Linear Control Systems (036012)
chapter 4

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Preliminary: linear algebra facts

- $(\ker M)^\perp = \operatorname{Im} M'$ and $(\operatorname{Im} M)^\perp = \ker M'$
- $\operatorname{Im} M_1 = \operatorname{Im} M_2 \iff \ker M_1' = \ker M_2'$
- if $M = M' \geq 0$, then $Mx = 0 \iff x'Mx = 0$
- if $M = M' \geq 0$, then left & right singular vectors coincide, $M = U\Sigma U'$
- if $f(x)$ is analytic, $M, T \in \mathbb{F}^{m\times m}$ with $\det T \neq 0$, then

$$f(TMT^{-1}) = Tf(M)T^{-1}$$

(in particular, $(TMT^{-1})^i = TM^iT^{-1}$ and $\mathrm{e}^{TMT^{-1}} = T\mathrm{e}^M T^{-1}$)

Preliminary: Cayley–Hamilton

In essence, each square matrix satisfies its own characteristic equation:

$$\chi_A(A) := A^n + \chi_{n-1}A^{n-1} + \cdots + \chi_1 A + \chi_0 I_n = 0.$$

Important consequence:

– A^k for all $k \geq n$ is a linear combination of A^i, $i = 0, \ldots, n-1$, like

$$\begin{aligned}
A^n &= -\chi_{n-1}A^{n-1} - \cdots - \chi_1 A - \chi_0 I_n \\
A^{n+1} &= -\chi_{n-1}A^n - \cdots - \chi_1 A^2 - \chi_0 A \\
&= \chi_{n-1}(\chi_{n-1}A^{n-1} + \cdots + \chi_1 A + \chi_0 I_n) - \cdots - \chi_1 A^2 - \chi_0 A \\
&= (\chi_{n-1}^2 - \chi_{n-2})A^{n-1} + \cdots + (\chi_{n-1}\chi_1 - \chi_0)A + \chi_{n-1}\chi_0 I_n \\
&\vdots
\end{aligned}$$

Preliminary: matrix Sylvester and Lyapunov equations

Given $A_1 \in \mathbb{F}^{p\times p}$, $A_2 \in \mathbb{F}^{m\times m}$, $Q \in \mathbb{F}^{p\times m}$, solve in $X \in \mathbb{F}^{p\times m}$

$$A_1 X + X A_2 + Q = 0.$$

If $\operatorname{spec}(A_1) \cap \operatorname{spec}(-A_2) = \varnothing$, X exists for all Q and is unique. Otherwise, there might be either no or infinitely many solutions, depending on Q.

Its special case for $A_2 = A_1'$ and $Q = Q'$ is known as the matrix Lyapunov equation,

$$AX + XA' + Q = 0.$$

If A is Hurwitz, then

$$X = \int_{\mathbb{R}^+} e^{At} Q e^{A't} \, dt.$$

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Outline

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State-space realizations

System interconnections in terms of state-space realizations

Structural properties

Kalman canonical decomposition and minimality

Coprime factorization via state-space realizations

Poles / zeros / directions via state-space realizations

System norms via state-space realizations

Model reduction by balanced truncation

State-space realizations

Let $G : \mathfrak{D}_G \subset \mathbb{R}^n \times L_2^m(\mathbb{R}_+) \to L_2^p(\mathbb{R}_+)$ be LTI, finite dimensional, and have a proper transfer function. There are $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ such that $u \mapsto y = Gu$ reads

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

The quadruple (A, B, C, D) is called a state-space realization of G.

If $x_0 = 0$, we have that $G : \mathfrak{D}_G \subset L_2^m(\mathbb{R}_+) \to L_2^p(\mathbb{R}_+)$ and we write

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Solution:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds.$$

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Similarity transformations

Let

$$\tilde{x}(t) := Tx(t)$$

for some nonsingular $T \in \mathbb{R}^{n \times n}$. We have:

$$\dot{\tilde{x}}(t) = T\dot{x}(t) = T\big(Ax(t) + Bu(t)\big) = TAT^{-1}\tilde{x}(t) + TBu(t)$$

and also

$$y(t) = Cx(t) + Du(t) = CT^{-1}\tilde{x}(t) + Du(t).$$

Hence,

$$(TAT^{-1}, TB, CT^{-1}, D)$$

is also a realization of the same G. This realization is said to be similar to (A, B, C, D).

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Impulse response and transfer function

The impulse response of G is

$$g(t) = D\delta(t) + C\mathrm{e}^{At}B.$$

The corresponding transfer function

$$G(s) = D + C(sI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

with $G(\infty) = D$, so that $G(s)$ is

- strictly proper iff $D = 0$ and bi-proper iff $\det(D) \neq 0$.

Readily seen that

$$D\delta(t) + CT^{-1}\mathrm{e}^{TAT^{-1}t}TB = D\delta(t) + C\mathrm{e}^{At}B$$

and

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

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Outline

Interconnecting in state space

For transfer functions, algebraic manipulations over complex functions:

- parallel: $G_1(s) + G_2(s)$
- series: $G_2(s)G_1(s)$
- inverse: $G^{-1}(s)$

For state-space realizations:

- can be done via matrix algebra.

Let

$$G_1 : \begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \\ y_1(t) = C_1 x_1(t) + D_1 u_1(t) \end{cases} \quad \text{and} \quad G_2 : \begin{cases} \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t) \\ y_2(t) = C_2 x_2(t) + D_2 u_2(t) \end{cases}$$

An efficient way of interconnecting such systems is to

- unite state vectors
- determine what inputs / outputs to connect

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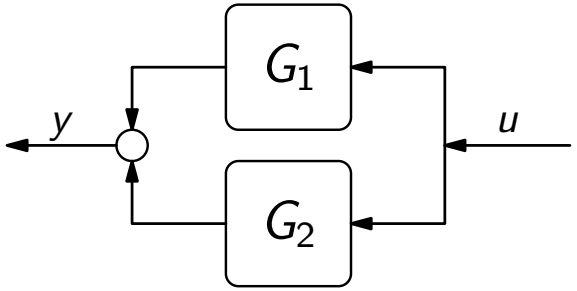
Parallel interconnection



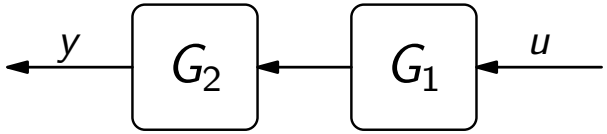
corresponds to

- $u_1 = u_2 = u$
- $y = y_1 + y_2$

Hence,

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + (D_1 + D_2)u(t) \end{cases}$$

Series / cascade interconnection



corresponds to

- $u_1 = u$
- $u_2 = y_1$
- $y = y_2$

Then

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + D_2 D_1 u(t) \end{cases}$$

Inversion

Let

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

be square (i.e. $p = m$) and bi-proper (i.e. $\det(D) \neq 0$). Its inverse is the system mapping $y \mapsto u$. Then

$$u(t) = -D^{-1}Cx(t) + D^{-1}y(t)$$

and

$$\dot{x}(t) = Ax(t) + B\big(-D^{-1}Cx(t) + D^{-1}y(t)\big)$$

Therefore,

$$G^{-1} : \begin{cases} \dot{x}(t) = (A - BD^{-1}C)x(t) + BD^{-1}y(t) \\ u(t) = -D^{-1}Cx(t) + D^{-1}y(t) \end{cases}$$

Summary

If

$$G_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad i \in \{1, 2\}$$

then

- $$G_1(s) + G_2(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ 0 & A_1 & B_1 \\ \hline C_2 & C_1 & D_1 + D_2 \end{array}\right]$$

- $$G_2(s)G_1(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 & D_2D_1 \end{array}\right]$$

- $$G_i^{-1}(s) = \left[\begin{array}{c|c} A_i - B_iD_i^{-1}C_i & B_iD_i^{-1} \\ \hline -D_i^{-1}C_i & D_i^{-1} \end{array}\right] = \left[\begin{array}{c|c} A_i - B_iD_i^{-1}C_i & -B_iD_i^{-1} \\ \hline D_i^{-1}C_i & D_i^{-1} \end{array}\right]$$

Partial fraction expansion

Partition

$$G_2(s)G_1(s) = H_1(s) + H_2(s)$$

with $H_i(s)$ having the "A" matrices of $G_i(s)$. Roth's removal rule:

$$\begin{bmatrix} A_1 & Q \\ 0 & A_2 \end{bmatrix} \text{ and } \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \text{ are similar iff } XA_1 - A_2X = -Q \text{ is solvable}$$

Thus, assuming the Sylvester equation $XA_1 - A_2X = -B_2C_1$ is solvable (it is enough to have $\mathrm{spec}(A_1) \cap \mathrm{spec}(A_2) = \emptyset$), use $T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ to get

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Moral: similarity transformations are a powerful tool.

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Outline

Controllability

We say that $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is controllable if the

– eigenvalues of $A + BK$ can be freely assigned by a choice of $K \in \mathbb{R}^{m\times n}$

(with the restriction that complex eigenvalues are in conjugate pairs).

Controllability criteria

The following statements are equivalent:

1. The pair (A, B) is controllable.
2. The matrix
$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \mathbb{C}^{n \times (n+m)}$$
has full rank $\forall \lambda \in \mathbb{C}$ (the PBH [Popov-Belevich-Hautus] test).
3. The matrix
$$W_c(t) := \int_0^t \mathrm{e}^{As} B B' \mathrm{e}^{A's} \mathrm{d}s \in \mathbb{R}^{n \times n}$$
is positive definite for all $t > 0$ (the Gramian-based test).
4. The controllability matrix
$$M_c := \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times (nm)}$$
has full rank (i.e. $\operatorname{rank}(M_c) = n$).

Technical Lemma 1

Lemma

$\operatorname{Im} W_c(t) = \operatorname{Im} M_c$ *for every* $t > 0$ *or, equivalently,* $\ker W_c(t) = \ker M_c'$.

Proof (outline).

$W_c(t) = [W_c(t)]' \geq 0$ implies that $\eta \in \ker W_c(t)$ iff

$$\eta' W_c(t)\eta = 0 \iff \int_0^t \|\eta' e^{As} B\|^2 ds = 0 \iff \eta' e^{As} B = 0, \quad \forall s \in [0, t]$$

As e^{At} is analytic (every Taylor series converges), the latter implies

$$\eta'(e^{As})^{(i)} B|_{s=0} = 0, \quad \forall i \in \mathbb{Z}_+ \iff \eta' \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = 0$$

By Cayley–Hamilton,

$$\ker \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}' = \ker M_c'.$$

Result follows because η is arbitrary. $\square$

Technical Lemma 2

Lemma

If $\operatorname{rank} W_c(t) = r < n$, *then there is a unitary matrix* U_c *such that*

$$(U_c A U_c', U_c B) = \left(\begin{bmatrix} A_c & \times \\ 0 & A_{\bar{c}} \end{bmatrix}, \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right),$$

where $(A_c, B_c) \in \mathbb{R}^{r\times r} \times \mathbb{R}^{r\times m}$ *is such that* $\int_0^t \mathrm{e}^{A_c s} B_c B_c' \mathrm{e}^{A_c' s} \mathrm{d}s > 0$.

We prove it for Hurwitz A. In this case $\operatorname{rank} W_c(t) = \operatorname{rank} P$, where

$$P := W_c(\infty) \geq 0, \quad \text{verifying Lyapunov eqn. } AP + PA' + BB' = 0$$

aka the controllability Gramian of (A, B). If A is not Hurwitz, $\hat{A} := A - \alpha I$ is Hurwitz for a sufficiently large $\alpha > 0$ and

$$\ker \int_0^t \mathrm{e}^{\hat{A}s} BB' \mathrm{e}^{\hat{A}'s} \mathrm{d}s = \bigcap_{s\in[0,t]} \ker B' \mathrm{e}^{(A-\alpha I)'s} = \bigcap_{s\in[0,t]} \ker B' \mathrm{e}^{A's} = \ker W_c(t)$$

so nothing changes if we prove the result for $\hat{A}$...

Technical Lemma 2 (contd)

Proof (outline).

If rank $P = r < m$, $\exists$ unitary U_c s.t. $U_c P U_c' = \left[\begin{smallmatrix} P_c & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for $r \times r$ $P_c > 0$. Let

$$(U_c A U_c', U_c B) = \left(\begin{bmatrix} A_c & A_{12} \\ A_{21} & A_{\bar{c}} \end{bmatrix}, \begin{bmatrix} B_c \\ B_2 \end{bmatrix} \right)$$

The Lyapunov equation for P reads then

$$\begin{bmatrix} A_c P_c + P_c A_c' + B_c B_c' & P_c A_{21}' + B_c B_2' \\ A_{21} P_c + B_2 B_c' & B_2 B_2' \end{bmatrix} = 0.$$

$$(2,2) \implies B_2 = 0 \overset{(1,2)}{\implies} A_{21} = 0 \implies A_c \text{ is Hurwitz} \overset{(1,1)}{\implies}$$

$$P_c = \int_{\mathbb{R}^+} \mathrm{e}^{A_c s} B_c B_c' \mathrm{e}^{A_c' s} \mathrm{d}s > 0$$

which leads to the last claim by already familiar arguments. $\square$

Equivalence of controllability conditions

$2 \implies 3$: Let rank $\begin{bmatrix} A - sI & B \end{bmatrix} = n$, $\forall s \in \mathbb{C}$, but rank $W_c(t) = r < n$. By TL2 there is a unitary U_c such that

$$U_c \begin{bmatrix} A - sI & B \end{bmatrix} \begin{bmatrix} U_c' & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{cc:c} A_c - sI_r & \times & B_c \\ 0 & A_{\bar{c}} - sI_{n-r} & 0 \end{array}\right],$$

whose rank drops at every $s \in \operatorname{spec}(A_{\bar{c}}) \implies$ contradiction.

$2 \impliedby 3$: Let rank $W_c(t) = n$, but rank $\begin{bmatrix} A - s_0 I & B \end{bmatrix} < n$ for $s_0 \in \mathbb{C}$. In this case $\exists \eta_0 \neq 0$ such that

$$\eta_0' \begin{bmatrix} A - s_0 I & B \end{bmatrix} = 0 \iff (\eta_0' A = s_0 \eta_0') \wedge (\eta_0' B = 0).$$

Hence, $\eta_0' e^{At} B = e^{s_0 t} \eta_0' B = 0$, $\forall t \implies \eta_0' W_c(t) = 0 \implies$ contradiction.

$3 \iff 4$: Follows by TL1.

Equivalence of controllability conditions

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$$U_c \begin{bmatrix} A - sI & B \end{bmatrix} \begin{bmatrix} U_c' & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{cc:c} A_c - sI_r & \times & B_c \\ 0 & A_{\bar{c}} - sI_{n-r} & 0 \end{array}\right],$$

whose rank drops at every $s \in \operatorname{spec}(A_{\bar{c}}) \implies$ contradiction.

$2 \impliedby 3$: Let rank $W_c(t) = n$, but rank $\begin{bmatrix} A - s_0 I & B \end{bmatrix} < n$ for $s_0 \in \mathbb{C}$. In this case $\exists \eta_0 \neq 0$ such that

$$\eta_0' \begin{bmatrix} A - s_0 I & B \end{bmatrix} = 0 \iff (\eta_0' A = s_0 \eta_0') \wedge (\eta_0' B = 0).$$

Hence, $\eta_0' e^{At} B = e^{s_0 t} \eta_0' B = 0$, $\forall t \implies \eta_0' W_c(t) = 0 \implies$ contradiction.

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Equivalence of controllability conditions (contd)

$1 \implies 2$: Let (A, B) be controllable, but rank $\begin{bmatrix} A - s_0 I & B \end{bmatrix} < n$ for $s_0 \in \mathbb{C}$. In this case $\exists \eta_0 \neq 0$ such that

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Hence, $\eta_0'(A + BK) = s_0 \eta_0' \implies s_0 \in \operatorname{spec}(A + BK)$ for all $K \implies (A, B)$ is not controllable $\implies$ contradiction.

$4 \implies 1$: If $m = 1$, then $\det(M_c) \neq 0$ and by Ackermann's formula

$$K = -e_n' M_c^{-1} \chi_{cl}(A)$$

assigns $\operatorname{spec}(A + BK)$ to roots of (an arbitrary) $\chi_{cl}(s) \implies$ controllability. If $m > 1$, then for any $0 \neq \tilde{b} \in \operatorname{Im} B$, $\exists \tilde{K} \in \mathbb{R}^{m \times n}$ such that $(A + B\tilde{K}, \tilde{b})$ is controllable (Heymann, 1968). Hence,

$$K = \tilde{K} - \tilde{u} e_n' \tilde{M}_c^{-1} \chi_{cl}(A + B\tilde{K})$$

does the trick, where $\tilde{u} \in \mathbb{R}^m$ is such that $B\tilde{u} = \tilde{b} \implies$ controllability. $\square$

Equivalence of controllability conditions (contd)

$1 \implies 2$: Let (A, B) be controllable, but rank $\begin{bmatrix} A - s_0 I & B \end{bmatrix} < n$ for $s_0 \in \mathbb{C}$. In this case $\exists \eta_0 \neq 0$ such that

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Controllability and similarity transformations

Let $(\tilde{A}, \tilde{B}, \tilde{C}, D) := (TAT^{-1}, TB, CT^{-1}, D)$. We have that

$$\begin{aligned}\tilde{M}_{\mathrm{c}} &= \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = T \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \\ &= TM_{\mathrm{c}}\end{aligned}$$

and

$$\begin{aligned}\tilde{W}_{\mathrm{c}}(t) &= \int_0^t \mathrm{e}^{\tilde{A}s}\tilde{B}\tilde{B}'\mathrm{e}^{\tilde{A}'s}\mathrm{d}s = \int_0^t T\mathrm{e}^{As}T^{-1}TBB'T'T^{-\prime}\mathrm{e}^{A's}T'\mathrm{d}s \\ &= T\int_0^t \mathrm{e}^{As}BB'\mathrm{e}^{A's}\mathrm{d}sT' \\ &= TW_{\mathrm{c}}(t)T'.\end{aligned}$$

Hence,

- controllability is invariant under similarity transformations.

Uncontrollable modes

If PBH fails at some $\lambda \in \mathbb{C}$, there is $0 \neq \eta \in \mathbb{C}^n$ such that

$$\eta' \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0 \iff (\eta' A = \lambda \eta') \wedge (\eta' B = 0)$$

Hence,

- PBH can fail only if $\lambda \in \operatorname{spec}(A)$
- PBH fails iff $\eta' B = 0$ for a left eigenvector of A

If PBH fails on $\lambda \in \operatorname{spec}(A)$ with corresponding left eigenvector η, then

$$\eta'(A + BK) = \lambda \eta' \implies \lambda \in \operatorname{spec}(A + BK), \quad \forall K$$

i.e. λ remains an eigenvalue (mode) of $A + BK$ for all K. Hence, every

- $\lambda \in \mathbb{C}$ at which PBH fails is called an *uncontrollable mode* of (A, B).

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Stabilizability

We say that $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is stabilizable if

- there is $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is Hurwitz.

The following statements are equivalent:

1. The pair (A, B) is stabilizable.
2. The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}_0$.

Controllable decomposition

There is a nonsingular matrix T_c such that

$$(T_c A T_c^{-1}, T_c B) = \left(\begin{bmatrix} A_c & \times \\ 0 & A_{\bar{c}} \end{bmatrix}, \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right),$$

where (A_c, B_c) is controllable and $\mathrm{spec}(A_{\bar{c}})$ comprises all uncontrollable modes of (A, B). Moreover, T_c brings (A, B) to this form iff

$$T_c W_c(t) T_c' = \begin{bmatrix} \tilde{W}_c(t) & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\tilde{W}_c(t) > 0$.

Observability & detectability

We say that $(C, A) \in \mathbb{R}^{p \times n} \times \mathbb{R}^{n \times n}$ is observable if the

– eigenvalues of $A + LC$ can be freely assigned by a choice of $L \in \mathbb{R}^{n \times p}$

(with the restriction that complex eigenvalues are in conjugate pairs).

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Because

$$\operatorname{spec}(A + LC) = \operatorname{spec}(A' + C'L'),$$

we have that

– (C, A) is observable iff (A', C') is controllable

– (C, A) is detectable iff (A', C') is stabilizable

and can use all tests (with the observability matrix, observability Gramians, PBH, observable decomposition, et cetera).

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and can use all tests (with the observability matrix, observability Gramians, PBH, observable decomposition, et cetera).

Observability criteria

The following statements are equivalent:

1. The pair (C, A) is observable.
2. The matrix $\begin{bmatrix} A - sI \\ C \end{bmatrix}$ has full column rank $\forall s \in \mathbb{C}$.
3. The matrix
$$W_o(t) := \int_0^t \mathrm{e}^{A'\tau} C' C \mathrm{e}^{A\tau} \mathrm{d}\tau$$
is positive definite for any $t > 0$.
4. The observability matrix
$$M_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$
has full rank (i.e. $\operatorname{rank}(M_o) = n$).

Observable decomposition

There is a nonsingular matrix T_o such that

$$(CT_o^{-1}, T_oAT_o^{-1}) = \left(\begin{bmatrix} C_o & 0 \end{bmatrix}, \begin{bmatrix} A_o & 0 \\ \times & A_{\bar{o}} \end{bmatrix} \right),$$

where (C_o, A_o) is observable and $\mathrm{spec}(A_{\bar{o}})$ comprises all unobservable modes of (C, A). Moreover, T_o brings (C, A) to this form iff

$$T_o^{-\prime} W_o(t) T_o^{-1} = \begin{bmatrix} \tilde{W}_o(t) & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\tilde{W}_c(t) > 0$.

As a matter of fact,

$$\tilde{W}_o(t) = T^{-\prime} W_o(t) T^{-1} \quad \text{and} \quad \tilde{M}_o = M_o T^{-1}.$$

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Outline

Uncontrollable modes and transfer functions

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{\mathrm{c}} & \times & B_{\mathrm{c}} \\ 0 & A_{\bar{\mathrm{c}}} & 0 \\ \hline C_{\mathrm{c}} & \times & D \end{array}\right],$$

where $(A_{\mathrm{c}}, B_{\mathrm{c}})$ is controllable. Now,

$$\begin{aligned} G(s) &= D + \begin{bmatrix} C_{\mathrm{c}} & \times \end{bmatrix} \left(sI - \begin{bmatrix} A_{\mathrm{c}} & \times \\ 0 & A_{\bar{\mathrm{c}}} \end{bmatrix} \right)^{-1} \begin{bmatrix} B_{\mathrm{c}} \\ 0 \end{bmatrix} \\ &= D + \begin{bmatrix} C_{\mathrm{c}} & \times \end{bmatrix} \begin{bmatrix} (sI - A_{\mathrm{c}})^{-1} & \times \\ 0 & (sI - A_{\bar{\mathrm{c}}})^{-1} \end{bmatrix} \begin{bmatrix} B_{\mathrm{c}} \\ 0 \end{bmatrix} \\ &= D + C_{\mathrm{c}}(sI - A_{\mathrm{c}})^{-1} B_{\mathrm{c}}. \end{aligned}$$

In other words,

- uncontrollable modes do not affect the corresponding transfer function.

The same conclusion holds for unobservable modes of a realization.

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Kalman canonical decomposition

There is a nonsingular matrix T such that

$$G(s) = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cccc|c} A_{c\bar{o}} & \times & \times & \times & B_{c\bar{o}} \\ 0 & A_{co} & 0 & \times & B_{co} \\ 0 & 0 & A_{\bar{c}\bar{o}} & \times & 0 \\ 0 & 0 & 0 & A_{\bar{c}o} & 0 \\ \hline 0 & C_{co} & 0 & C_{\bar{c}o} & D \end{array}\right] = \left[\begin{array}{c|c} A_{co} & B_{co} \\ \hline C_{co} & D \end{array}\right],$$

where (A_{co}, B_{co}) is controllable and (C_{co}, A_{co}) is observable, so that the

- $\mathrm{spec}(A_{co})$ contains controllable-and-observable
- $\mathrm{spec}(A_{c\bar{o}})$ contains controllable-but-unobservable
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modes of the triple (C, A, B), respectively. Again, neither uncontrollable nor unobservable modes (aka hidden modes) affect the transfer function.

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Minimality

A realization (A, B, C, D) of a given system G is said to be

- minimal if the dimension of A is smallest among all realizations of G.

Theorem

A realization (A, B, C, D) is minimal iff (A, B) is controllable and (C, A) is observable.

Proof (outline of the "only if" part).

Follows from the Kalman canonical decomposition. $\square$

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Follows from the Kalman canonical decomposition. □

Minimality (contd)

Proof (outline of the "if" part).

Let (A, B) be controllable, (C, A) be observable, but the realization be not minimal for $A \in \mathbb{R}^{n\times n}$. I.e. there are $A_\mathrm{r} \in \mathbb{R}^{n_\mathrm{r}\times n_\mathrm{r}}$, B_r, and C_r so that

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Hence, $C\mathrm{e}^{At}B = C_\mathrm{r}\mathrm{e}^{A_\mathrm{r}t}B_\mathrm{r}$, $\forall t$, or $C\mathrm{e}^{A\sigma}\mathrm{e}^{A\tau}B = C_\mathrm{r}\mathrm{e}^{A_\mathrm{r}\sigma}\mathrm{e}^{A_\mathrm{r}\tau}B_\mathrm{r}$. This yields

$$\mathrm{e}^{A'\sigma}C'C\mathrm{e}^{A\sigma}\mathrm{e}^{A\tau}BB'\mathrm{e}^{A'\tau} = \mathrm{e}^{A'\sigma}C'C_\mathrm{r}\mathrm{e}^{A_\mathrm{r}\sigma}\mathrm{e}^{A_\mathrm{r}\tau}B_\mathrm{r}B'\mathrm{e}^{A'\tau}.$$

Integrating both sides from 0 to t over both σ and τ,

$$\underbrace{W_\mathrm{o}(t)W_\mathrm{c}(t)}_{\mathrm{rank}=n} = \int_0^t \mathrm{e}^{A'\sigma}C'C_\mathrm{r}\mathrm{e}^{A_\mathrm{r}\sigma}\,\mathrm{d}\sigma \int_0^t \mathrm{e}^{A_\mathrm{r}\tau}B_\mathrm{r}B'\mathrm{e}^{A'\tau}\,\mathrm{d}\tau =: \underbrace{W_{\mathrm{r}1}(t)W_{\mathrm{r}2}(t)}_{\mathrm{rank}\leq n_\mathrm{r}}$$

with $W_{\mathrm{r}1}(t) \in \mathbb{R}^{n\times n_\mathrm{r}}$ and $W_{\mathrm{r}2}(t) \in \mathbb{R}^{n_\mathrm{r}\times n}$. Ranks do not agree then. $\square$

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All minimal realizations

Theorem

Any two minimal realizations of a finite-dimensional LTI system are similar.

Proof (outline).

Obviously, any realization, similar to another minimal realization, is minimal.

Now, let (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ be minimal. Hence,

$$Ce^{At}B = \tilde{C}e^{\tilde{A}t}\tilde{B} \iff CA^iB = \tilde{C}\tilde{A}^i\tilde{B} \implies M_oA^iM_c = \tilde{M}_o\tilde{A}^i\tilde{M}_c, \quad \forall i$$

Construct now

$$T := (\tilde{M}_o'\tilde{M}_o)^{-1}\tilde{M}_o'M_o \quad \text{and} \quad S := M_c\tilde{M}_c'(\tilde{M}_c\tilde{M}_c')^{-1}.$$

It can be shown that $T = S^{-1}$, $TM_c = \tilde{M}_c$, $M_oS = M_oT^{-1} = \tilde{M}_o$, and then $TAS = TAT^{-1} = \tilde{A}$. $\square$

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Proof (outline).

Obviously, any realization, similar to another minimal realization, is minimal. Now, let (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ be minimal. Hence,

$$C e^{At} B = \tilde{C} e^{\tilde{A}t} \tilde{B} \iff CA^i B = \tilde{C} \tilde{A}^i \tilde{B} \implies M_o A^i M_c = \tilde{M}_o \tilde{A}^i \tilde{M}_c, \quad \forall i$$

Construct now

$$T := (\tilde{M}_o' \tilde{M}_o)^{-1} \tilde{M}_o' M_o \quad \text{and} \quad S := M_c \tilde{M}_c' (\tilde{M}_c \tilde{M}_c')^{-1}.$$

It can be shown that $T = S^{-1}$, $TM_c = \tilde{M}_c$, $M_o S = M_o T^{-1} = \tilde{M}_o$, and then $TAS = TAT^{-1} = \tilde{A}$. $\square$

All minimal realizations

Theorem

Any two minimal realizations of a finite-dimensional LTI system are similar.

Proof (outline).

Obviously, any realization, similar to another minimal realization, is minimal. Now, let (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ be minimal. Hence,

$$C\mathrm{e}^{At}B = \tilde{C}\mathrm{e}^{\tilde{A}t}\tilde{B} \iff CA^iB = \tilde{C}\tilde{A}^i\tilde{B} \implies M_\mathrm{o}A^iM_\mathrm{c} = \tilde{M}_\mathrm{o}\tilde{A}^i\tilde{M}_\mathrm{c}, \quad \forall i$$

Construct now

$$T := (\tilde{M}_\mathrm{o}'\tilde{M}_\mathrm{o})^{-1}\tilde{M}_\mathrm{o}'M_\mathrm{o} \quad \text{and} \quad S := M_\mathrm{c}\tilde{M}_\mathrm{c}'(\tilde{M}_\mathrm{c}\tilde{M}_\mathrm{c}')^{-1}.$$

It can be shown that $T = S^{-1}$, $TM_\mathrm{c} = \tilde{M}_\mathrm{c}$, $M_\mathrm{o}S = M_\mathrm{o}T^{-1} = \tilde{M}_\mathrm{o}$, and then $TAS = TAT^{-1} = \tilde{A}$. $\square$

Gilbert's realization

If $G(s)$ is proper and such that

$$G(s) = \frac{1}{d(s)} N_G(s), \quad \text{for } d(s) = (s - a_1) \cdots (s - a_r) \text{ with } a_j \neq a_i$$

and polynomial matrix $N_G(s)$, then

$$G(s) = G(\infty) + \sum_{i=1}^{r} \frac{1}{s - a_i} G_i \quad \text{for } G_i := \lim_{s \to a_i} (s - a_i) G(s).$$

If rank $G_i = n_i$, then $\exists B_i \in \mathbb{R}^{n_i \times m}$, $C_i \in \mathbb{R}^{p \times n_i}$ having rank n_i and such that

$$G_i = C_i B_i$$

(rank decomposition).

Gilbert's realization (contd)

Theorem

The realization

$$G(s) = \left[\begin{array}{ccc|c} a_1 I_{n_1} & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & a_r I_{n_r} & B_r \\ \hline C_1 & \cdots & C_r & G(\infty) \end{array}\right]$$

is minimal (its dimension is $\sum_{i=1}^{r} n_i$).

Proof (outline): If $\lambda = a_i$ is uncontrollable mode, then $\exists \eta \neq 0$ such that

$$\begin{bmatrix} \eta_1' & \cdots & \eta_r' \end{bmatrix} \begin{bmatrix} (a_1-\lambda)I_{n_1} & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & (a_r-\lambda)I_{n_r} & B_r \end{bmatrix}$$
$$= \begin{bmatrix} (a_1-\lambda)\eta_1' & \cdots & (a_r-\lambda)\eta_r' & \sum_{i=1}^{r} \eta_i' B_i \end{bmatrix} = 0$$

So $\eta_j = 0$ for all $j \neq i$ and then $\eta_i' B_i = 0 \implies \eta_i = 0$ (contradiction). $\square$

Outline

Reminder: doubly coprime factorization over RH_∞

Given a real-rational proper $G(s)$, there are right coprime $N, M \in RH_\infty$ and left coprime $\tilde{N}, \tilde{M} \in RH_\infty$ such that

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

and their Bézout factors verifying

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The question is

- how to construct these functions ?

State-space way

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with (A, B) stabilizable and (C, A) detectable. Let K and L be any matrices such that $A + BK$ and $A + LC$ are Hurwitz. Then

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & B + LD & -L \\ \hline -K & I & 0 \\ -C & -D & I \end{array}\right]$$

and

$$\begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right].$$

State-space way (contd)

Remember,

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & \bar{B}\bar{D}^{-1} \\ \hline -\bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{array}\right].$$

Then

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix}^{-1} = \left[\begin{array}{c|cc} A + LC & B + LD & -L \\ \hline -K & I & 0 \\ -C & -D & I \end{array}\right]^{-1} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right]$$

$$= \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix}$$

because

$$\begin{bmatrix} I & 0 \\ -D & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ D & I \end{bmatrix}, \quad \begin{bmatrix} B + LD & -L \end{bmatrix} \begin{bmatrix} I & 0 \\ D & I \end{bmatrix} \begin{bmatrix} K \\ C \end{bmatrix} = BK - LC.$$

The fact that $N(s)M^{-1}(s) = G(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ is also easy to show . . .

State-space way (contd)

Remember,

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & \bar{B}\bar{D}^{-1} \\ \hline -\bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{array}\right].$$

Then

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix}^{-1} = \left[\begin{array}{c|cc} A + LC & B + LD & -L \\ \hline -K & I & 0 \\ -C & -D & I \end{array}\right]^{-1} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right]$$

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The fact that $N(s)M^{-1}(s) = G(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ is also easy to show . . .

Outline

Poles of the realization

Eigenvalues of A are called poles of the realization (A, B, C, D). Because

$$G(s) = D + C(sI - A)^{-1}B = D + \frac{1}{\det(sI - A)} C \operatorname{adj}(sI - A)B$$

poles of $G(s)$ are also poles of its realization. Then

Theorem

The McMillan degree of $G(s)$ is equal to the order of its minimal realization (A, B, C, D) and the set of poles of $G(s)$ coincides with $\operatorname{spec}(A)$.

Proof (outline).

It follows by the Kalman canonical decomposition that hidden modes don't affect transfer functions. The proof that every pole of a minimal realization is a pole of $G(s)$ is too technical . . . □

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Reminder: pole directions

Remember the Smith–McMillan form:

$$U(s)G(s)V(s) = \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\alpha_i(s)$ divides $\alpha_{i+1}(s)$, $\beta_{i+1}(s)$ divides $\beta_i(s)$. Then

$$\mathrm{pdir_i}(G, p_i) = \left(\mathrm{Im}\, V(p_i) \begin{bmatrix} e_{\mu_i+1} & \cdots & e_m \end{bmatrix}\right)^{\perp} = \ker \begin{bmatrix} e'_{\mu_i+1} \\ \vdots \\ e'_m \end{bmatrix} [V(p_i)]'$$

$$\mathrm{pdir_o}(G, p_i) = \ker \begin{bmatrix} \tilde{e}'_{\mu_i+1} \\ \vdots \\ \tilde{e}'_p \end{bmatrix} U(p_i) = \left(\mathrm{Im}[U(p_i)]' \begin{bmatrix} \tilde{e}_{\mu_i+1} & \cdots & \tilde{e}_p \end{bmatrix}\right)^{\perp}$$

When Smith–McMillan meets Jordan

Example:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -s & 1 \\ -s & 1 & 0 \end{bmatrix}}_{U(s)} \underbrace{\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}^{-1}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & s & s^2 \end{bmatrix}}_{V(s)} = \begin{bmatrix} 1/s^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [V(0)]' = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \ker(0I - A)',$$

$$\ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U(0) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \ker(0I - A).$$

Pole directions via state-space realizations

It can be shown that

- the geometric multiplicity of a pole of $\Phi(s) := (sI - A)^{-1}$ at p_i equals the geometric multiplicity of an eigenvalue of A at p_i

and then $\mathrm{pdir}_{\mathrm{i}}(\Phi, p_i) = \ker[(p_i I - A)]'$ and $\mathrm{pdir}_{\mathrm{o}}(\Phi, p_i) = \ker(p_i I - A)$.

Motivated by that, for $G(s) = D + C(sI - A)^{-1}B$

$$\mathrm{pdir}_{\mathrm{i}}(G, p_i) = B' \ker[(p_i I - A)]' \quad \text{and} \quad \mathrm{pdir}_{\mathrm{o}}(G, p_i) = C \ker(p_i I - A).$$

In Gilbert's realization

$$G(s) \doteq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|c} a_1 I_{n_1} & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & a_r I_{n_r} & B_r \\ \hline C_1 & \cdots & C_r & D \end{array}\right],$$

$\mathrm{pdir}_{\mathrm{i}}(G, a_i) = \mathrm{Im}\, B_i'$ and $\mathrm{pdir}_{\mathrm{o}}(G, a_i) = \mathrm{Im}\, C_i$.

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Rosenbrock system matrix

The polynomial matrix

$$R_G(s) := \begin{bmatrix} A - sI_n & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - s \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

is called the Rosenbrock system matrix of G given in terms of (A, B, C, D). Because

$$\begin{aligned} R_G(s) &= \begin{bmatrix} A - sI & 0 \\ C & G(s) \end{bmatrix} \begin{bmatrix} I & -(sI - A)^{-1}B \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -C(sI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ 0 & G(s) \end{bmatrix}, \end{aligned}$$

we have that

$$\operatorname{rank}(R_G(s_0)) = n + \operatorname{rank}(G(s_0)), \quad \forall s_0 \notin \operatorname{spec}(A).$$

and then $\operatorname{nrank}(R_G(s)) = n + \operatorname{nrank}(G(s))$.

Invariant zeros of the realization

Every $z_i \in \mathbb{C}$ at which

$$\operatorname{rank}(R_G(z_i)) < \operatorname{nrank}(R_G(s))$$

is called an invariant zero of the realization (A, B, C, D). Because

$$\begin{bmatrix} TAT^{-1} - sI & TB \\ CT^{-1} & D \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I_m \end{bmatrix},$$

they are invariant under similarity. Finding them is a generalized eigenvalue problem (e.g. `eig([A,B;C,D],[eye(n,n+m);zeros(m,n+m)])` if $p = m$).

Theorem

Invariant zeros of (A, B, C, D) comprise all its hidden modes, as well as the transmission zeros of $G(s) = D + C(sI - A)^{-1}B$.

Proof (observations).

Straightforward if invariant zeros are not in $\operatorname{spec}(A)$, nasty otherwise. □

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Proof (observations).

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Zero directions

Remember, zero directions for transfer functions (if zeros are not poles):

$$\mathrm{zdir_i}(G, z_i) = \ker G(z_i) \subset \mathbb{C}^m \quad \text{and} \quad \mathrm{zdir_o}(G, z_i) = \ker[G(z_i)]' \subset \mathbb{C}^p.$$

Then

$$\begin{aligned} 0 &= \begin{bmatrix} A - z_i I & 0 \\ C & G(z_i) \end{bmatrix} \begin{bmatrix} 0 \\ u_i \end{bmatrix} = \begin{bmatrix} A - z_i I & B \\ C & D \end{bmatrix} \begin{bmatrix} I & (z_i I - A)^{-1} B \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ u_i \end{bmatrix} \\ &= \begin{bmatrix} A - z_i I & B \\ C & D \end{bmatrix} \begin{bmatrix} (z_i I - A)^{-1} B u_i \\ u_i \end{bmatrix}. \end{aligned}$$

so that $\mathrm{zdir_i}(G, z_i) \in \begin{bmatrix} 0 & I_m \end{bmatrix} \ker R_G(z_i)$. The other direction is also true:

$$0 = \begin{bmatrix} A - z_i I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} = \begin{bmatrix} A - z_i I & 0 \\ C & G(z_i) \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ u_i \end{bmatrix},$$

where $\tilde{x}_i := x_i - (z_i I - A)^{-1} B u_i = 0$ then, because $\det(A - z_i I) \neq 0$.

Zero directions (contd)

Thus, we have that

$$\mathrm{zdir_i}(G, z_i) = \begin{bmatrix} 0 & I_m \end{bmatrix} \ker R_G(z_i)$$

and, by similar arguments, that

$$\mathrm{zdir_o}(G, z_i) = \begin{bmatrix} 0 & I_p \end{bmatrix} \ker [R_G(z_i)]'.$$

These relations should also hold true if $z_i \in \mathrm{spec}(A)$, perhaps. At least if

$$\begin{bmatrix} A - z_i I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} = 0,$$

then $u_i \neq 0$ (by observability) and $(A - z_i I)x_i + Bu_i = 0$ implies that

- $\mathrm{zdir_i}(G, z_i) \perp \mathrm{pdir_i}(G, z_i)$ whenever $z_i \in \mathrm{spec}(A)$

($\mathrm{zdir_o}(G, z_i) \perp \mathrm{pdir_o}(G, z_i)$ then too), which is a circumstantial evidence.

Zero directions (contd)

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Example 1

Let

$$G(s) = \frac{1}{s}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{s}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Its minimal (Gilbert's) realization is

$$G(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right].$$

It has one pole at the origin and because $\ker(s-0)|_{s=0} = \mathbb{C}$, we have that

$$\mathrm{pdir_i}(G,0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathbb{C} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\mathrm{pdir_o}(G,0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathbb{C} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

Example 1 (contd)

The Rosenbrock system matrix

$$R_G(s) = \begin{bmatrix} -s & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is such that $\text{rank}(R_G(s)) = 2$ for all $s \in \mathbb{C}$. Thus, the system has no zeros. All these results agree with those derived in Chapter 3.

Example 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{s}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{s}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Its minimal (Gilbert's) realization is

$$G(s) = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

It has one pole at the origin and because $\ker(s-0)|_{s=0} = \mathbb{C}$, we have that

$$\mathrm{pdir_i}(G,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbb{C} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

and

$$\mathrm{pdir_o}(G,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathbb{C} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

Example 2 (contd)

The Rosenbrock system matrix

$$R_G(s) = \begin{bmatrix} -s & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has full normal rank and $\det(R_G(s)) = -s$. Thus, the system has a zero at the origin too. Because

$$\ker R_G(0) = \operatorname{span}\left(\left[\begin{array}{c} -1 \\ \hdashline 1 \\ 0 \end{array}\right]\right) \quad \text{and} \quad \ker[R_G(0)]' = \operatorname{span}\left(\left[\begin{array}{c} -1 \\ \hdashline 0 \\ 1 \end{array}\right]\right),$$

we have that

$$\operatorname{zdir}_{\mathrm{i}}(G, 0) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \operatorname{zdir}_{\mathrm{o}}(G, 0) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

All these results agree with those derived in Chapter 3.

Invariant zeros: filtering inputs

In the SISO case, if $G(s)$ has a zero at z_i, then

$$u(t) = \mathrm{e}^{z_i t}\mathbb{1}(t)$$

is filtered out by G ($Y(s) = \frac{1}{s-z_i}G(s)$ is well defined at $s = z_i$, so that $y(t)$ does not contain a component with $\mathrm{e}^{z_i t}$).

In the MIMO case, let

$$u(t) = u_i \mathrm{e}^{z_i t}\mathbb{1}(t)$$

for $u_i \neq 0$ such that which the Sylvester equation $-x_i z_i + Ax_i + Bu_i = 0$ is solvable in $x_i \in \mathbb{C}^n$. This happens

- for all $u_i \in \mathbb{C}^m$ if $z_i \notin \mathrm{spec}(A)$
- for all $u_i \perp \mathrm{pdir}_1(G, z_i) \subset \mathbb{C}^m$ if $z_i \in \mathrm{spec}(A)$

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In the SISO case, if $G(s)$ has a zero at z_i, then

$$u(t) = \mathrm{e}^{z_i t}\mathbb{1}(t)$$

is filtered out by G ($Y(s) = \frac{1}{s-z_i}G(s)$ is well defined at $s = z_i$, so that $y(t)$ does not contain a component with $\mathrm{e}^{z_i t}$).

In the MIMO case, let

$$u(t) = u_i \mathrm{e}^{z_i t}\mathbb{1}(t)$$

for $u_i \neq 0$ such that which the Sylvester equation $-x_i z_i + Ax_i + Bu_i = 0$ is solvable in $x_i \in \mathbb{C}^n$. This happens

- for all $u_i \in \mathbb{C}^m$ if $z_i \notin \mathrm{spec}(A)$
- for all $u_i \perp \mathrm{pdir_i}(G, z_i) \subset \mathbb{C}^m$ if $z_i \in \mathrm{spec}(A)$

Invariant zeros: filtering inputs (contd)

Then (remember the partial fraction expansion formula from Slide 15)

$$Y(s) = G(s)u_i\frac{1}{s-z_i} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} z_i & 1 \\ \hline u_i & 0 \end{array}\right] = \left[\begin{array}{cc|c} A & Bu_i & 0 \\ 0 & z_i & 1 \\ \hline C & Du_i & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A & Bu_i & -x_i \\ 0 & z_i & 1 \\ \hline C & Cx_i + Du_i & 0 \end{array}\right] = -C(sI-A)^{-1}x_i + (Cx_i + Du_i)\frac{1}{s-z_i}$$

Hence,

$$y(t) = \underbrace{-C\,\mathrm{e}^{At}x_i\mathbb{1}(t)}_{\text{transients}} + \underbrace{(Cx_i + Du_i)\,\mathrm{e}^{z_i t}\mathbb{1}(t)}_{\text{steady-state effect of } u(t)}$$

If

$$Cx_i + Du_i = 0 \iff \begin{bmatrix} A - z_i I & B \\ C & D \end{bmatrix}\begin{bmatrix} x_i \\ u_i \end{bmatrix} = 0 \iff u_i \in \mathrm{zdir}(G, z_i),$$

then the response to $u(t)$ includes only transients. In addition, if

$$x(0) = x_i \implies y(t) \equiv 0, \text{ i.e. no response to } u(t) = u_i\mathrm{e}^{z_i t}\mathbb{1}(t) \text{ at all.}$$

Invariant zeros: filtering inputs (contd)

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Realization poles and coprime factors

Remember, $G = \tilde{M}^{-1}\tilde{N} = NM^{-1}$ with

$$\tilde{M}(s) = \left[\begin{array}{c|c} A + LC - sI & L \\ \hline C & I \end{array}\right] \quad \text{and} \quad M(s) = \left[\begin{array}{c|c} A + BK - sI & B \\ \hline K & I \end{array}\right].$$

Now,

$$R_{\tilde{M}}(s) = \begin{bmatrix} A + LC - sI & L \\ C & I \end{bmatrix} = \begin{bmatrix} A - sI & L \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & I \end{bmatrix},$$

$$R_M(s) = \begin{bmatrix} A + BK - sI & B \\ K & I \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} A - sI & 0 \\ K & I \end{bmatrix}$$

Hence

- $z_i \in \mathbb{C}$ is an invariant zero of $\tilde{M}$ iff it is a realization pole of G, with $\mathrm{zdir}_\ell(\tilde{M}, z_i) = \mathrm{pdir}_o(G, z_i)$;
- $z_i \in \mathbb{C}$ is an invariant zero of M iff it is a realization pole of G, with $\mathrm{zdir}_o(M, z_i) = \mathrm{pdir}_\ell(G, z_i)$.

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Invariant zeros and coprime factors

Again, because $G = \tilde{M}^{-1}\tilde{N} = NM^{-1}$ with

$$\tilde{N}(s) = \left[\begin{array}{c|c} A + LC & B + LD \\ \hline C & D \end{array}\right] \quad \text{and} \quad N(s) = \left[\begin{array}{c|c} A + BK & B \\ \hline C + DK & D \end{array}\right],$$

we have that

$$\begin{aligned} R_G(s) &= \begin{bmatrix} I & -L \\ 0 & I \end{bmatrix} \begin{bmatrix} A + LC - sI & B + LD \\ C & D \end{bmatrix} = \begin{bmatrix} I & -L \\ 0 & I \end{bmatrix} R_{\tilde{N}}(s) \\ &= \begin{bmatrix} A + BK - sI & B \\ C + DK & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix} \qquad = R_N(s) \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix} \end{aligned}$$

Hence,

- $z_i \in \mathbb{C}$ is an invariant zero of $\tilde{N}$ iff it is an invariant zero of G, with $\mathrm{zdir}_i(\tilde{N}, z_i) = \mathrm{zdir}_i(G, z_i)$;
- $z_i \in \mathbb{C}$ is an invariant zero of N iff it is an invariant zero of G, with $\mathrm{zdir}_o(N, z_i) = \mathrm{zdir}_o(G, z_i)$.

Invariant zeros and coprime factors

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Outline

Computing H_2 norm

Proposition

If A is Hurwitz and $D = 0$, then

$$\|G\|_2^2 = \operatorname{tr}(B'QB) = \operatorname{tr}(CPC'),$$

where Q and P are the observability and controllability Gramians of (C, A) and (A, B), respectively.

Proof.

The impulse response of G is $g(t) = C\mathrm{e}^{At}B\mathbb{1}(t)$. By Parseval,

$$\begin{aligned}\|G\|_2^2 = \|g\|_2^2 &= \int_{\mathbb{R}_+} \operatorname{tr}\big(g(t)'g(t)\big)\,\mathrm{d}t = \int_{\mathbb{R}_+} \operatorname{tr}\big(B'\mathrm{e}^{A't}C'C\mathrm{e}^{At}B\big)\,\mathrm{d}t \\ &= \operatorname{tr}\left(B'\int_{\mathbb{R}_+} \mathrm{e}^{A't}C'C\mathrm{e}^{At}\mathrm{d}t\,B\right) = \operatorname{tr}(B'QB),\end{aligned}$$

The other formula is derived similarly, because $\operatorname{tr}(M'M) = \operatorname{tr}(MM')$. $\square$

Computing H_∞ norm

Proposition

If A is Hurwitz, then $\|G\|_\infty < \gamma$ for a given $\gamma > 0$ iff $\overline{\sigma}(D) < \gamma$ and

$$H_G := \begin{bmatrix} A & 0 \\ C'C & -A' \end{bmatrix} - \begin{bmatrix} B \\ C'D \end{bmatrix} (\gamma^2 I - D'D)^{-1} \begin{bmatrix} -D'C & B' \end{bmatrix}$$

has no pure imaginary eigenvalues.

Proof (outline).

Because $G \in RH_\infty$,

$$\|G\|_\infty < \gamma \iff \gamma^2 I - [G(\mathrm{j}\omega)]' G(\mathrm{j}\omega) > 0, \quad \forall \omega \in \mathbb{R} \cup \{\pm\infty\}.$$

As $G(\mathrm{j}\infty) = D$, $\overline{\sigma}(D) < \gamma$ follows (and assumed hereafter). Thus,

$$\|G\|_\infty < \gamma \iff \Phi(s) := \gamma^2 I - G^\sim(s) G(s) \text{ has no pure imaginary zeros}$$

How to verify that?

Computing H_∞ norm (contd)

Proof (outline, contd).

Now,

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \implies G^{\sim}(s) = \left[\begin{array}{c|c} -A' & C' \\ \hline -B' & D' \end{array}\right].$$

Hence,

$$\Phi(s) = \gamma^2 I - \left[\begin{array}{c|c} -A' & C' \\ \hline -B' & D' \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ C'C & -A' & C'D \\ \hline -D'C & B' & \gamma^2 I - D'D \end{array}\right].$$

As $\operatorname{spec}(A) \cap \mathrm{j}\mathbb{R} = \varnothing$, imaginary zeros of $\Phi(s)$ are its invariant zeros. Then

$$R_\Phi(\mathrm{j}\omega) = \begin{bmatrix} A - \mathrm{j}\omega I & 0 & B \\ C'C & -A' - \mathrm{j}\omega I & C'D \\ -D'C & B' & \gamma^2 I - D'D \end{bmatrix}$$

and H_G is the Schur complement of $\gamma^2 I - D'D$ in it. $\square$

KYP (Kalman–Yakubovich–Popov) lemma

Consider $p \times m$ *system* $G(s) = D + C(sI - A)^{-1}B$, *with* $\operatorname{spec}(A) \cap \mathrm{j}\mathbb{R} = \varnothing$, *and let* $M_{KYP} = M'_{KYP} \in \mathbb{R}^{(m+p)\times(m+p)}$. *The frequency-dependent inequality*

$$\begin{bmatrix} [G(\mathrm{j}\omega)]' & I_m \end{bmatrix} M_{KYP} \begin{bmatrix} G(\mathrm{j}\omega) \\ I_m \end{bmatrix} < 0, \quad \forall \omega$$

holds iff there is $X = X' \in \mathbb{R}^{n\times n}$ *verifying the linear matrix inequality (LMI)*

$$\begin{bmatrix} C' & 0 \\ D' & I_m \end{bmatrix} M_{KYP} \begin{bmatrix} C & D \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} I_n & A' \\ 0 & B' \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix} < 0.$$

KYP implies that

- infinite set of inequalities $\iff$ finite number of LMIs (solvable)

Many important special cases, e.g.

$$M_{KYP} = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \implies \text{calculate the } L_\infty(\mathrm{j}\mathbb{R})\text{-norm of } G$$

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Outline

Model reduction problem

Complexity vs. accuracy is one of the key tradeoffs in (control) engineering. "Complexity" is understood as "order" in the LTI case. Then:

- given an n-order $p \times m$ LTI G and $n_\mathrm{r} < n$, find an n_r-order $p \times m$ LTI G_r, which is "close" to G,

say in the sense that $\|G - G_\mathrm{r}\|_\infty$ is "small."

In what follows, an approach based on

- structural properties of state-space realizations

is considered. It is both practical (for relatively small n's) and enlightening. We consider model reduction for stable systems only.

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Classical control recipes

Thinking in terms of pole dominance, i.e.

- all poles are equal, but some poles are more equal than others.

Example 1:

$$G(s) = \frac{1}{(s+1)(\tau s+1)} \quad \text{for } \tau \in (0,1) \quad \Longrightarrow \quad G_r(s) = \frac{1}{s+1},$$

justifiable if $\tau \ll 1$ (far right), $\|G - G_r\|_\infty = \tau/(1+\tau)$.

Example 2:

$$G(s) = \frac{2s+1}{(s+1)((2-\epsilon)s+1)} \quad \text{for } \epsilon \in (0,1) \quad \Longrightarrow \quad G_r(s) = \frac{1}{s+1},$$

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MIMO extensions

Dominant poles ideas are

- ☹ overly hand-waving
- ☹ messy if directional properties have to be accounted for

Alternative thinking:

- hidden modes can be detected and eliminated w/o consequences
- what about "almost hidden" modes?
  - detect?
  - costs of eliminating?

Controllability and observability Gramians are $P = P' \geq 0$ and $Q = Q' \geq 0$ satisfying

$$AP + PA' + BB' = 0 \quad \text{and} \quad A'Q + QA + C'C = 0.$$

$P > 0$ iff (A, B) is controllable and $Q > 0$ iff (C, A) is observable.

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First try

We (maybe) remember that if (A, B) is uncontrollable, there is T_c such that

$$\left[\begin{array}{c|c} T_c A T_c^{-1} & T_c B \\ \hline C T_c^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_c & \times & B_c \\ 0 & A_{\bar{c}} & 0 \\ \hline C_c & C_{\bar{c}} & 0 \end{array}\right] = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right].$$

and this T_c can be constructed via the Gramian, $T_c P T_c' = \left[\begin{smallmatrix} P_c & 0 \\ 0 & 0 \end{smallmatrix}\right]$. So if

$$TPT' = \begin{bmatrix} \Sigma_{P_1} & 0 \\ 0 & \Sigma_{P_2} \end{bmatrix} \quad \text{with } \|\Sigma_{P_1}\| \gg \|\Sigma_{P_2}\|,$$

is

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right] \approx \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right]$$

if A_{21} and B_2 are "small"?

First try: example 3

Let

$$G(s) = \frac{18}{5s^2 + 12s + 9} = \left[\begin{array}{cc|c} -2 & -1/\alpha & 1 \\ \alpha & -0.4 & \alpha \\ \hline -1 & 1/\alpha & 0 \end{array}\right],$$

which is true for all $\alpha \neq 0$ and its controllability Gramian,

$$P = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25\alpha^2 \end{bmatrix},$$

is of the requires form if $\alpha \ll 1$. Yet the choice

$$G_r(s) = \left[\begin{array}{c|c} -2 & 1 \\ \hline -1 & 0 \end{array}\right] = \frac{1}{s+2}$$

is not what we need, as

$$\|G - G_r\|_\infty = 2.5 > \|G - 0\|_\infty = \|G\|_\infty = 2.$$

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First try: example 3 (contd)

Observability Gramian

$$Q = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25/\alpha^2 \end{bmatrix} \quad \text{compare with } P = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25\alpha^2 \end{bmatrix}$$

indicates that

- the second state becomes in a sense "over-observable" if $\alpha \ll 1$.

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Similarity transformations and Gramians

If $(\tilde{A}, \tilde{B}, \tilde{C}, 0) = (TAT^{-1}, TB, CT^{-1}, 0)$, then

$$\tilde{P} = TPT' \quad \text{and} \quad \tilde{Q} = T^{-\prime}QT^{-1}.$$

Hence,

- eigenvalues of P and Q are not preserved under similarity.

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$$\tilde{P}\tilde{Q} = TPQT^{-1}$$

is similar to PQ, so its eigenvalues are invariant under similarity. Moreover,

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- PQ is diagonalizable $\quad UQ^{1/2}PQ^{1/2}U' = (UQ^{1/2})PQ(UQ^{1/2})^{-1}$

Balanced realization

Theorem

If (A, B, C, D) is a minimal realization of an n-dimensional stable G, then there is T such that $(\tilde{A}, \tilde{B}, \tilde{C}, D) := (TAT^{-1}, TB, CT^{-1}, D)$ has[1]

$$\tilde{P} = \tilde{Q} = \Sigma := \begin{bmatrix} \sigma_1 I_{n_1} & & \\ & \ddots & \\ & & \sigma_l I_{n_l} \end{bmatrix},$$

where $\sigma_1 > \cdots > \sigma_l > 0$ and $n_i \in \mathbb{N}$ with $\sum_i n_i = n$.

Some facts about σ_i:

- known as Hankel singular values of G
- square roots of the singular values of PQ
- $\|G\|_H := \sigma_1 = \sqrt{\rho(PQ)}$ is known as the Hankel norm of G
- $L_2(\mathbb{R}_-) \to L_2(\mathbb{R}_+)$ induced norm of G

[1] Matlab command: `[Gb,Sig]=balreal(G)`.

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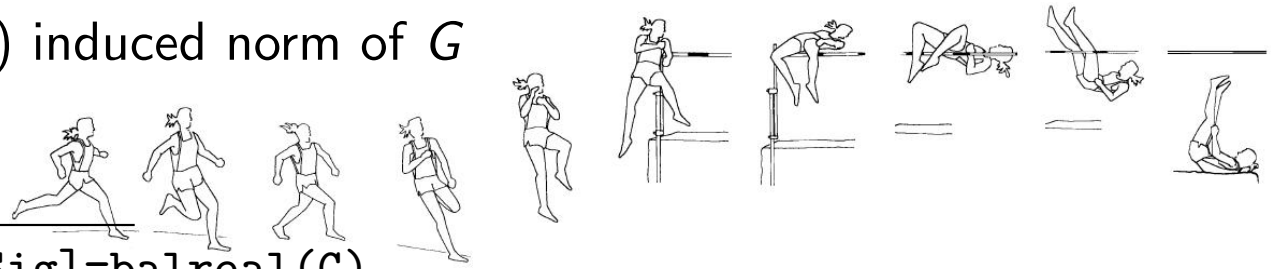

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Second try: balanced truncation

Let G be stable and (A, D, C, D) be its balanced realization. Partition

$$P = Q = \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where $\Sigma_1 = \mathrm{diag}\{\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}\}$ and $\Sigma_2 = \mathrm{diag}\{\sigma_{r+1} I_{n_{r+1}}, \ldots, \sigma_l I_{n_l}\}$ for $\sigma_1 > \cdots > \sigma_r > \sigma_{r+1} > \cdots > \sigma_l$. The correspondent state partition is

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

The system G_r with the transfer function

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is called the balanced truncation of G.

Balanced truncation properties

If G_r is the balanced truncation of of G, then

- $P_1 = Q_1 = \Sigma_1 > 0$ are Gramians of (A_{11}, B_1, C_1, D)
- $G_r \in RH_\infty$
- $\|G - G_r\|_\infty \leq 2(\sigma_{r+1} + \cdots + \sigma_l)$
- if $r = l - 1$, then the bound above is achieved, i.e. $\|G - G_{l-1}\|_\infty = 2\sigma_l$

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Balanced truncation: example 3

Let

$$G(s) = \frac{18}{5s^2 + 12s + 9} = \left[\begin{array}{cc|c} -2 & -1/\alpha & 1 \\ \alpha & -0.4 & \alpha \\ \hline -1 & 1/\alpha & 0 \end{array}\right],$$

with

$$P = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25\alpha^2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.25/\alpha^2 \end{bmatrix}.$$

Its HSVs are $\sigma_1 = 1.25$ and $\sigma_2 = 0.25$ and balanced realization (for $\alpha = 1$)

$$G(s) = \left[\begin{array}{cc|c} -0.4 & 1 & 1 \\ -1 & -2 & 1 \\ \hline 1 & -1 & 0 \end{array}\right].$$

Balanced truncation for $r = 1$:

$$G_1(s) = \left[\begin{array}{c|c} -0.4 & 1 \\ \hline 1 & 0 \end{array}\right] = \frac{5}{5s + 2} \quad \Longrightarrow \quad \|G - G_1\|_\infty = 2 \times 0.25 = 0.5,$$

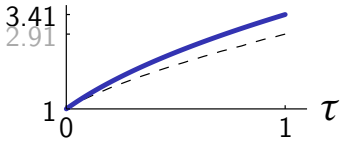
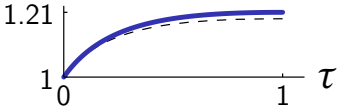
which is smaller than $\|G\|_\infty = 2$.

Balanced truncation: example 1 (contd)

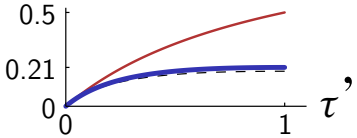
If

$$G(s) = \frac{1}{(s+1)(\tau s+1)},$$

then balanced truncation to $r = 1$ results in

$$G_1(s) = \frac{k_1}{\tau_1 s + 1} \quad \text{with } \tau_1 = \text{  } \text{ and } k_1 = \text{  }$$

which is different from keeping the rightmost pole at -1. Also,

$$\|G - G_1\|_\infty = \text{  },$$

where

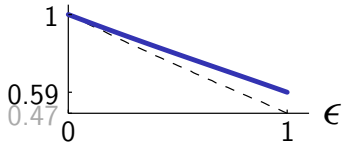
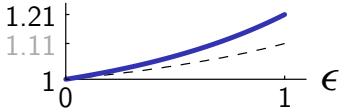
- red line corresponds to the naïve modal truncation
- dashed lines correspond to the (brute force) optimal H_∞ reduction

Balanced truncation: example 2 (contd)

If

$$G(s) = \frac{2s+1}{(s+1)((2-\epsilon)s+1)},$$

then balanced truncation to $r = 1$ results in

$$G_1(s) = \frac{k_2}{\tau_2 s + 1} \quad \text{with } \tau_2 = \text{  } \text{ and } k_2 = \text{  }$$

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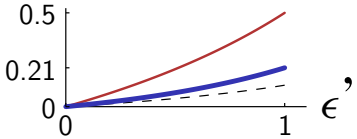
$$\|G - G_1\|_\infty = \text{  },$$

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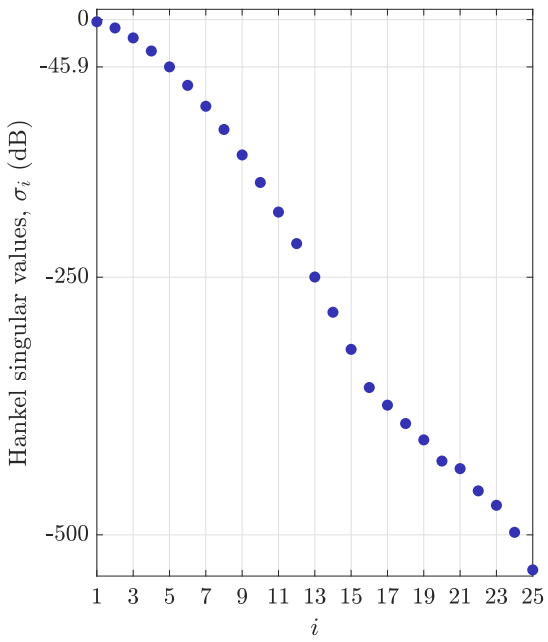
It has



Then

$$G_4(s) = \frac{24.986(s+3.196)(s^2+3.165s+19.48)}{(s^2+5.629s+24.58)(s^2+14.69s+63.86)}$$

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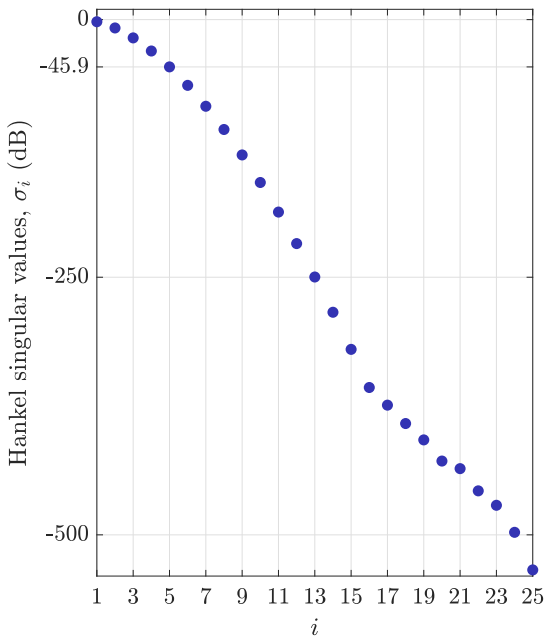

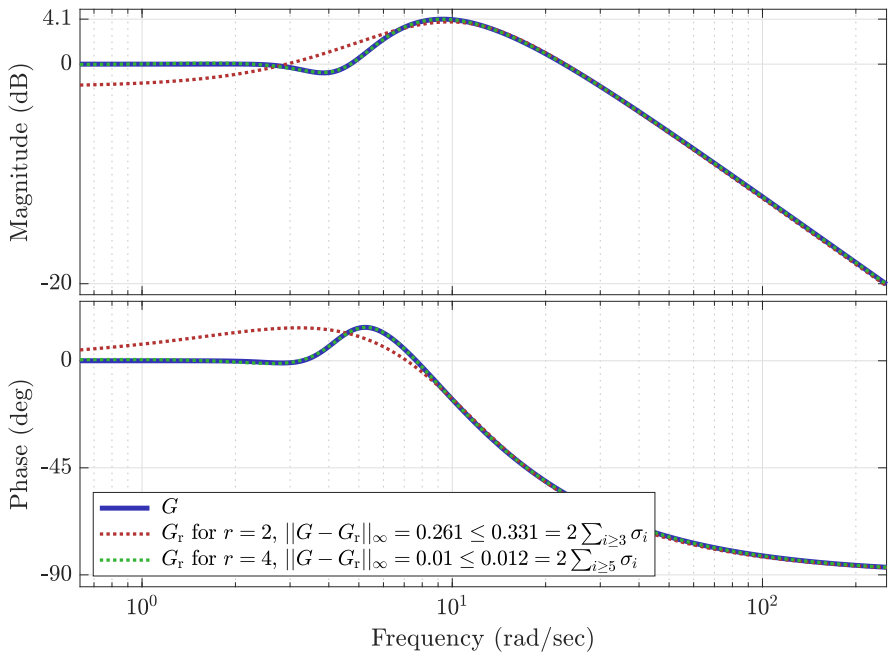

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Balanced truncation: example 5 (need for $\sigma_r > \sigma_{r+1}$)

Let

$$G(s) = \frac{(s-1)^2}{(s+1)^2}.$$

Its balance realization

$$G(s) = \left[\begin{array}{c:c|c} -1+\cos 2\theta & 1-\sin 2\theta & 2\sin\theta \\ \hdashline -1-\sin 2\theta & -1-\cos 2\theta & 2\cos\theta \\ \hline -2\sin\theta & -2\cos\theta & 1 \end{array}\right].$$

for every θ and $P = Q = I_2$. But

$$A_{11} = -1 + \cos 2\theta$$

is not Hurwitz if $\theta = \pi k$ for $k \in \mathbb{Z}$.