directional properties complicate matters.

Example

Let

$$G_1(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix} \quad \text{and} \quad G_2(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/(s^2+1) \end{bmatrix}$$

In this case

$$G_e(s) = \begin{bmatrix} (1 + R_{11}(s))/s & R_{12}(s)/(s^2+1) \\ R_{21}(s)/s & 1/s + R_{22}(s)/(s^2+1) \end{bmatrix}$$

and stability conditions are $R_{11}(0) = -1$, $R_{21}(0) = 0$, $R_{12}(\pm j) = 0$, but no stabilizing $R_{22}(s)$ exists.

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Estimation stabilization: stabilizability

Lemma

Let $G_2 = N_2 M_2^{-1} = \tilde{M}_2^{-1} \tilde{N}_2$ be coprime factorizations over RH_∞. The following conditions are equivalent:

1. *there is $R \in RH_\infty$ such that $G_e = G_1 + RG_2 \in RH_\infty$,*
2. *$G_1 M_2 \in RH_\infty$ (equivalently, $\mathfrak{D}_{G_1} \subset \mathfrak{D}_{G_2}$),*
3. *there is a left coprime factorization of the form*

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_1 \\ 0 & \tilde{M}_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix} = \begin{bmatrix} \tilde{N}_1 - \tilde{M}_1 \tilde{M}_2^{-1} \tilde{N}_2 \\ \tilde{M}_2^{-1} \tilde{N}_2 \end{bmatrix}$$

for some $\tilde{N}_1, \tilde{M}_1 \in RH_\infty$.

Estimation stabilization: all stabilizing R

Theorem

If a factorization of the form

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_1 \\ 0 & \tilde{M}_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix}$$

exists, then $R \in RH_\infty$ *stabilizes* G_e *iff there is* $Q \in RH_\infty$ *such that*

$$R = \tilde{M}_1 + Q\tilde{M}_2$$

and then

$$G_e = \tilde{N}_1 + Q\tilde{N}_2$$

is the set of all attainable stable error systems.

Estimation stabilization: state-space condition

Lemma

If the realization

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right]$$

is stabilizable, then G admits a left coprime factorization of form

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_1 \\ 0 & \tilde{M}_2 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix}$$

iff (C_2, A) is detectable, in which case

$$\begin{bmatrix} \tilde{M}_1(s) & \tilde{N}_1(s) \\ \tilde{M}_2(s) & \tilde{N}_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + L_2 C_2 & L_2 & B + L_2 D_2 \\ \hline C_1 & 0 & D_1 \\ C_2 & I & D_2 \end{array}\right]$$

for any L_2 such that $A + L_2 C_2$ is Hurwitz.

Estimation stabilization: state-space condition (contd)

Example

Return to tracking, which can be written as

$$G_e(s) = \frac{1}{s} - \frac{R(s)}{s(s+1)} = \left[\, 1 \;\vdots\; R(s) \,\right] \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 1 \\ \hline 1 & 1 & 0 \\ \hdashline -1 & 0 & 0 \end{array}\right].$$

With $L_2 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, for which $\operatorname{spec}(A + L_2 C_2) = \{-1, -1\}$,

$$\begin{bmatrix} \tilde{M}_1(s) & \tilde{N}_1(s) \\ \tilde{M}_2(s) & \tilde{N}_2(s) \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} 1/(s+1) & (s+2)/(s+1)^2 \\ s/(s+1) & -1/(s+1)^2 \end{bmatrix}$$

so

$$R(s) = \frac{1 + sQ(s)}{s+1} \quad \text{and} \quad G_e(s) = \frac{s+2-Q(s)}{(s+1)^2}$$

and $G_e \in RH_\infty$, with $1 - P(s)R(s) = s(s+2-Q(s))/(s+1)^2$.

Estimation stabilization: state-space condition (contd)

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Integral action in R

Assume an integral action is required in the controller (for all good reasons we know). We may think of two approaches:

1. augmented design — immediate
   1.1 augment an integrator to the plant
   1.2 design a stabilizer for the augmented plant
   1.3 move the integrator to the stabilizer
2. constrained YK parametrization — need to figure out how
   2.1 start with all stabilizing controllers, say $R = (X + Q\tilde{N})^{-1}(-Y + Q\tilde{M})$
   2.2 find under what conditions on Q it contains an integral action

For the sake of simplicity, we consider the SISO case only.

Integral action in R: YK condition

So let

$$R(s) = \frac{-Y(s) + Q(s)\tilde{M}(s)}{X(s) + Q(s)\tilde{N}(s)}$$

We know that unstable poles of $R(s)$ are unstable zeros of its denominator. Hence,

$$R(s) = \frac{\tilde{R}(s)}{s} \iff X(0) + Q(0)\tilde{N}(0) = 0 \iff (X + Q\tilde{N})W \in RH_\infty$$

where $W(s) = 1/s$. But this is

- the estimation stability setup with $G_1 = XW$ and $G_2 = \tilde{N}W$.

If $\tilde{N}(0) = 0$, then $X(0) \neq 0$ and

- $G_2 = \tilde{N}W \in RH_\infty$, while $G_1 = XW \notin RH_\infty \xrightarrow{\text{by Lemma}}$ not stabilizable

(naturally). We thus assume that $\tilde{N}(0) \neq 0$.

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Integral action in R: YK condition (contd)

If $\tilde{N}(0) \neq 0$, then $G_2(s) = \tilde{N}(s)/s$ has its only unstable pole at $s = 0$ and a possible choice $\tilde{M}_2 = s/(s+1)$, for which

$$\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} 1 & -X(0)/\tilde{N}(0) \\ 0 & s/(s+1) \end{bmatrix}^{-1} \begin{bmatrix} (\tilde{N}(0)X(s) - \tilde{N}(s)X(0))/(\tilde{N}(0)s) \\ \tilde{N}(s)/(s+1) \end{bmatrix}$$

is the required *lcf*. Hence, all $Q \in RH_\infty$ for which R has an integral action are given as

$$Q(s) = -\frac{X(0)}{\tilde{N}(0)} + Q_0(s)\frac{s}{s+1}$$

All such Q's ensure that

$$Q(0) = -\frac{X(0)}{\tilde{N}(0)}$$

irrespective of Q_0.

Example

Consider

$$G(s) = \frac{1}{s}$$

Its coprime factors and their Bézout coefficients are

$$\tilde{N}(s) = \frac{1}{s+1}, \quad \tilde{M}(s) = \frac{s}{s+1}, \quad X(s) = 1, \quad \text{and} \quad Y(s) = 1$$

All stabilizing:

$$R(s) = \frac{-s-1+sQ(s)}{s+1+Q(s)}$$

All Q's guaranteeing an integral action in $R(s)$:

$$Q(s) = -1 + \frac{sQ_0(s)}{s+1}$$

All stabilizing controllers with an integral action:

$$R(s) = \frac{-2s^2-3s-1+s^2Q_0(s)}{s(s+1+Q_0(s))}$$

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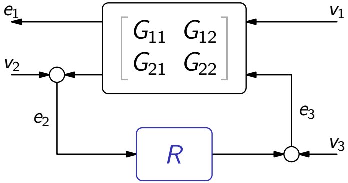
Outline

Internal stability



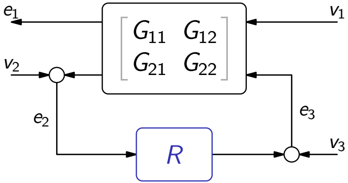
is said to be internally stable if all nine systems $v_i \mapsto e_j$ are stable (in H_∞).

$$\begin{bmatrix} I & 0 & -G_{12} \\ 0 & I & -G_{22} \\ 0 & -R & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

If $T_{\text{aux}} : (v_1, v_2, v_3) \mapsto (e_1, e_2, e_3)$, then

$$T_{\text{aux}} = \begin{bmatrix} G_{11} & 0 & G_{12} \\ G_{21} & I & G_{22} \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} G_{12} \\ G_{22} \\ I \end{bmatrix} R(I - G_{22}R)^{-1} \begin{bmatrix} G_{21} & I & G_{22} \end{bmatrix}.$$

Internal stability



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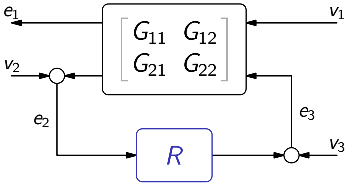
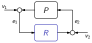
Stabilizability conditions



Theorem

There is an internally stabilizing R iff there are coprime factorizations of the form

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_{12} \\ 0 & \tilde{M}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ M_{21} & M_{22} \end{bmatrix}^{-1}$$

with right coprime $\tilde{N}_{22}$, $\tilde{M}_{22}$ and left coprime N_{22}, M_{22}. If factorizations as above exist, then R internally stabilizes the system iff it internally stabilizes  *under $P = G_{22}$.*

Stabilizability conditions (contd)

Proof (rough outline) : First, note that

$$T_{\mathrm{aux}} \begin{bmatrix} I & 0 & 0 \\ 0 & \tilde{X}_{22} & -N_{22} \\ 0 & \tilde{Y}_{22} & M_{22} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12}\tilde{Y}_{22} & G_{12}M_{22} \\ G_{21} & \tilde{M}_{22}^{-1} & 0 \\ 0 & \tilde{Y}_{22} & M_{22} \end{bmatrix} + \begin{bmatrix} G_{12} \\ G_{22} \\ I \end{bmatrix} T_{\mathrm{c}} \begin{bmatrix} G_{21} & M_{22}^{-1} & 0 \end{bmatrix}$$

Hence, necessity of $G_{12}M_{22} \in RH_\infty$ (cf. open-loop stabilizability). Second,

$$\begin{bmatrix} I & \tilde{M}_{12} & -\tilde{N}_{12} \\ 0 & \tilde{M}_{22} & -\tilde{N}_{22} \\ 0 & Y_{22} & X_{22} \end{bmatrix} T_{\mathrm{aux}} = \begin{bmatrix} G_{11} + \tilde{M}_{12}G_{21} & \tilde{M}_{12} & 0 \\ \tilde{M}_{22}G_{21} & \tilde{M}_{22} & 0 \\ Y_{22}G_{21} & Y_{22} & M_{22}^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_{22}^{-1} \end{bmatrix} T_{\mathrm{c}} \begin{bmatrix} G_{21} & I & G_{22} \end{bmatrix},$$

Hence, necessity of $G_{11} + \tilde{M}_{12}G_{21} \in RH_\infty$ and $\tilde{M}_{22}G_{21} \in RH_\infty$. Then a lot of hard labor and the result follows ... $\square$

Stabilizability conditions: interpretation

Conditions

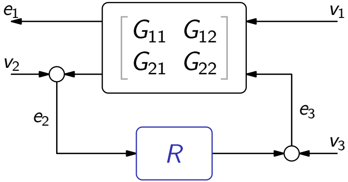
$$
\begin{aligned}
\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} &= \begin{bmatrix} \tilde{N}_{11} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{N}_{21} & \tilde{N}_{12} - \tilde{M}_{12}\tilde{M}_{22}^{-1}\tilde{N}_{22} \\ \tilde{M}_{22}^{-1}\tilde{N}_{21} & \tilde{M}_{22}^{-1}\tilde{N}_{22} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{M}_{12} \\ -I \end{bmatrix} \tilde{M}_{22}^{-1} \begin{bmatrix} \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} \\
&= \begin{bmatrix} N_{11} - N_{12}M_{22}^{-1}M_{21} & N_{12}M_{22}^{-1} \\ N_{21} - N_{22}M_{22}^{-1}M_{21} & N_{22}M_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} N_{11} & 0 \\ N_{21} & 0 \end{bmatrix} - \begin{bmatrix} N_{12} \\ N_{22} \end{bmatrix} M_{22}^{-1} \begin{bmatrix} M_{21} & -I \end{bmatrix}
\end{aligned}
$$

imply that

- all unstable modes of G must be present in G_{22},

around which the feedback loop is closed.

All stable closed-loop maps



From

$$T_{\mathrm{aux}} = \begin{bmatrix} G_{11} & 0 & G_{12} \\ G_{21} & I & G_{22} \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} G_{12} \\ G_{22} \\ I \end{bmatrix} R(I - G_{22}R)^{-1} \begin{bmatrix} G_{21} & I & G_{22} \end{bmatrix}$$

and $R(I - G_{22}R)^{-1} = -\tilde{Y}_{22}\tilde{M}_{22} + M_{22}Q\tilde{M}_{22}$, we have that

$$\begin{aligned} T_{e_1 v_1} &= G_{11} - G_{12}\tilde{Y}_{22}\tilde{M}_{22}G_{21} + G_{12}M_{22}Q\tilde{M}_{22}G_{21} \\ &= \tilde{N}_{11} - (\tilde{M}_{12} + N_{12}Y_{22})\tilde{M}_{22}^{-1}\tilde{N}_{21} + N_{12}Q\tilde{N}_{21} \\ &= \tilde{N}_{11} + \tilde{X}_{12}\tilde{N}_{21} + N_{12}Q\tilde{N}_{21} \end{aligned}$$

(with the help of $M_{22}^{-1}\tilde{Y}_{22} = Y_{22}\tilde{M}_{22}^{-1}$ and $\tilde{X}_{12}\tilde{M}_{22} = -\tilde{M}_{12} - N_{12}Y_{22}$).

Stabilizability conditions in state space

Proposition

Let

$$\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

It is stabilizable iff (A, B_2) *is stabilizable and* (C_2, A) *is detectable. If these conditions hold, then all stabilizing controllers are given by*

$$R(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} A + B_2K + LC_2 + LD_{22}K & -L & B_2 + LD_{22} \\ \hline K & 0 & I \\ -C_2 - D_{22}K & I & -D_{22} \end{array}\right], Q(s)\right),$$

where K *and* L *are such that* $A + B_2K$ *and* $A + LC_2$ *are Hurwitz and* $Q \in RH_\infty$ *and such that* $\det(I - Q(\infty)D_{22}) \neq 0$*, but otherwise arbitrary.*