Linear Control Systems (036012)
chapter 4

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Preliminary: linear algebra facts

- $(\ker M)^\perp = \operatorname{Im} M'$ and $(\operatorname{Im} M)^\perp = \ker M'$
- $\operatorname{Im} M_1 = \operatorname{Im} M_2 \iff \ker M_1' = \ker M_2'$
- if $M = M' \geq 0$, then $Mx = 0 \iff x'Mx = 0$
- if $M = M' \geq 0$, then left & right singular vectors coincide, $M = U\Sigma U'$
- if $f(x)$ is analytic, $M, T \in \mathbb{F}^{m\times m}$ with $\det T \neq 0$, then

$$f(TMT^{-1}) = Tf(M)T^{-1}$$

(in particular, $(TMT^{-1})^i = TM^iT^{-1}$ and $\mathrm{e}^{TMT^{-1}} = T\mathrm{e}^{M}T^{-1}$)

Preliminary: Cayley–Hamilton

In essence, each square matrix satisfies its own characteristic equation:

$$\chi_A(A) := A^n + \chi_{n-1}A^{n-1} + \cdots + \chi_1 A + \chi_0 I_n = 0.$$

Important consequence:

- A^k for all $k \geq n$ is a linear combination of A^i, $i = 0, \ldots, n-1$, like

$$\begin{aligned} A^n &= -\chi_{n-1}A^{n-1} - \cdots - \chi_1 A - \chi_0 I_n \\ A^{n+1} &= -\chi_{n-1}A^{n} - \cdots - \chi_1 A^2 - \chi_0 A \\ &= \chi_{n-1}(\chi_{n-1}A^{n-1} + \cdots + \chi_1 A + \chi_0 I_n) - \cdots - \chi_1 A^2 - \chi_0 A \\ &= (\chi_{n-1}^2 - \chi_{n-2})A^{n-1} + \cdots + (\chi_{n-1}\chi_1 - \chi_0)A + \chi_{n-1}\chi_0 I_n \end{aligned}$$

$\vdots$

Preliminary: matrix Sylvester and Lyapunov equations

Given $A_1 \in \mathbb{F}^{p\times p}$, $A_2 \in \mathbb{F}^{m\times m}$, $Q \in \mathbb{F}^{p\times m}$, solve in $X \in \mathbb{F}^{p\times m}$

$$A_1 X + X A_2 + Q = 0.$$

If $\operatorname{spec}(A_1) \cap \operatorname{spec}(-A_2) = \varnothing$, X exists for all Q and is unique. Otherwise, there might be either no or infinitely many solutions, depending on Q.

Its special case for $A_2 = A_1'$ and $Q = Q'$ is known as the matrix Lyapunov equation,

$$AX + XA' + Q = 0.$$

If A is Hurwitz, then

$$X = \int_{\mathbb{R}^+} \mathrm{e}^{At} Q \mathrm{e}^{A't} \mathrm{d}t,$$

Outline

State-space realizations

System interconnections in terms of state-space realizations

Structural properties

Kalman canonical decomposition and minimality

Coprime factorization via state-space realizations

Poles / zeros / directions via state-space realizations

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Model reduction by balanced truncation

State-space realizations

Let $G : \mathfrak{D}_G \subset \mathbb{R}^n \times L_2^m(\mathbb{R}_+) \to L_2^p(\mathbb{R}_+)$ be LTI, finite dimensional, and have a proper transfer function. There are $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ such that $u \mapsto y = Gu$ reads

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

The quadruple (A, B, C, D) is called a state-space realization of G.

If $x_0 = 0$, we have that $G : \mathfrak{D}_G \subset L_2^m(\mathbb{R}_+) \to L_2^p(\mathbb{R}_+)$ and we write

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Solution:

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s.$$

Similarity transformations

Let

$$\tilde{x}(t) := Tx(t)$$

for some nonsingular $T \in \mathbb{R}^{n\times n}$. We have:

$$\dot{\tilde{x}}(t) = T\dot{x}(t) = T\bigl(Ax(t) + Bu(t)\bigr) = TAT^{-1}\tilde{x}(t) + TBu(t)$$

and also

$$y(t) = Cx(t) + Du(t) = CT^{-1}\tilde{x}(t) + Du(t).$$

Hence,

$$(TAT^{-1}, TB, CT^{-1}, D)$$

is also a realization of the same G. This realization is said to be similar to (A, B, C, D).

Impulse response and transfer function

The impulse response of G is

$$g(t) = D\delta(t) + C\mathrm{e}^{At}B.$$

The corresponding transfer function

$$G(s) = D + C(sI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

with $G(\infty) = D$, so that $G(s)$ is

– strictly proper iff $D = 0$ and bi-proper iff $\det(D) \neq 0$.

Readily seen that

$$D\delta(t) + CT^{-1}\mathrm{e}^{TAT^{-1}t}TB = D\delta(t) + C\mathrm{e}^{At}B$$

and

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Outline

Interconnecting in state space

For transfer functions, algebraic manipulations over complex functions:

- parallel: $G_1(s) + G_2(s)$
- series: $G_2(s)G_1(s)$
- inverse: $G^{-1}(s)$

For state-space realizations:

- can be done via matrix algebra.

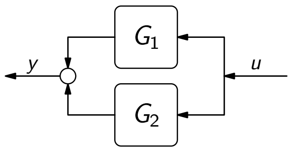
Let

$$G_1 : \begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \\ y_1(t) = C_1 x_1(t) + D_1 u_1(t) \end{cases} \quad \text{and} \quad G_2 : \begin{cases} \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t) \\ y_2(t) = C_2 x_2(t) + D_2 u_2(t) \end{cases}$$

An efficient way of interconnecting such systems is to

- unite state vectors
- determine what inputs / outputs to connect

Parallel interconnection



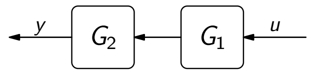
corresponds to

- $u_1 = u_2 = u$
- $y = y_1 + y_2$

Hence,

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + (D_1 + D_2) u(t) \end{cases}$$

Series / cascade interconnection



corresponds to

- $u_1 = u$
- $u_2 = y_1$
- $y = y_2$

Then

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + D_2 D_1 u(t) \end{cases}$$

Inversion

Let

$$G: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

be square (i.e. $p = m$) and bi-proper (i.e. $\det(D) \neq 0$). Its inverse is the system mapping $y \mapsto u$. Then

$$u(t) = -D^{-1}Cx(t) + D^{-1}y(t)$$

and

$$\dot{x}(t) = Ax(t) + B\big(-D^{-1}Cx(t) + D^{-1}y(t)\big)$$

Therefore,

$$G^{-1}: \begin{cases} \dot{x}(t) = (A - BD^{-1}C)x(t) + BD^{-1}y(t) \\ u(t) = -D^{-1}Cx(t) + D^{-1}y(t) \end{cases}$$

Summary

If

$$G_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad i \in \{1, 2\}$$

then

- $G_1(s) + G_2(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ 0 & A_1 & B_1 \\ \hline C_2 & C_1 & D_1 + D_2 \end{array}\right]$
- $G_2(s)G_1(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 & D_2D_1 \end{array}\right]$
- $G_i^{-1}(s) = \left[\begin{array}{c|c} A_i - B_iD_i^{-1}C_i & B_iD_i^{-1} \\ \hline -D_i^{-1}C_i & D_i^{-1} \end{array}\right] = \left[\begin{array}{c|c} A_i - B_iD_i^{-1}C_i & -B_iD_i^{-1} \\ \hline D_i^{-1}C_i & D_i^{-1} \end{array}\right]$

Partial fraction expansion

Partition

$$G_2(s)G_1(s) = H_1(s) + H_2(s)$$

with $H_i(s)$ having the "A" matrices of $G_i(s)$. Roth's removal rule:

- $\begin{bmatrix} A_1 & Q \\ 0 & A_2 \end{bmatrix}$ and $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ are similar iff $XA_1 - A_2X = -Q$ is solvable

Thus, assuming the Sylvester equation $XA_1 - A_2X = -B_2C_1$ is solvable (it is enough to have $\operatorname{spec}(A_1) \cap \operatorname{spec}(A_2) = \varnothing$), use $T = \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ to get

$$G_2(s)G_1(s) = \left[\begin{array}{cc|c} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 & D_2D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2D_1 + XB_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 - C_2X & D_2D_1 \end{array}\right]$$

$$= \underbrace{\left[\begin{array}{c|c} A_1 & B_1 \\ \hline D_2C_1 - C_2X & 0 \end{array}\right]}_{H_1(s)} + \underbrace{\left[\begin{array}{c|c} A_2 & B_2D_1 + XB_1 \\ \hline C_2 & D_2D_1 \end{array}\right]}_{H_2(s)}$$

Moral: similarity transformations are a powerful tool.

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Controllability

We say that $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is controllable if the

- eigenvalues of $A + BK$ can be freely assigned by a choice of $K \in \mathbb{R}^{m\times n}$

(with the restriction that complex eigenvalues are in conjugate pairs).

Controllability criteria

The following statements are equivalent:

1. The pair (A, B) is controllable.
2. The matrix
$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \mathbb{C}^{n\times(n+m)}$$
has full rank $\forall \lambda \in \mathbb{C}$ (the PBH [Popov-Belevich-Hautus] test).
3. The matrix
$$W_c(t) := \int_0^t e^{As} BB' e^{A's} ds \in \mathbb{R}^{n\times n}$$
is positive definite for all $t > 0$ (the Gramian-based test).
4. The controllability matrix
$$M_c := \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times(nm)}$$
has full rank (i.e. $\text{rank}(M_c) = n$).

Technical Lemma 1

Lemma

$\text{Im}\, W_c(t) = \text{Im}\, M_c$ *for every* $t > 0$ *or, equivalently,* $\ker W_c(t) = \ker M_c'$.

Proof (outline).

$W_c(t) = [W_c(t)]' \geq 0$ implies that $\eta \in \ker W_c(t)$ iff

$$\eta' W_c(t) \eta = 0 \iff \int_0^t \|\eta' e^{As} B\|^2 ds = 0 \iff \eta' e^{As} B = 0, \quad \forall s \in [0, t]$$

As e^{At} is analytic (every Taylor series converges), the latter implies

$$\eta' (e^{As})^{(i)} B|_{s=0} = 0, \quad \forall i \in \mathbb{Z}_+ \iff \eta' \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = 0$$

By Cayley–Hamilton,

$$\ker \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}' = \ker M_c'.$$

Result follows because η is arbitrary. □

Technical Lemma 2

Lemma

If $\text{rank}\, W_c(t) = r < n$*, then there is a unitary matrix* U_c *such that*

$$(U_c A U_c', U_c B) = \left(\begin{bmatrix} A_c & \times \\ 0 & A_{\bar{c}} \end{bmatrix}, \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right),$$

where $(A_c, B_c) \in \mathbb{R}^{r\times r} \times \mathbb{R}^{r\times m}$ *is such that* $\int_0^t e^{A_c s} B_c B_c' e^{A_c' s} ds > 0$.

We prove it for Hurwitz A. In this case $\text{rank}\, W_c(t) = \text{rank}\, P$, where

$$P := W_c(\infty) \geq 0, \quad \text{verifying Lyapunov eqn. } AP + PA' + BB' = 0$$

aka the controllability Gramian of (A, B). If A is not Hurwitz, $\hat{A} := A - \alpha I$ is Hurwitz for a sufficiently large $\alpha > 0$ and

$$\ker \int_0^t e^{\hat{A}s} BB' e^{\hat{A}'s} ds = \bigcap_{s\in[0,t]} \ker B' e^{(A-\alpha I)'s} = \bigcap_{s\in[0,t]} \ker B' e^{A's} = \ker W_c(t)$$

so nothing changes if we prove the result for $\hat{A}$...

Technical Lemma 2 (contd)

Proof (outline).

If $\operatorname{rank} P = r < m$, $\exists$ unitary U_c s.t. $U_c P U_c' = \left[\begin{smallmatrix} P_c & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for $r \times r$ $P_c > 0$. Let

$$(U_c A U_c', U_c B) = \left(\begin{bmatrix} A_c & A_{12} \\ A_{21} & A_{\bar c} \end{bmatrix}, \begin{bmatrix} B_c \\ B_2 \end{bmatrix} \right)$$

The Lyapunov equation for P reads then

$$\begin{bmatrix} A_c P_c + P_c A_c' + B_c B_c' & P_c A_{21}' + B_c B_2' \\ A_{21} P_c + B_2 B_c' & B_2 B_2' \end{bmatrix} = 0.$$

$$(2,2) \implies B_2 = 0 \overset{(1,2)}{\implies} A_{21} = 0 \implies A_c \text{ is Hurwitz} \overset{(1,1)}{\implies}$$

$$P_c = \int_{\mathbb{R}^+} e^{A_c s} B_c B_c' e^{A_c' s} ds > 0$$

which leads to the last claim by already familiar arguments. □

Equivalence of controllability conditions

$2 \implies 3:$ Let $\operatorname{rank} \begin{bmatrix} A - sI & B \end{bmatrix} = n$, $\forall s \in \mathbb{C}$, but $\operatorname{rank} W_c(t) = r < n$. By TL2 there is a unitary U_c such that

$$U_c \begin{bmatrix} A - sI & B \end{bmatrix} \begin{bmatrix} U_c' & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{cc|c} A_c - sI_r & \times & B_c \\ 0 & A_{\bar c} - sI_{n-r} & 0 \end{array}\right],$$

whose rank drops at every $s \in \operatorname{spec}(A_{\bar c}) \implies$ contradiction.

$2 \impliedby 3:$ Let $\operatorname{rank} W_c(t) = n$, but $\operatorname{rank} \begin{bmatrix} A - s_0 I & B \end{bmatrix} < n$ for $s_0 \in \mathbb{C}$. In this case $\exists \eta_0 \neq 0$ such that

$$\eta_0' \begin{bmatrix} A - s_0 I & B \end{bmatrix} = 0 \iff (\eta_0' A = s_0 \eta_0') \wedge (\eta_0' B = 0).$$

Hence, $\eta_0' e^{At} B = e^{s_0 t} \eta_0' B = 0$, $\forall t \implies \eta_0' W_c(t) = 0 \implies$ contradiction.

$3 \iff 4:$ Follows by TL1.

Equivalence of controllability conditions (contd)

$1 \implies 2:$ Let (A, B) be controllable, but $\operatorname{rank} \begin{bmatrix} A - s_0 I & B \end{bmatrix} < n$ for $s_0 \in \mathbb{C}$. In this case $\exists \eta_0 \neq 0$ such that

$$\eta_0' \begin{bmatrix} A - s_0 I & B \end{bmatrix} = 0 \iff (\eta_0' A = s_0 \eta_0') \wedge (\eta_0' B = 0).$$

Hence, $\eta_0'(A + BK) = s_0 \eta_0' \implies s_0 \in \operatorname{spec}(A + BK)$ for all $K \implies (A, B)$ is not controllable $\implies$ contradiction.

$4 \implies 1:$ If $m = 1$, then $\det(M_c) \neq 0$ and by Ackermann's formula

$$K = -e_n' M_c^{-1} \chi_{cl}(A)$$

assigns $\operatorname{spec}(A + BK)$ to roots of (an arbitrary) $\chi_{cl}(s) \implies$ controllability. If $m > 1$, then for any $0 \neq \tilde b \in \operatorname{Im} B$, $\exists \tilde K \in \mathbb{R}^{m \times n}$ such that $(A + B\tilde K, \tilde b)$ is controllable (Heymann, 1968). Hence,

$$K = \tilde K - \tilde u e_n' \tilde M_c^{-1} \chi_{cl}(A + B\tilde K)$$

does the trick, where $\tilde u \in \mathbb{R}^m$ is such that $B\tilde u = \tilde b \implies$ controllability. □

Controllability and similarity transformations

Let $(\tilde A, \tilde B, \tilde C, D) := (TAT^{-1}, TB, CT^{-1}, D)$. We have that

$$\tilde M_c = \begin{bmatrix} \tilde B & \tilde A \tilde B & \cdots & \tilde A^{n-1} \tilde B \end{bmatrix} = T \begin{bmatrix} B & AB & \cdots & A^{n-1} B \end{bmatrix} = T M_c$$

and

$$\tilde W_c(t) = \int_0^t e^{\tilde A s} \tilde B \tilde B' e^{\tilde A' s} ds = \int_0^t T e^{As} T^{-1} T B B' T' T^{-\prime} e^{A' s} T' ds = T \int_0^t e^{As} B B' e^{A' s} ds T' = T W_c(t) T'.$$

Hence,

- controllability is invariant under similarity transformations.

Uncontrollable modes

If PBH fails at some $\lambda \in \mathbb{C}$, there is $0 \neq \eta \in \mathbb{C}^n$ such that

$$\eta' \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0 \iff (\eta' A = \lambda \eta') \wedge (\eta' B = 0)$$

Hence,

- PBH can fail only if $\lambda \in \mathrm{spec}(A)$
- PBH fails iff $\eta' B = 0$ for a left eigenvector of A

If PBH fails on $\lambda \in \mathrm{spec}(A)$ with corresponding left eigenvector η, then

$$\eta'(A + BK) = \lambda \eta' \implies \lambda \in \mathrm{spec}(A + BK), \quad \forall K$$

i.e. λ remains an eigenvalue (mode) of $A + BK$ for all K. Hence, every

- $\lambda \in \mathbb{C}$ at which PBH fails is called an uncontrollable mode of (A, B).

Stabilizability

We say that $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ is stabilizable if

- there is $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is Hurwitz.

The following statements are equivalent:

1. The pair (A, B) is stabilizable.
2. The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}_0$.

Controllable decomposition

There is a nonsingular matrix T_c such that

$$(T_c A T_c^{-1}, T_c B) = \left(\begin{bmatrix} A_c & \times \\ 0 & A_{\bar{c}} \end{bmatrix}, \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right),$$

where (A_c, B_c) is controllable and $\mathrm{spec}(A_{\bar{c}})$ comprises all uncontrollable modes of (A, B). Moreover, T_c brings (A, B) to this form iff

$$T_c W_c(t) T_c' = \begin{bmatrix} \tilde{W}_c(t) & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\tilde{W}_c(t) > 0$.

Observability & detectability

We say that $(C, A) \in \mathbb{R}^{p \times n} \times \mathbb{R}^{n \times n}$ is observable if the

- eigenvalues of $A + LC$ can be freely assigned by a choice of $L \in \mathbb{R}^{n \times p}$

(with the restriction that complex eigenvalues are in conjugate pairs).

We say that $(C, A) \in \mathbb{R}^{p \times n} \times \mathbb{R}^{n \times n}$ is detectable if

- there is $L \in \mathbb{R}^{n \times p}$ such that $A + LC$ is Hurwitz.

Because

$$\mathrm{spec}(A + LC) = \mathrm{spec}(A' + C'L'),$$

we have that

- (C, A) is observable iff (A', C') is controllable
- (C, A) is detectable iff (A', C') is stabilizable

and can use all tests (with the observability matrix, observability Gramians, PBH, observable decomposition, et cetera).

Observability criteria

The following statements are equivalent:

1. The pair (C, A) is observable.
2. The matrix $\begin{bmatrix} A - sI \\ C \end{bmatrix}$ has full column rank $\forall s \in \mathbb{C}$.
3. The matrix
$$W_\mathrm{o}(t) := \int_0^t \mathrm{e}^{A'\tau} C' C \mathrm{e}^{A\tau} \mathrm{d}\tau$$
is positive definite for any $t > 0$.
4. The observability matrix
$$M_\mathrm{o} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$
has full rank (i.e. $\mathrm{rank}(M_\mathrm{o}) = n$).

Observable decomposition

There is a nonsingular matrix T_o such that

$$(CT_\mathrm{o}^{-1}, T_\mathrm{o}AT_\mathrm{o}^{-1}) = \left(\begin{bmatrix} C_\mathrm{o} & 0 \end{bmatrix}, \begin{bmatrix} A_\mathrm{o} & 0 \\ \times & A_{\bar{\mathrm{o}}} \end{bmatrix} \right),$$

where $(C_\mathrm{o}, A_\mathrm{o})$ is observable and $\mathrm{spec}(A_{\bar{\mathrm{o}}})$ comprises all unobservable modes of (C, A). Moreover, T_o brings (C, A) to this form iff

$$T_\mathrm{o}^{-\prime} W_\mathrm{o}(t) T_\mathrm{o}^{-1} = \begin{bmatrix} \tilde{W}_\mathrm{o}(t) & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\tilde{W}_\mathrm{c}(t) > 0$.

As a matter of fact,

$$\tilde{W}_\mathrm{o}(t) = T^{-\prime} W_\mathrm{o}(t) T^{-1} \quad \text{and} \quad \tilde{M}_\mathrm{o} = M_\mathrm{o} T^{-1}.$$

Outline

Uncontrollable modes and transfer functions

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_\mathrm{c} & \times & B_\mathrm{c} \\ 0 & A_{\bar{\mathrm{c}}} & 0 \\ \hline C_\mathrm{c} & \times & D \end{array}\right],$$

where $(A_\mathrm{c}, B_\mathrm{c})$ is controllable. Now,

$$\begin{aligned} G(s) &= D + \begin{bmatrix} C_\mathrm{c} & \times \end{bmatrix} \left(sI - \begin{bmatrix} A_\mathrm{c} & \times \\ 0 & A_{\bar{\mathrm{c}}} \end{bmatrix} \right)^{-1} \begin{bmatrix} B_\mathrm{c} \\ 0 \end{bmatrix} \\ &= D + \begin{bmatrix} C_\mathrm{c} & \times \end{bmatrix} \begin{bmatrix} (sI - A_\mathrm{c})^{-1} & \times \\ 0 & (sI - A_{\bar{\mathrm{c}}})^{-1} \end{bmatrix} \begin{bmatrix} B_\mathrm{c} \\ 0 \end{bmatrix} \\ &= D + C_\mathrm{c}(sI - A_\mathrm{c})^{-1} B_\mathrm{c}. \end{aligned}$$

In other words,

- uncontrollable modes do not affect the corresponding transfer function.

The same conclusion holds for unobservable modes of a realization.

Kalman canonical decomposition

There is a nonsingular matrix T such that

$$G(s) = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cccc|c} A_{c\bar{o}} & \times & \times & \times & B_{c\bar{o}} \\ 0 & A_{co} & 0 & \times & B_{co} \\ 0 & 0 & A_{\bar{c}\bar{o}} & \times & 0 \\ 0 & 0 & 0 & A_{\bar{c}o} & 0 \\ \hline 0 & C_{co} & 0 & C_{\bar{c}o} & D \end{array}\right] = \left[\begin{array}{c|c} A_{co} & B_{co} \\ \hline C_{co} & D \end{array}\right],$$

where (A_{co}, B_{co}) is controllable and (C_{co}, A_{co}) is observable, so that the

- spec(A_{co}) contains controllable-and-observable
- spec$(A_{c\bar{o}})$ contains controllable-but-unobservable
- spec$(A_{\bar{c}o})$ contains observable-but-uncontrollable
- spec$(A_{\bar{c}\bar{o}})$ contains uncontrollable-and-unobservable

modes of the triple (C, A, B), respectively. Again, neither uncontrollable nor unobservable modes (aka hidden modes) affect the transfer function.

Minimality

A realization (A, B, C, D) of a given system G is said to be

- minimal if the dimension of A is smallest among all realizations of G.

Theorem

A realization (A, B, C, D) is minimal iff (A, B) is controllable and (C, A) is observable.

Proof (outline of the "only if" part).

Follows from the Kalman canonical decomposition. □

Minimality (contd)

Proof (outline of the "if" part).

Let (A, B) be controllable, (C, A) be observable, but the realization be not minimal for $A \in \mathbb{R}^{n\times n}$. I.e. there are $A_r \in \mathbb{R}^{n_r\times n_r}$, B_r, and C_r so that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D \end{array}\right] \quad \text{with } n_r < n.$$

Hence, $Ce^{At}B = C_r e^{A_r t} B_r$, $\forall t$, or $Ce^{A\sigma}e^{A\tau}B = C_r e^{A_r\sigma}e^{A_r\tau}B_r$. This yields

$$e^{A'\sigma}C'Ce^{A\sigma}e^{A\tau}BB'e^{A'\tau} = e^{A'\sigma}C'C_r e^{A_r\sigma}e^{A_r\tau}B_r B'e^{A'\tau}.$$

Integrating both sides from 0 to t over both σ and τ,

$$\underbrace{W_o(t)W_c(t)}_{\text{rank}=n} = \int_0^t e^{A'\sigma}C'C_r e^{A_r\sigma}\,d\sigma \int_0^t e^{A_r\tau}B_r B'e^{A'\tau}\,d\tau =: \underbrace{W_{r1}(t)W_{r2}(t)}_{\text{rank}\le n_r}$$

with $W_{r1}(t) \in \mathbb{R}^{n\times n_r}$ and $W_{r2}(t) \in \mathbb{R}^{n_r\times n}$. Ranks do not agree then. □

All minimal realizations

Theorem

Any two minimal realizations of a finite-dimensional LTI system are similar.

Proof (outline).

Obviously, any realization, similar to another minimal realization, is minimal. Now, let (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ be minimal. Hence,

$$Ce^{At}B = \tilde{C}e^{\tilde{A}t}\tilde{B} \iff CA^iB = \tilde{C}\tilde{A}^i\tilde{B} \implies M_oA^iM_c = \tilde{M}_o\tilde{A}^i\tilde{M}_c, \quad \forall i$$

Construct now

$$T := (\tilde{M}_o'\tilde{M}_o)^{-1}\tilde{M}_o'M_o \quad \text{and} \quad S := M_c\tilde{M}_c'(\tilde{M}_c\tilde{M}_c')^{-1}.$$

It can be shown that $T = S^{-1}$, $TM_c = \tilde{M}_c$, $M_oS = M_oT^{-1} = \tilde{M}_o$, and then $TAS = TAT^{-1} = \tilde{A}$. □

Gilbert's realization

If $G(s)$ is proper and such that

$$G(s) = \frac{1}{d(s)} N_G(s), \quad \text{for } d(s) = (s-a_1)\cdots(s-a_r) \text{ with } a_j \neq a_i$$

and polynomial matrix $N_G(s)$, then

$$G(s) = G(\infty) + \sum_{i=1}^{r} \frac{1}{s-a_i} G_i \quad \text{for } G_i := \lim_{s \to a_i} (s-a_i)G(s).$$

If rank $G_i = n_i$, then $\exists B_i \in \mathbb{R}^{n_i \times m}$, $C_i \in \mathbb{R}^{p \times n_i}$ having rank n_i and such that

$$G_i = C_i B_i$$

(rank decomposition).

Gilbert's realization (contd)

Theorem

The realization

$$G(s) = \left[\begin{array}{ccc|c} a_1 I_{n_1} & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & a_r I_{n_r} & B_r \\ \hline C_1 & \cdots & C_r & G(\infty) \end{array}\right]$$

is minimal (its dimension is $\sum_{i=1}^{r} n_i$).

Proof (outline): If $\lambda = a_i$ is uncontrollable mode, then $\exists \eta \neq 0$ such that

$$\begin{bmatrix} \eta_1' & \cdots & \eta_r' \end{bmatrix} \begin{bmatrix} (a_1-\lambda)I_{n_1} & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & (a_r-\lambda)I_{n_r} & B_r \end{bmatrix} = \begin{bmatrix} (a_1-\lambda)\eta_1' & \cdots & (a_r-\lambda)\eta_r' & \sum_{i=1}^{r} \eta_i' B_i \end{bmatrix} = 0$$

So $\eta_j = 0$ for all $j \neq i$ and then $\eta_i' B_i = 0 \implies \eta_i = 0$ (contradiction). □

Outline

Reminder: doubly coprime factorization over RH_∞

Given a real-rational proper $G(s)$, there are right coprime $N, M \in RH_\infty$ and left coprime $\tilde{N}, \tilde{M} \in RH_\infty$ such that

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

and their Bézout factors verifying

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The question is

- how to construct these functions ?

State-space way

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with (A, B) stabilizable and (C, A) detectable. Let K and L be any matrices such that $A + BK$ and $A + LC$ are Hurwitz. Then

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & B + LD & -L \\ \hline -K & I & 0 \\ -C & -D & I \end{array}\right]$$

and

$$\begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right].$$

State-space way (contd)

Remember,

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & \bar{B}\bar{D}^{-1} \\ \hline -\bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{array}\right].$$

Then

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix}^{-1} = \left[\begin{array}{c|cc} A + LC & B + LD & -L \\ \hline -K & I & 0 \\ -C & -D & I \end{array}\right]^{-1} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right]$$
$$= \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix}$$

because

$$\begin{bmatrix} I & 0 \\ -D & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ D & I \end{bmatrix}, \quad \begin{bmatrix} B + LD & -L \end{bmatrix} \begin{bmatrix} I & 0 \\ D & I \end{bmatrix} \begin{bmatrix} K \\ C \end{bmatrix} = BK - LC.$$

The fact that $N(s)M^{-1}(s) = G(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ is also easy to show ...

Outline

Poles of the realization

Eigenvalues of A are called poles of the realization (A, B, C, D). Because

$$G(s) = D + C(sI - A)^{-1}B = D + \frac{1}{\det(sI - A)} C \operatorname{adj}(sI - A)B$$

poles of $G(s)$ are also poles of its realization. Then

Theorem

The McMillan degree of $G(s)$ is equal to the order of its minimal realization (A, B, C, D) and the set of poles of $G(s)$ coincides with $\operatorname{spec}(A)$.

Proof (outline).

It follows by the Kalman canonical decomposition that hidden modes don't affect transfer functions. The proof that every pole of a minimal realization is a pole of $G(s)$ is too technical ... □

Reminder: pole directions

Remember the Smith–McMillan form:

$$U(s)G(s)V(s) = \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\alpha_i(s)$ divides $\alpha_{i+1}(s)$, $\beta_{i+1}(s)$ divides $\beta_i(s)$. Then

$$\mathrm{pdir_i}(G, p_i) = \left(\mathrm{Im}\, V(p_i) \begin{bmatrix} e_{\mu_i+1} & \cdots & e_m \end{bmatrix}\right)^{\perp} = \ker \begin{bmatrix} e'_{\mu_i+1} \\ \vdots \\ e'_m \end{bmatrix} [V(p_i)]'$$

$$\mathrm{pdir_o}(G, p_i) = \ker \begin{bmatrix} \tilde{e}'_{\mu_i+1} \\ \vdots \\ \tilde{e}'_p \end{bmatrix} U(p_i) = \left(\mathrm{Im} [U(p_i)]' \begin{bmatrix} \tilde{e}_{\mu_i+1} & \cdots & \tilde{e}_p \end{bmatrix}\right)^{\perp}$$

When Smith–McMillan meets Jordan

Example:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -s & 1 \\ -s & 1 & 0 \end{bmatrix}}_{U(s)} \underbrace{\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}^{-1}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & s & s^2 \end{bmatrix}}_{V(s)} = \begin{bmatrix} 1/s^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \ker(0I - A)',$$

$$\ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \ker(0I - A).$$

Pole directions via state-space realizations

It can be shown that

- the geometric multiplicity of a pole of $\Phi(s) := (sI - A)^{-1}$ at p_i equals the geometric multiplicity of an eigenvalue of A at p_i

and then $\mathrm{pdir_i}(\Phi, p_i) = \ker[(p_i I - A)]'$ and $\mathrm{pdir_o}(\Phi, p_i) = \ker(p_i I - A)$.

Motivated by that, for $G(s) = D + C(sI - A)^{-1}B$

$$\mathrm{pdir_i}(G, p_i) = B' \ker[(p_i I - A)]' \quad \text{and} \quad \mathrm{pdir_o}(G, p_i) = C \ker(p_i I - A).$$

In Gilbert's realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|c} a_1 I_{n_1} & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & a_r I_{n_r} & B_r \\ \hline C_1 & \cdots & C_r & D \end{array}\right],$$

$\mathrm{pdir_i}(G, a_i) = \mathrm{Im}\, B'_i$ and $\mathrm{pdir_o}(G, a_i) = \mathrm{Im}\, C_i$.

Rosenbrock system matrix

The polynomial matrix

$$R_G(s) := \begin{bmatrix} A - sI_n & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - s \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

is called the Rosenbrock system matrix of G given in terms of (A, B, C, D). Because

$$R_G(s) = \begin{bmatrix} A - sI & 0 \\ C & G(s) \end{bmatrix} \begin{bmatrix} I & -(sI - A)^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -C(sI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ 0 & G(s) \end{bmatrix},$$

we have that

$$\mathrm{rank}(R_G(s_0)) = n + \mathrm{rank}(G(s_0)), \quad \forall s_0 \notin \mathrm{spec}(A).$$

and then $\mathrm{nrank}(R_G(s)) = n + \mathrm{nrank}(G(s))$.

Invariant zeros of the realization

Every $z_i \in \mathbb{C}$ at which

$$\operatorname{rank}(R_G(z_i)) < \operatorname{nrank}(R_G(s))$$

is called an invariant zero of the realization (A, B, C, D). Because

$$\begin{bmatrix} TAT^{-1} - sI & TB \\ CT^{-1} & D \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I_m \end{bmatrix},$$

they are invariant under similarity. Finding them is a generalized eigenvalue problem (e.g. `eig([A,B;C,D],[eye(n,n+m);zeros(m,n+m)])` if $p = m$).

Theorem
Invariant zeros of (A, B, C, D) comprise all its hidden modes, as well as the transmission zeros of $G(s) = D + C(sI - A)^{-1}B$.

Proof (observations).
Straightforward if invariant zeros are not in $\operatorname{spec}(A)$, nasty otherwise. □

Zero directions

Remember, zero directions for transfer functions (if zeros are not poles):

$$\operatorname{zdir}_{\mathrm{i}}(G, z_i) = \ker G(z_i) \subset \mathbb{C}^m \quad \text{and} \quad \operatorname{zdir}_{\mathrm{o}}(G, z_i) = \ker[G(z_i)]' \subset \mathbb{C}^p.$$

Then

$$0 = \begin{bmatrix} A - z_iI & 0 \\ C & G(z_i) \end{bmatrix} \begin{bmatrix} 0 \\ u_i \end{bmatrix} = \begin{bmatrix} A - z_iI & B \\ C & D \end{bmatrix} \begin{bmatrix} I & (z_iI - A)^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ u_i \end{bmatrix}$$
$$= \begin{bmatrix} A - z_iI & B \\ C & D \end{bmatrix} \begin{bmatrix} (z_iI - A)^{-1}Bu_i \\ u_i \end{bmatrix}.$$

so that $\operatorname{zdir}_{\mathrm{i}}(G, z_i) \in \begin{bmatrix} 0 & I_m \end{bmatrix} \ker R_G(z_i)$. The other direction is also true:

$$0 = \begin{bmatrix} A - z_iI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} = \begin{bmatrix} A - z_iI & 0 \\ C & G(z_i) \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ u_i \end{bmatrix},$$

where $\tilde{x}_i := x_i - (z_iI - A)^{-1}Bu_i = 0$ then, because $\det(A - z_iI) \neq 0$.

Zero directions (contd)

Thus, we have that

$$\operatorname{zdir}_{\mathrm{i}}(G, z_i) = \begin{bmatrix} 0 & I_m \end{bmatrix} \ker R_G(z_i)$$

and, by similar arguments, that

$$\operatorname{zdir}_{\mathrm{o}}(G, z_i) = \begin{bmatrix} 0 & I_p \end{bmatrix} \ker[R_G(z_i)]'.$$

These relations should also hold true if $z_i \in \operatorname{spec}(A)$, perhaps. At least if

$$\begin{bmatrix} A - z_iI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} = 0,$$

then $u_i \neq 0$ (by observability) and $(A - z_iI)x_i + Bu_i = 0$ implies that

- $\operatorname{zdir}_{\mathrm{i}}(G, z_i) \perp \operatorname{pdir}_{\mathrm{i}}(G, z_i)$ whenever $z_i \in \operatorname{spec}(A)$

($\operatorname{zdir}_{\mathrm{o}}(G, z_i) \perp \operatorname{pdir}_{\mathrm{o}}(G, z_i)$ then too), which is a circumstantial evidence.

Example 1

Let

$$G(s) = \frac{1}{s} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{s} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Its minimal (Gilbert's) realization is

$$G(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right].$$

It has one pole at the origin and because $\ker(s - 0)|_{s=0} = \mathbb{C}$, we have that

$$\operatorname{pdir}_{\mathrm{i}}(G, 0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbb{C} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\operatorname{pdir}_{\mathrm{o}}(G, 0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbb{C} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

Example 1 (contd)

The Rosenbrock system matrix

$$R_G(s) = \begin{bmatrix} -s & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is such that $\operatorname{rank}(R_G(s)) = 2$ for all $s \in \mathbb{C}$. Thus, the system has no zeros. All these results agree with those derived in Chapter 3.

Example 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{s}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{s}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Its minimal (Gilbert's) realization is

$$G(s) = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

It has one pole at the origin and because $\ker(s-0)|_{s=0} = \mathbb{C}$, we have that

$$\operatorname{pdir}_{\mathrm{i}}(G, 0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbb{C} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

and

$$\operatorname{pdir}_{\mathrm{o}}(G, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathbb{C} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

Example 2 (contd)

The Rosenbrock system matrix

$$R_G(s) = \begin{bmatrix} -s & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has full normal rank and $\det(R_G(s)) = -s$. Thus, the system has a zero at the origin too. Because

$$\ker R_G(0) = \operatorname{span}\left(\left[\begin{array}{c} -1 \\ \hdashline 1 \\ 0 \end{array}\right]\right) \quad \text{and} \quad \ker[R_G(0)]' = \operatorname{span}\left(\left[\begin{array}{c} -1 \\ \hdashline 0 \\ 1 \end{array}\right]\right),$$

we have that

$$\operatorname{zdir}_{\mathrm{i}}(G, 0) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \operatorname{zdir}_{\mathrm{o}}(G, 0) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

All these results agree with those derived in Chapter 3.

Invariant zeros: filtering inputs

In the SISO case, if $G(s)$ has a zero at z_i, then

$$u(t) = \mathrm{e}^{z_i t}\mathbb{1}(t)$$

is filtered out by G ($Y(s) = \frac{1}{s-z_i}G(s)$ is well defined at $s = z_i$, so that $y(t)$ does not contain a component with $\mathrm{e}^{z_i t}$).

In the MIMO case, let

$$u(t) = u_i \mathrm{e}^{z_i t}\mathbb{1}(t)$$

for $u_i \neq 0$ such that which the Sylvester equation $-x_i z_i + A x_i + B u_i = 0$ is solvable in $x_i \in \mathbb{C}^n$. This happens

- for all $u_i \in \mathbb{C}^m$ if $z_i \notin \operatorname{spec}(A)$
- for all $u_i \perp \operatorname{pdir}_{\mathrm{i}}(G, z_i) \subset \mathbb{C}^m$ if $z_i \in \operatorname{spec}(A)$

Invariant zeros: filtering inputs (contd)

Then (remember the partial fraction expansion formula from Slide 15)

$$Y(s)=G(s)u_i\frac{1}{s-z_i}=\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]\left[\begin{array}{c|c}z_i&1\\\hline u_i&0\end{array}\right]=\left[\begin{array}{cc|c}A&Bu_i&0\\0&z_i&1\\\hline C&Du_i&0\end{array}\right]$$

$$=\left[\begin{array}{cc|c}A&Bu_i&-x_i\\0&z_i&1\\\hline C&Cx_i+Du_i&0\end{array}\right]=-C(sI-A)^{-1}x_i+(Cx_i+Du_i)\frac{1}{s-z_i}$$

Hence,

$$y(t)=\underbrace{-C\mathrm{e}^{At}x_i\mathbb{1}(t)}_{\text{transients}}+\underbrace{(Cx_i+Du_i)\,\mathrm{e}^{z_it}\mathbb{1}(t)}_{\text{steady-state effect of }u(t)}$$

If

$$Cx_i+Du_i=0\iff\begin{bmatrix}A-z_iI&B\\C&D\end{bmatrix}\begin{bmatrix}x_i\\u_i\end{bmatrix}=0\iff u_i\in\mathrm{zdir}_\mathrm{i}(G,z_i),$$

then the response to $u(t)$ includes only transients. In addition, if

- $x(0)=x_i \implies y(t)\equiv 0$, i.e. no response to $u(t)=u_i\mathrm{e}^{z_it}\mathbb{1}(t)$ at all.

Realization poles and coprime factors

Remember, $G=\tilde M^{-1}\tilde N=NM^{-1}$ with

$$\tilde M(s)=\left[\begin{array}{c|c}A+LC-sI&L\\\hline C&I\end{array}\right]\quad\text{and}\quad M(s)=\left[\begin{array}{c|c}A+BK-sI&B\\\hline K&I\end{array}\right].$$

Now,

$$R_{\tilde M}(s)=\begin{bmatrix}A+LC-sI&L\\C&I\end{bmatrix}=\begin{bmatrix}A-sI&L\\0&I\end{bmatrix}\begin{bmatrix}I&0\\C&I\end{bmatrix},$$

$$R_M(s)=\begin{bmatrix}A+BK-sI&B\\K&I\end{bmatrix}=\begin{bmatrix}I&B\\0&I\end{bmatrix}\begin{bmatrix}A-sI&0\\K&I\end{bmatrix}$$

Hence

- $z_i\in\mathbb{C}$ is an invariant zero of $\tilde M$ iff it is a realization pole of G, with $\mathrm{zdir}_\mathrm{i}(\tilde M,z_i)=\mathrm{pdir}_\mathrm{o}(G,z_i)$;
- $z_i\in\mathbb{C}$ is an invariant zero of M iff it is a realization pole of G, with $\mathrm{zdir}_\mathrm{o}(M,z_i)=\mathrm{pdir}_\mathrm{i}(G,z_i)$.

Invariant zeros and coprime factors

Again, because $G=\tilde M^{-1}\tilde N=NM^{-1}$ with

$$\tilde N(s)=\left[\begin{array}{c|c}A+LC&B+LD\\\hline C&D\end{array}\right]\quad\text{and}\quad N(s)=\left[\begin{array}{c|c}A+BK&B\\\hline C+DK&D\end{array}\right],$$

we have that

$$R_G(s)=\begin{bmatrix}I&-L\\0&I\end{bmatrix}\begin{bmatrix}A+LC-sI&B+LD\\C&D\end{bmatrix}=\begin{bmatrix}I&-L\\0&I\end{bmatrix}R_{\tilde N}(s)$$

$$=\begin{bmatrix}A+BK-sI&B\\C+DK&D\end{bmatrix}\begin{bmatrix}I&0\\-K&I\end{bmatrix}=R_N(s)\begin{bmatrix}I&0\\-K&I\end{bmatrix}$$

Hence,

- $z_i\in\mathbb{C}$ is an invariant zero of $\tilde N$ iff it is an invariant zero of G, with $\mathrm{zdir}_\mathrm{i}(\tilde N,z_i)=\mathrm{zdir}_\mathrm{i}(G,z_i)$;
- $z_i\in\mathbb{C}$ is an invariant zero of N iff it is an invariant zero of G, with $\mathrm{zdir}_\mathrm{o}(N,z_i)=\mathrm{zdir}_\mathrm{o}(G,z_i)$.

Outline

State-space realizations

System interconnections in terms of state-space realizations

Structural properties

Kalman canonical decomposition and minimality

Coprime factorization via state-space realizations

Poles / zeros / directions via state-space realizations

System norms via state-space realizations

Model reduction by balanced truncation

Computing H_2 norm

Proposition

If A is Hurwitz and $D = 0$, then

$$\|G\|_2^2 = \operatorname{tr}(B'QB) = \operatorname{tr}(CPC'),$$

where Q and P are the observability and controllability Gramians of (C, A) and (A, B), respectively.

Proof.

The impulse response of G is $g(t) = Ce^{At}B\mathbb{1}(t)$. By Parseval,

$$\|G\|_2^2 = \|g\|_2^2 = \int_{\mathbb{R}_+} \operatorname{tr}\bigl(g(t)'g(t)\bigr)\,dt = \int_{\mathbb{R}_+} \operatorname{tr}\bigl(B'e^{A't}C'Ce^{At}B\bigr)\,dt$$
$$= \operatorname{tr}\left(B' \int_{\mathbb{R}_+} e^{A't}C'Ce^{At}\,dt\, B\right) = \operatorname{tr}(B'QB),$$

The other formula is derived similarly, because $\operatorname{tr}(M'M) = \operatorname{tr}(MM')$. □

Computing H_∞ norm

Proposition

If A is Hurwitz, then $\|G\|_\infty < \gamma$ for a given $\gamma > 0$ iff $\bar{\sigma}(D) < \gamma$ and

$$H_G := \begin{bmatrix} A & 0 \\ C'C & -A' \end{bmatrix} - \begin{bmatrix} B \\ C'D \end{bmatrix} (\gamma^2 I - D'D)^{-1} \begin{bmatrix} -D'C & B' \end{bmatrix}$$

has no pure imaginary eigenvalues.

Proof (outline).

Because $G \in RH_\infty$,

$$\|G\|_\infty < \gamma \iff \gamma^2 I - [G(j\omega)]'G(j\omega) > 0, \quad \forall \omega \in \mathbb{R} \cup \{\pm\infty\}.$$

As $G(j\infty) = D$, $\bar{\sigma}(D) < \gamma$ follows (and assumed hereafter). Thus,

$$\|G\|_\infty < \gamma \iff \Phi(s) := \gamma^2 I - G^{\sim}(s)G(s) \text{ has no pure imaginary zeros}$$

How to verify that?

Computing H_∞ norm (contd)

Proof (outline, contd).

Now,

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \implies G^{\sim}(s) = \left[\begin{array}{c|c} -A' & C' \\ \hline -B' & D' \end{array}\right].$$

Hence,

$$\Phi(s) = \gamma^2 I - \left[\begin{array}{c|c} -A' & C' \\ \hline -B' & D' \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ C'C & -A' & C'D \\ \hline -D'C & B' & \gamma^2 I - D'D \end{array}\right].$$

As $\operatorname{spec}(A) \cap j\mathbb{R} = \varnothing$, imaginary zeros of $\Phi(s)$ are its invariant zeros. Then

$$R_\Phi(j\omega) = \begin{bmatrix} A - j\omega I & 0 & B \\ C'C & -A' - j\omega I & C'D \\ -D'C & B' & \gamma^2 I - D'D \end{bmatrix}$$

and H_G is the Schur complement of $\gamma^2 I - D'D$ in it. □

KYP (Kalman–Yakubovich–Popov) lemma

Consider $p \times m$ system $G(s) = D + C(sI - A)^{-1}B$, with $\operatorname{spec}(A) \cap j\mathbb{R} = \varnothing$, and let $M_{KYP} = M_{KYP}' \in \mathbb{R}^{(m+p)\times(m+p)}$. The frequency-dependent inequality

$$\begin{bmatrix} [G(j\omega)]' & I_m \end{bmatrix} M_{KYP} \begin{bmatrix} G(j\omega) \\ I_m \end{bmatrix} < 0, \quad \forall \omega$$

holds iff there is $X = X' \in \mathbb{R}^{n\times n}$ verifying the linear matrix inequality (LMI)

$$\begin{bmatrix} C' & 0 \\ D' & I_m \end{bmatrix} M_{KYP} \begin{bmatrix} C & D \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} I_n & A' \\ 0 & B' \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix} < 0.$$

KYP implies that

- infinite set of inequalities $\iff$ finite number of LMIs (solvable)

Many important special cases, e.g.

$$M_{KYP} = \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right] \implies \text{calculate the } L_\infty(j\mathbb{R})\text{-norm of } G$$

Outline

State-space realizations

System interconnections in terms of state-space realizations

Structural properties

Kalman canonical decomposition and minimality

Coprime factorization via state-space realizations

Poles / zeros / directions via state-space realizations

System norms via state-space realizations

Model reduction by balanced truncation

Model reduction problem

Complexity vs. accuracy is one of the key tradeoffs in (control) engineering. "Complexity" is understood as "order" in the LTI case. Then:

- given an n-order $p \times m$ LTI G and $n_r < n$, find an n_r-order $p \times m$ LTI G_r, which is "close" to G,

say in the sense that $\|G - G_r\|_\infty$ is "small."

In what follows, an approach based on

- structural properties of state-space realizations

is considered. It is both practical (for relatively small n's) and enlightening. We consider model reduction for stable systems only.

Classical control recipes

Thinking in terms of pole dominance, i.e.

- all poles are equal, but some poles are more equal than others.

Example 1:

$$G(s) = \frac{1}{(s+1)(\tau s+1)} \quad \text{for } \tau \in (0,1) \quad \Longrightarrow \quad G_r(s) = \frac{1}{s+1},$$

justifiable if $\tau \ll 1$ (far right), $\|G - G_r\|_\infty = \tau/(1+\tau)$.

Example 2:

$$G(s) = \frac{2s+1}{(s+1)((2-\epsilon)s+1)} \quad \text{for } \epsilon \in (0,1) \quad \Longrightarrow \quad G_r(s) = \frac{1}{s+1},$$

justifiable if $\epsilon \ll 1$ (almost cancels the zero), $\|G - G_r\|_\infty = \epsilon/(3-\epsilon)$.

MIMO extensions

Dominant poles ideas are

- $\ddot\frown$ overly hand-waving
- $\ddot\frown$ messy if directional properties have to be accounted for

Alternative thinking:

- hidden modes can be detected and eliminated w/o consequences
- what about "almost hidden" modes?
  - detect?
  - costs of eliminating?

Controllability and observability Gramians are $P = P' \geq 0$ and $Q = Q' \geq 0$ satisfying

$$AP + PA' + BB' = 0 \quad \text{and} \quad A'Q + QA + C'C = 0.$$

$P > 0$ iff (A, B) is controllable and $Q > 0$ iff (C, A) is observable.

First try

We (maybe) remember that if (A, B) is uncontrollable, there is T_c such that

$$\left[\begin{array}{c|c} T_c A T_c^{-1} & T_c B \\ \hline C T_c^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_c & \times & B_c \\ 0 & A_{\bar{c}} & 0 \\ \hline C_c & C_{\bar{c}} & 0 \end{array}\right] = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right].$$

and this T_c can be constructed via the Gramian, $T_c P T_c' = \left[\begin{smallmatrix} P_c & 0 \\ 0 & 0 \end{smallmatrix}\right]$. So if

$$TPT' = \begin{bmatrix} \Sigma_{P_1} & 0 \\ 0 & \Sigma_{P_2} \end{bmatrix} \quad \text{with } \|\Sigma_{P_1}\| \gg \|\Sigma_{P_2}\|,$$

is

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right] \approx \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right]$$

if A_{21} and B_2 are “small”?

First try: example 3

Let

$$G(s) = \frac{18}{5s^2 + 12s + 9} = \left[\begin{array}{cc|c} -2 & -1/\alpha & 1 \\ \alpha & -0.4 & \alpha \\ \hline -1 & 1/\alpha & 0 \end{array}\right],$$

which is true for all $\alpha \neq 0$ and its controllability Gramian,

$$P = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25\alpha^2 \end{bmatrix},$$

is of the requires form if $\alpha \ll 1$. Yet the choice

$$G_r(s) = \left[\begin{array}{c|c} -2 & 1 \\ \hline -1 & 0 \end{array}\right] = -\frac{1}{s+2}$$

is not what we need, as

$$\|G - G_r\|_\infty = 2.5 > \|G - 0\|_\infty = \|G\|_\infty = 2.$$

First try: example 3 (contd)

Observability Gramian

$$Q = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25/\alpha^2 \end{bmatrix} \quad \text{compare with } P = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25\alpha^2 \end{bmatrix}$$

indicates that

- the second state becomes in a sense “over-observable” if $\alpha \ll 1$.

Moral:

- P (or Q) alone is not an accurate indication of the relative importance of the system modes in the input / output behavior.

Remedy:

- balance “degrees” of controllability and observability of each mode.

Similarity transformations and Gramians

If $(\tilde{A}, \tilde{B}, \tilde{C}, 0) = (TAT^{-1}, TB, CT^{-1}, 0)$, then

$$\tilde{P} = TPT' \quad \text{and} \quad \tilde{Q} = T^{-\prime} Q T^{-1}.$$

Hence,

- eigenvalues of P and Q are not preserved under similarity.

But

$$\tilde{P}\tilde{Q} = TPQT^{-1}$$

is similar to PQ, so its eigenvalues are invariant under similarity. Moreover,

$$\operatorname{spec}(PQ) = \operatorname{spec}(Q^{1/2} PQQ^{-1/2}) = \operatorname{spec}(Q^{1/2} PQ^{1/2}),$$

implying

- eigenvalues of PQ are real and nonnegative $\quad Q^{1/2}PQ^{1/2}$ is symmetric
- PQ is diagonalizable $\quad UQ^{1/2}PQ^{1/2}U' = (UQ^{1/2})PQ(UQ^{1/2})^{-1}$

Balanced realization

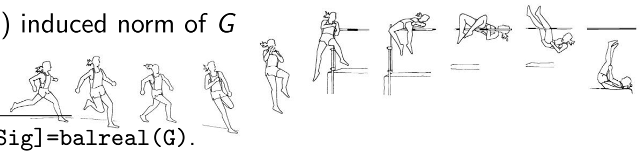
Theorem
If (A, B, C, D) is a minimal realization of an n-dimensional stable G, then there is T such that $(\tilde{A}, \tilde{B}, \tilde{C}, D) := (TAT^{-1}, TB, CT^{-1}, D)$ has[1]

$$\tilde{P} = \tilde{Q} = \Sigma := \begin{bmatrix} \sigma_1 I_{n_1} & & \\ & \ddots & \\ & & \sigma_l I_{n_l} \end{bmatrix},$$

where $\sigma_1 > \cdots > \sigma_l > 0$ and $n_i \in \mathbb{N}$ with $\sum_i n_i = n$.

Some facts about σ_i:

- known as Hankel singular values of G
- square roots of the singular values of PQ
- $\|G\|_{\mathrm{H}} := \sigma_1 = \sqrt{\rho(PQ)}$ is known as the Hankel norm of G
  - $L_2(\mathbb{R}_-) \to L_2(\mathbb{R}_+)$ induced norm of G



[1] Matlab command: `[Gb,Sig]=balreal(G)`.

Second try: balanced truncation

Let G be stable and (A, D, C, D) be its balanced realization. Partition

$$P = Q = \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where $\Sigma_1 = \operatorname{diag}\{\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}\}$ and $\Sigma_2 = \operatorname{diag}\{\sigma_{r+1} I_{n_{r+1}}, \ldots, \sigma_l I_{n_l}\}$ for $\sigma_1 > \cdots > \sigma_r > \sigma_{r+1} > \cdots > \sigma_l$. The correspondent state partition is

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

The system G_r with the transfer function

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is called the balanced truncation of G.

Balanced truncation properties

If G_r is the balanced truncation of of G, then

- $P_1 = Q_1 = \Sigma_1 > 0$ are Gramians of (A_{11}, B_1, C_1, D)
- $G_r \in RH_\infty$
- $\|G - G_r\|_\infty \le 2(\sigma_{r+1} + \cdots + \sigma_l)$
- if $r = l - 1$, then the bound above is achieved, i.e. $\|G - G_{l-1}\|_\infty = 2\sigma_l$

Balanced truncation: example 3

Let

$$G(s) = \frac{18}{5s^2 + 12s + 9} = \left[\begin{array}{cc|c} -2 & -1/\alpha & 1 \\ \alpha & -0.4 & \alpha \\ \hline -1 & 1/\alpha & 0 \end{array}\right],$$

with

$$P = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25\alpha^2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25/\alpha^2 \end{bmatrix}.$$

Its HSVs are $\sigma_1 = 1.25$ and $\sigma_2 = 0.25$ and balanced realization (for $\alpha = 1$)

$$G(s) = \left[\begin{array}{cc|c} -0.4 & 1 & 1 \\ -1 & -2 & 1 \\ \hline 1 & -1 & 0 \end{array}\right].$$

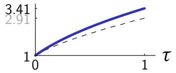
Balanced truncation for $r = 1$:

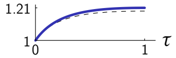
$$G_1(s) = \left[\begin{array}{c|c} -0.4 & 1 \\ \hline 1 & 0 \end{array}\right] = \frac{5}{5s+2} \quad \Longrightarrow \quad \|G - G_1\|_\infty = 2 \times 0.25 = 0.5,$$

which is smaller than $\|G\|_\infty = 2$.

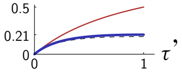
Balanced truncation: example 1 (contd)

If
$$G(s) = \frac{1}{(s+1)(\tau s+1)},$$
then balanced truncation to $r=1$ results in

$$G_1(s) = \frac{k_1}{\tau_1 s + 1}$$ with $\tau_1 =$ 

and $k_1 =$ 

which is different from keeping the rightmost pole at -1. Also,

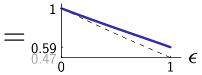
$\|G - G_1\|_\infty =$ ,

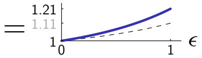
where

- red line corresponds to the naïve modal truncation
- dashed lines correspond to the (brute force) optimal H_∞ reduction

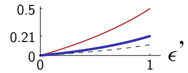
Balanced truncation: example 2 (contd)

If
$$G(s) = \frac{2s+1}{(s+1)((2-\epsilon)s+1)},$$
then balanced truncation to $r=1$ results in

$$G_1(s) = \frac{k_2}{\tau_2 s + 1}$$ with $\tau_2 =$ 

and $k_2 =$ 

which is different from keeping the pole at -1. Also,

$\|G - G_1\|_\infty =$ ,

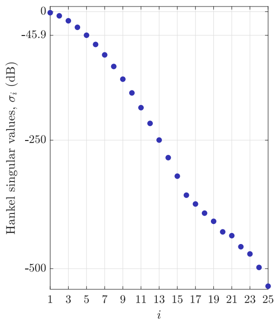
where

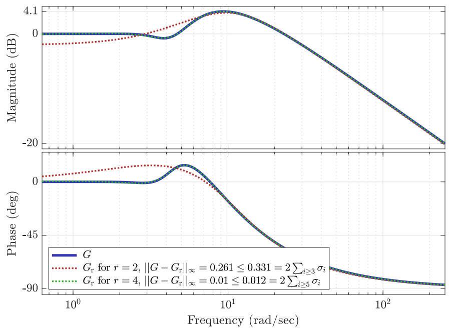
- red line corresponds to the naïve modal truncation
- dashed lines correspond to the (brute force) optimal H_∞ reduction

Balanced truncation: example 4

Let
$$G(s) = 1 - \left(\frac{s+1}{s+2}\right)^{25}.$$

It has





Then
$$G_4(s) = \frac{24.986(s+3.196)(s^2+3.165s+19.48)}{(s^2+5.629s+24.58)(s^2+14.69s+63.86)}$$
is quite accurate (and its poles are not related to those of $G(s)$).

Balanced truncation: example 5 (need for $\sigma_r > \sigma_{r+1}$)

Let
$$G(s) = \frac{(s-1)^2}{(s+1)^2}.$$

Its balance realization

$$G(s) = \left[\begin{array}{cc|c} -1+\cos 2\theta & 1-\sin 2\theta & 2\sin\theta \\ -1-\sin 2\theta & -1-\cos 2\theta & 2\cos\theta \\ \hline -2\sin\theta & -2\cos\theta & 1 \end{array}\right].$$

for every θ and $P = Q = I_2$. But

$$A_{11} = -1 + \cos 2\theta$$

is not Hurwitz if $\theta = \pi k$ for $k \in \mathbb{Z}$.