Linear Control Systems (036012) chapter 5

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Outline

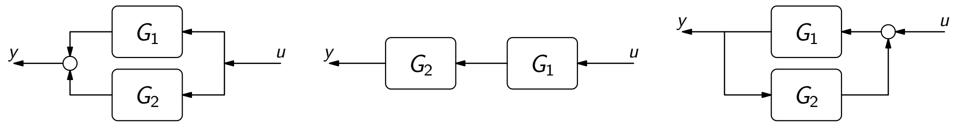
System interconnections

Linear fractional transformations

Control as interconnections

- control is about changing behavior of systems
- two options
  1. redesign
  2. interact

Basic interconnections



parallel series feedback

Main questions:

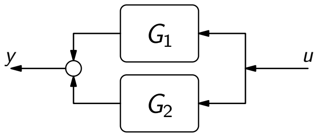
- how interactions change properties?
- when degrees are preserved?

Assume:

$$G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \quad \text{and} \quad G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

are minimal realizations of order n_1 and n_2, respectively.

Parallel interconnections



$$= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

Realization modes are the union of those of its components. Minimality?

Unobservable modes:

$$0 = \begin{bmatrix} A_1 - \lambda I & 0 \\ 0 & A_2 - \lambda I \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} (A_1 - \lambda I)\eta_1 \\ (A_2 - \lambda I)\eta_2 \\ C_1\eta_1 + C_2\eta_2 \end{bmatrix}$$

- $\eta_1 \neq 0$ and $\eta_2 \neq 0$ by observability of (C_1, A_1) and (C_2, A_2)
- $\lambda \in \operatorname{spec}(A_1) \cap \operatorname{spec}(A_2)$
- $C_1 \ker(\lambda I - A_1) \cap C_2 \ker(\lambda I - A_2) \neq \{0\}$

Parallel interconnections (contd)

Proposition

Suppose that both (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are minimal. The realization of their parallel interconnection is controllable iff

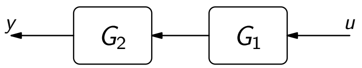
$$\operatorname{pdir}_i(G_1, \lambda) \cap \operatorname{pdir}_i(G_2, \lambda) = \{0\}$$

and is observable iff

$$\operatorname{pdir}_o(G_1, \lambda) \cap \operatorname{pdir}_o(G_2, \lambda) = \{0\},$$

both for all $\lambda \in \operatorname{spec}(A_1) \cap \operatorname{spec}(A_2)$.

Cascade interconnections



$$= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right]$$

Realization modes are the union of those of its components. Minimality?

Unobservable modes:

$$0 = \begin{bmatrix} A_1 - \lambda I & 0 \\ B_2C_1 & A_2 - \lambda I \\ D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & A_2 - \lambda I & B_2 \\ 0 & C_2 & D_2 \end{bmatrix} \begin{bmatrix} (A_1 - \lambda I)\eta_1 \\ \eta_2 \\ C_1\eta_1 \end{bmatrix}$$

- $\eta_1 \neq 0$ by observability of (C_2, A_2)
- $\lambda \in \operatorname{spec}(A_1)$
- $(A_1 - \lambda I)\eta_1 = 0$ and $\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} \eta_2 \\ C_1\eta_1 \end{bmatrix} = 0$
- $C_1 \ker(\lambda I - A_1) \cap \begin{bmatrix} 0 & I \end{bmatrix} \ker R_{G_2}(\lambda) \neq \{0\}$

Cascade interconnections (contd)

Proposition

Suppose that both (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are minimal. The realization of their cascade interconnection is controllable iff

$$\operatorname{pdir}_i(G_2, \lambda) \cap \operatorname{zdir}_o(G_1, \lambda) = \{0\}$$

for all $\lambda \in \operatorname{spec}(A_2)$ and is observable iff

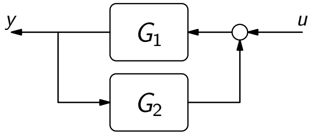
$$\operatorname{zdir}_i(G_2, \lambda) \cap \operatorname{pdir}_o(G_1, \lambda) = \{0\},$$

for all $\lambda \in \operatorname{spec}(A_1)$.

Remark: $\operatorname{zdir}_o(G_1, \lambda)$ and $\operatorname{zdir}_i(G_2, \lambda)$ might be nontrivial even if λ is not a zero of G_1 and G_2, respectively. If $\operatorname{nrank}(R_{G_1}(s)) < n_1 + p_1$, be it because $m_1 < p_1$ or because of its normal rank deficiency, $\operatorname{zdir}_o(G_1, s)$ is nontrivial for all s. Likewise, if $\operatorname{nrank}(R_{G_2}(s)) < n_2 + m_2$ for whatever reason, $\operatorname{zdir}_i(G_2, s)$ is nontrivial for all s too. Hence, if $G(s)$ has its McMillan degree below $n_1 + n_2$, we call it just "cancellation," rather than "pole-zero cancellation."

Feedback interconnections

If $D_2 = 0$ (simplicity), then



$$= \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 \\ B_2 C_1 & A_2 + B_2 D_1 C_2 & B_2 D_1 \\ \hline C_1 & D_1 C_2 & D_1 \end{array}\right]$$

Realization modes are unrelated to those of its components. Minimality?

Observability PBH:

$$\begin{bmatrix} A_1 - \lambda I & B_1 C_2 \\ B_2 C_1 & A_2 + B_2 D_1 C_2 - \lambda I \\ C_1 & D_1 C_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & B_2 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_1 - \lambda I & B_1 C_2 \\ 0 & A_2 - \lambda I \\ C_1 & D_1 C_2 \end{bmatrix}$$

Hence

– observability here is lost iff it's lost in $G_1 G_2$

Feedback interconnections (contd)

Proposition

Suppose that both (A_1, B_1, C_1, D_1) *and* (A_2, B_2, C_2, D_2) *are minimal and that* $\det(I - D_1 D_2) \neq 0$. *The realization of their feedback interconnection is controllable iff*

$$\mathrm{pdir}_i(G_2, \lambda) \cap \mathrm{zdir}_o(G_1, \lambda) = \{0\}$$

and is observable iff

$$\mathrm{zdir}_i(G_1, \lambda) \cap \mathrm{pdir}_o(G_2, \lambda) = \{0\},$$

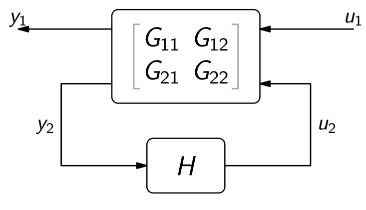
both for all $\lambda \in \mathrm{spec}(A_2)$.

Outline

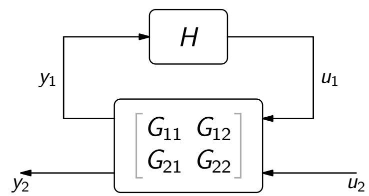
System interconnections

Linear fractional transformations

LFT



lower, $\mathcal{F}_l(G, H)$



upper, $\mathcal{F}_u(G, H)$

The lower LFT $\mathcal{F}_l(H, G) : u_1 \mapsto y_1$ reads

$$\mathcal{F}_l(H, G) = G_{11} + G_{12} H (I - G_{22} H)^{-1} G_{21} = G_{11} + G_{12} (I - H G_{22})^{-1} H G_{21}$$

The upper LFT $\mathcal{F}_u(H, G) : u_2 \mapsto y_2$ reads

$$\mathcal{F}_u(G, H) = G_{22} + G_{21} (I - H G_{11})^{-1} H G_{12} = G_{22} + G_{21} H (I - G_{11} H)^{-1} G_{12}$$

Moreover,

$$\mathcal{F}_u(G, H) = \mathcal{F}_l\left(\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] G \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], H\right)$$

Special cases

Parallel:

$$G_1 + G_2 = \mathcal{F}_\mathrm{l}\left(\begin{bmatrix} G_1 & I \\ I & 0 \end{bmatrix}, G_2\right)$$

Series:

$$G_2 G_1 = \mathcal{F}_\mathrm{l}\left(\begin{bmatrix} 0 & I \\ G_1 & 0 \end{bmatrix}, G_2\right)$$

Feedback:

$$G_1(I - G_2 G_1)^{-1} = \mathcal{F}_\mathrm{l}\left(\begin{bmatrix} G_1 & G_1 \\ G_1 & G_1 \end{bmatrix}, G_2\right)$$

GoF (positive feedback):

$$\begin{bmatrix} T_\mathrm{c} & T_\mathrm{i} \\ S_\mathrm{o} & T_\mathrm{d} \end{bmatrix} := \begin{bmatrix} R \\ I \end{bmatrix}(I - PR)^{-1}\begin{bmatrix} I & P \end{bmatrix} = \mathcal{F}_\mathrm{l}\left(\left[\begin{array}{cc:c} 0 & 0 & I \\ I & P & P \\ \hdashline I & P & P \end{array}\right], R\right)$$

State-space realization:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \mathcal{F}_\mathrm{u}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{s}I\right)$$

Well posedness

Clearly,

$$I - G_{22}H \text{ is invertible} \implies \mathcal{F}_\mathrm{l}(G, H) \text{ is well defined}$$

Example

The LFT

$$\mathcal{F}_\mathrm{l}\left(\left[\begin{array}{c:cc} 1 & 0 & 1 \\ \hdashline 1 & \alpha & 0 \\ 0 & 0 & 0 \end{array}\right], I\right) = 1, \quad \forall\alpha$$

although

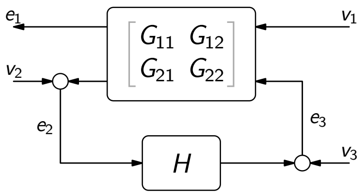
$$I - G_{22}H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1-\alpha & 0 \\ 0 & 1 \end{bmatrix}$$

is singular for $\alpha = 1$. But (internal signal) u_2, satisfyng

$$\begin{bmatrix} 1-\alpha & 0 \\ 0 & 1 \end{bmatrix} u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1,$$

is not well defined under $\alpha = 1$.

Well posedness (contd)



LFT is well posed if all mappings $v_i \mapsto e_j$ are well defined.

$$\left[\begin{array}{c:cc} I & 0 & -G_{12} \\ \hdashline 0 & I & -G_{22} \\ 0 & -H & I \end{array}\right] \left[\begin{array}{c} e_1 \\ \hdashline e_2 \\ e_3 \end{array}\right] = \left[\begin{array}{c:cc} G_{11} & 0 & 0 \\ \hdashline G_{21} & I & 0 \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{c} v_1 \\ \hdashline v_2 \\ v_3 \end{array}\right]$$

$$\Downarrow$$

$$\begin{bmatrix} I & -G_{22} \\ -H & I \end{bmatrix} \text{ is invertible}$$

$$\Downarrow$$

$$I - G_{22}H \text{ is invertible} \iff I - HG_{22} \text{ is invertible}$$

I/O inversion

Proposition

If $\mathcal{F}_l(G, H)$ is square and G_{11} is nonsingular, then

$$[\mathcal{F}_l(G, H)]^{-1} = \mathcal{F}_l\left(\begin{bmatrix} G_{11}^{-1} & -G_{11}^{-1}G_{12} \\ G_{21}G_{11}^{-1} & G_{22} - G_{21}G_{11}^{-1}G_{12} \end{bmatrix}, H\right).$$

If $\mathcal{F}_u(G, H)$ is square and G_{22} is nonsingular, then

$$[\mathcal{F}_u(G, H)]^{-1} = \mathcal{F}_u\left(\begin{bmatrix} G_{11} - G_{12}G_{22}^{-1}G_{21} & G_{12}G_{22}^{-1} \\ -G_{22}^{-1}G_{21} & G_{22}^{-1} \end{bmatrix}, H\right).$$

Proof (outline): The lower LFT relation follows by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \iff \begin{bmatrix} u_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11}^{-1} & -G_{11}^{-1}G_{12} \\ G_{21}G_{11}^{-1} & G_{22} - G_{21}G_{11}^{-1}G_{12} \end{bmatrix}\begin{bmatrix} y_1 \\ u_2 \end{bmatrix}$$

The upper LFT relation swaps y_2 and u_2. □ □

T–H inversion

Proposition

If G is invertible, with square and invertible G_{12} and G_{21}, then

$$T = \mathcal{F}_l(G, H) \iff H = \mathcal{F}_u(G^{-1}, T) = \mathcal{F}_l\left(\left[\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right] G^{-1} \left[\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right], T\right).$$

Proof (outline): Relation follows by

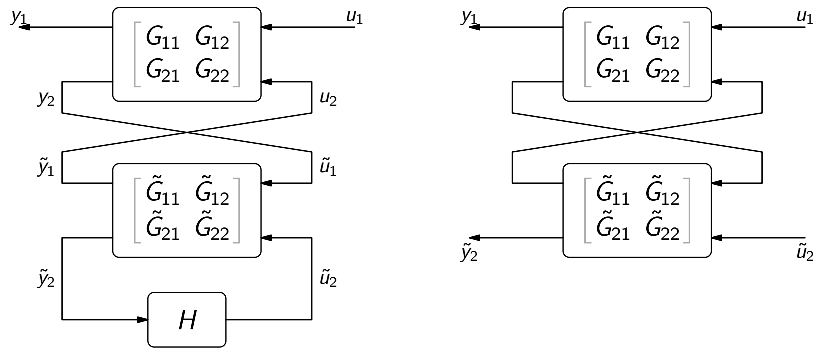
$$T : u_1 \mapsto y_1 \quad \text{in} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = G \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad u_2 = H y_2,$$

so

$$H : y_2 \mapsto u_2 \quad \text{in} \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = G^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad y_1 = T\, u_1.$$

The invertibility of G_{12} and G_{21} is required to show the equivalence of the well-posedness properties of $\mathcal{F}_l(G, H)$ and $\mathcal{F}_u(G^{-1}, T)$. □

Redheffer star product



$\mathcal{F}_l(G, \mathcal{F}_l(\tilde{G}, H))$ $\qquad$ $G \star \tilde{G}$

so

$$\mathcal{F}_l(G, \mathcal{F}_l(\tilde{G}, H)) = \mathcal{F}_l(G \star \tilde{G}, H).$$