Linear Control Systems (036012)
chapter 3

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Outline

Basics

Signals are mappings from $\mathbb{R}$ (time) to $\mathbb{F}^n$ for $n \in \mathbb{N}$, denoted $x : \mathbb{R} \to \mathbb{F}^n$. The value of x at a time instance t is denoted $x(t) \in \mathbb{F}^n$. We say that x is

- scalar-valued if $n = 1$
- vector-valued if $n > 1$

Set of signals is a vector space, with

addition: $x = y + z$ reads $x(t) = y(t) + z(t)$ for at t

multiplication by scalar: $x = \alpha y$ reads $x(t) = \alpha y(t)$ for all t

Another important operation is

shift: $\mathbb{S}_\tau x$ reads $(\mathbb{S}_\tau x)(t) = x(t + \tau)$ for all t

The set

$$\text{supp}(x) := \{t \in \mathbb{R} \mid x(t) \neq 0\}$$

is called the support of x.

Normed (Banach) spaces

Admissibility $\to$ signal spaces:

$$L_1^n(\mathbb{R}) := \left\{ x : \mathbb{R} \to \mathbb{F}^n \mid \|x\|_1 := \int_{\mathbb{R}} \|x(t)\|_1 \mathrm{d}t < \infty \right\}$$

$$L_\infty^n(\mathbb{R}) := \left\{ x : \mathbb{R} \to \mathbb{F}^n \mid \|x\|_\infty := \sup_{t \in \mathbb{R}} \|x(t)\|_\infty < \infty \right\}$$

$$L_2^n(\mathbb{R}) := \left\{ x : \mathbb{R} \to \mathbb{F}^n \mid \|x\|_2^2 := \int_{\mathbb{R}} \|x(t)\|^2 \mathrm{d}t < \infty \right\}$$

$$L_{2+}^n(\mathbb{R}) := \left\{ x \in L_2^n(\mathbb{R}) \mid x(t) = 0 \text{ for all } t < 0 \right\}$$

$$L_{2-}^n(\mathbb{R}) := \left\{ x \in L_2^n(\mathbb{R}) \mid x(t) = 0 \text{ for all } t > 0 \right\}$$

L_2 is Hilbert, with

$$\langle x, y \rangle_2 := \int_{\mathbb{R}} y'(t) x(t) \mathrm{d}t$$

Fourier transform

Defined

$$\mathfrak{F}\{x\} = X(j\omega) := \int_{\mathbb{R}} x(t)e^{-j\omega t}dt,$$

where $\omega \in \mathbb{R}$ called the (angular) frequency and measured in rad/sec. $\mathfrak{F}\{x\}$ is the frequency-domain representation or spectrum of x.

$$\mathfrak{F}^{-1}\{X\} = x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} X(j\omega)e^{j\omega t}d\omega,$$

i.e. x is a superposition, weighted by $X(j\omega)$, of harmonic signals $\exp_{j\omega}$ with frequencies ω.

Fourier transform (contd)

Existence:
  – well defined (absolute convergence) for $x \in L_1$
  – extendible (weaker convergence) for $x \in L_2$ by the Plancherel Theorem

$\mathfrak{F}\{\cdot\}$ is a unitary mapping $L_2(\mathbb{R}) \to L_2(j\mathbb{R})$, i.e. it preserves sizes:

$$\underbrace{\int_{\mathbb{R}} \|x(t)\|^2 dt}_{\|x\|_2^2} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \|X(j\omega)\|^2 d\omega}_{\|X\|_2^2}$$

It also preserves angles:

$$\underbrace{\int_{\mathbb{R}} [x_2(t)]'x_1(t)dt}_{\langle x_1, x_2 \rangle} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} [X_2(j\omega)]'X_1(j\omega)d\omega}_{\langle X_1, X_2 \rangle_2}$$

(Parseval's theorem, in engineering literature).

Laplace transform

The two-sided (bilateral) Laplace transform:

$$\mathfrak{L}\{x\} = F(s) := \int_{\mathbb{R}} x(t)e^{-st}dt$$

at all $s \in \mathbb{C}$ where the integral converges (region of convergence, aka RoC). We need $x \exp_{-s} \in L_1$ for absolute convergence.

Normally used for signals supported on semi-axes:
  – if $\mathrm{supp}(x) \subset \mathbb{R}_+$, then RoC is typically $\mathbb{C}_\alpha$ (may be $\bar{\mathbb{C}}_\alpha$)
  – if $\mathrm{supp}(x) \subset \mathbb{R}_-$, then RoC is typically $\mathbb{C} \setminus \bar{\mathbb{C}}_\alpha$ (may be $\mathbb{C} \setminus \mathbb{C}_\alpha$)
for some $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$.

Laplace transform: mind RoC

Readily verifiable that
  – $x(t) = x_+(t) := e^{-t}\mathbb{1}(t) =$ $\implies X(s) = \dfrac{1}{s+1}$
  – $x(t) = x_-(t) := -e^{-t}\mathbb{1}(-t) =$ $\implies X(s) = \dfrac{1}{s+1}$

i.e.
  – X alone does not define x unambiguously.

If the knowledge of X is complemented by its RoC:
  – $X(s) = \dfrac{1}{s+1}$ and RoC $= \mathbb{C}_{-1}$ $\implies x = x_+$
  – $X(s) = \dfrac{1}{s+1}$ and RoC $= \mathbb{C} \setminus \mathbb{C}_{-1}$ $\implies x = x_-$

Paley–Wiener and H_2 space

$\mathfrak{L}\{\cdot\}$ is a unitary mapping $L_{2+}(\mathbb{R}) \to H_2$, where

$$H_2^n := \left\{ X : \mathbb{C}_0 \to \mathbb{C}^n \,\middle|\, X(s) \text{ is holomorphic in } \mathbb{C}_0 \text{ and} \right.$$
$$\left. \|X\|_2 := \sup_{\sigma > 0} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|X(\sigma + j\omega)\|^2 \, d\omega \right)^{1/2} < \infty \right\}$$

Important:

 – the boundary function $\tilde{X}(j\omega) := \lim_{\sigma \downarrow 0} X(\sigma + j\omega)$ exists for almost all ω and $\|\tilde{X}\|_2 = \|X\|_2$ (so $\tilde{X} \in L_2(j\mathbb{R})$)
 – H_2 functions are identified with boundary functions $\implies H_2 \subset L_2(j\mathbb{R})$
 – H_2-norm (provided $X \in H_2$):

$$\|X\|_2 = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|X(j\omega)\|^2 \, d\omega \right)^{1/2}$$

Outline

Linear systems on L_2

Linear operators

$$G : \mathfrak{D}_G \subset L_2^m \to L_2^p$$

for some domain $\mathfrak{D}_G$.

Sufficiently general class of $G : u \mapsto y$ satisfies

$$y(t) = \int_{\mathbb{R}} g(t, s) u(s) \, ds$$

where $g : \mathbb{R}^2 \to \mathbb{R}^{p \times m}$ is the impulse response (kernel) of G, whose
 – $g_{\bullet j}(t, s)$ is $y(t)$ under $u(t) = e_j \delta(t - s)$.

Some systems $u \mapsto y$

G_{int} (integrator) $\dot{y}(t) = u(t) \implies g_{\text{int}}(t, s) = \mathbb{1}(t - s)$

G_{dint} (discrete integrator)

$$y(t) = y(t - T) + u(t) \implies g_{\text{dint}}(t, s) = \sum_{i \in \mathbb{N}} \delta(t - s - (i - 1)T)$$

G_{fmint} (finite-memory integrator)

$$y(t) = \int_{t-T}^{t} u(s) \, ds \implies g_{\text{fmint}}(t, s) = \mathbb{1}(t - s) - \mathbb{1}(t - s - T)$$

D_τ (τ-delay operator) $y(t) = u(t - \tau) \implies d_\tau(t, s) = \delta(t - s - \tau)$

F_{ilp} (ideal low-pass filter)

$$f_{\text{ilp}}(t, s) = \frac{\omega_b}{\pi} \operatorname{sinc}(\omega_b(t - s))$$

Causality

$G : u \mapsto y$ is causal if for every $t_c \in \mathbb{R}$

  – $y(t) = 0$ for all $t \le t_c$ whenever $u(t) = 0$ for all $t \le t_c$.

Roughly, it says that

  – $y(t)$ may only depend on past and present inputs u for all t.

Consequently,

  – causal systems may be thought of as $\mathfrak{D}_G \subset L_{2+}^m \to L_{2+}^p$.

Criterion:

$$y(t) = \int_{\mathbb{R}} g(t,s)u(s)\mathrm{d}s = \int_{t_c}^{\infty} g(t,s)u(s)\mathrm{d}s = 0, \quad \forall t < t_c, u \in \mathfrak{D}_G$$

whence

$$G \text{ is causal} \iff g(t,s) = 0 \text{ for all } s > t.$$

Remark: G is said to be anti-causal if y may only depend on future and present inputs u.
A linear G is anti-causal $\iff g(t,s) = 0$ for all $s < t$.

Time (shift) invariance

Linear $G : \mathfrak{D}_G \subset L_2^m \to L_2^p$ is time invariant (shift invariant) if

  – $G\mathbb{S}_\tau = \mathbb{S}_\tau G$ for all $\tau \in \mathbb{R}$

If G LTI, its impulse response $g(t,s) = (G\mathbb{S}_{-s}\delta)(t)$ and then

$$y(t) = \int_{\mathbb{R}} (G\mathbb{S}_{-s}\delta)(t)u(s)\mathrm{d}s = \int_{\mathbb{R}} (\mathbb{S}_{-s}G\delta)(t)u(s)\mathrm{d}s$$
$$= \int_{\mathbb{R}} (G\delta)(t-s)u(s)\mathrm{d}s,$$

i.e. only the response of G to δ applied at $t = 0$ matters. We then treat

  – $g : \mathbb{R} \to \mathbb{R}^{p \times m}$ (i.e. $g(t)$)

  – can write the response as the convolution integral

$$y(t) = \int_{\mathbb{R}} g(t-s)u(s)\mathrm{d}s =: g * u$$

Adjoint system

L_2 is Hilbert $\implies G$ has its adjoint G', defined via $\langle Gu, v \rangle = \langle u, G'v \rangle$. If

$$(Gu)(t) = \int_{\mathbb{R}} g(t,s)u(s)\mathrm{d}s,$$

then

$$\langle Gu, v \rangle = \int_{\mathbb{R}} v'(t)(Gu)(t)\mathrm{d}t = \int_{\mathbb{R}} v'(t) \int_{\mathbb{R}} g(t,s)u(s)\mathrm{d}s\,\mathrm{d}t$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} (v'(t)g(t,s))u(s)\mathrm{d}s\,\mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}} (g'(t,s)v(t))'u(s)\mathrm{d}t\,\mathrm{d}s$$
$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g'(t,s)v(t)\mathrm{d}t \right)' u(s)\mathrm{d}s = \int_{\mathbb{R}} \underbrace{\left(\int_{\mathbb{R}} g'(s,t)v(s)\mathrm{d}s \right)'}_{(G'v)(t)} u(t)\mathrm{d}t$$
$$= \langle u, G'v \rangle.$$

Thus, the impulse response of G' is $[g(s,t)]'$, or $[g(-t)]'$ if G is LTI.

  – G is causal $\implies G'$ is anti-causal.

Stability

Linear $G : \mathfrak{D}_G \subset L_2^m \to L_2^p$ is stable (L_2-stable)

  – $\mathfrak{D}_G = L_2^m$ and

  – $\|G\| := \sup_{\|u\|_2 = 1} \|Gu\|_2 < \infty$ (L_2-induced norm)

It is known (Young's convolution inequality) that

$$g \in L_1, u \in L_2 \implies g * u \in L_2 \text{ and } \|g * u\|_2 \le \|g\|_1 \|u\|_2.$$

Hence, if G is LTI, then it is

  – stable whenever $g \in L_1$.

But $g, u \in L_2$ might not imply that $g * u \in L_2$, so

  – $g \in L_2$ does not necessarily imply the stability of G.

Unfortunately,

  ∴ no N&S stability test in terms of the impulse response in general.

Outline

Basic property

Because

$$y = g * u \iff \mathfrak{L}\{y\} = \mathfrak{L}\{g\}\mathfrak{L}\{u\} \iff \mathfrak{F}\{y\} = \mathfrak{F}\{g\}\mathfrak{F}\{u\}$$

convolution representations become product in transformed domains, i.e. if G is LTI, then

$$y = Gu \iff Y(j\omega) = G(j\omega)U(j\omega)$$

whenever both g and u are Fourier transformable and

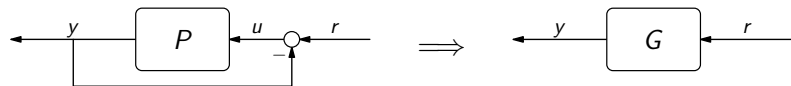$$y = Gu \iff Y(s) = G(s)U(s)$$

for all $s \in \text{RoC}(g) \cap \text{RoC}(u)$. Here

- $G(s) = (\mathfrak{L}\{g\})(s)$ is the transfer function of G
- $G(j\omega) = (\mathfrak{F}\{g\})(j\omega)$ is the frequency response of G

Beware of frequency response analysis: example

Consider

for

$$p(t) = -2\,e^{-t}\mathbb{1}(t) = $$

Then

$$y(t) = -2\int_{-\infty}^{t} e^{-(t-s)}u(s)\,ds \iff \dot{y}(t) = -y(t) - 2u(t).$$
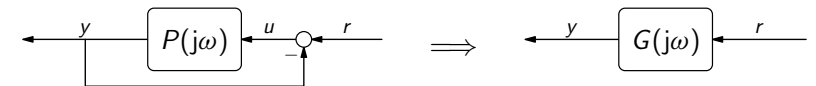
As $u(t) = r(t) - y(t)$,

$$\dot{y}(t) = y(t) - 2r(t) \iff y(t) = -2\int_{-\infty}^{t} e^{t-s}u(s)\,ds,$$

so G is causal, with $g(t) = -2e^t\mathbb{1}(t)$, and unstable.

Beware of frequency response analysis: example (contd)

In the frequency domain,

with $Y(j\omega) = P(j\omega)(R(j\omega) - Y(j\omega))$. Hence

$$G(j\omega) = \frac{P(j\omega)}{1 + P(j\omega)} = \frac{2}{1 - j\omega} \implies g(t) = 2\,e^t\mathbb{1}(-t) = $$

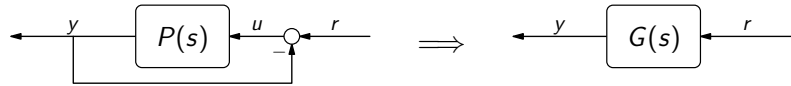This G is anti-causal and stable (for this $g \in L_1$), which makes no sense.

Hazards of analyzing systems in the Fourier domain:

- ☹ hard to cope with exponentially growing signals
- ☹ hard to trace causality

Feedback in Laplace domain: example

In the Laplace domain,

for

$$p(t) = -2\,e^{-t}\mathbb{1}(t) \quad \Longrightarrow \quad P(s) = -\frac{2}{s+1}$$

whose RoC $= \mathbb{C}_{-1}$ (includes $j\mathbb{R}$). Then, via $Y(s) = P(s)(R(s) - Y(s))$,

$$G(s) = \frac{P(s)}{1 + P(s)} = \frac{2}{1-s} \quad \Longrightarrow \quad g(t) = \begin{cases} 2\,e^t\mathbb{1}(-t) & \text{if RoC} = \mathbb{C} \setminus \bar{\mathbb{C}}_1 \\ -2\,e^t\mathbb{1}(t) & \text{if RoC} = \mathbb{C}_1 \end{cases}$$

It is not unreasonable to assume that

- causality is preserved under this feedback $\Longrightarrow$ RoC must remain a RHP

The correct impulse response can then be obtained immediately.

Systems in Laplace domain

Typically,

- control applications are concerned with causal systems
- impulse responses are supported in $\mathbb{R}_+$
- signals are assumed to have support in $\mathbb{R}_+$ too
- RoC's are $\mathbb{C}_\alpha$ for some $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$
- causal LTI systems treated as operators $G : \mathfrak{D}_G \subset L_{2+}^m \to L_{2+}^p$

Outcomes:

- ☺ causal LTI systems = transfer functions
- ☺ dynamical systems can be manipulated algebraically

Transfer function: examples

Systems we already saw:

- $g_{\text{int}}(t) = \mathbb{1}(t) \quad\Longrightarrow\quad G_{\text{int}}(s) = \frac{1}{s}$

- $g_{\text{dint}}(t) = \sum \delta(t - iT) \implies G_{\text{dint}}(s) = \sum e^{-siT} = \frac{1}{1 - e^{-sT}}$

- $g_{\text{fmint}}(t) = \mathbb{1}_{[0,T]}(t) \quad\Longrightarrow\quad G_{\text{fmint}}(s) = \frac{1 - e^{-sT}}{s}$

- $d_\tau(t) = \delta(t - \tau) \quad\Longrightarrow\quad D_\tau(s) = e^{-s\tau}$

whose RoC's are $\mathbb{C}_0$, $\mathbb{C}_0$, $\mathbb{C}$, and $\mathbb{C}$, respectively.

Causality + stability in Laplace domain

An LTI G is causal and stable iff its transfer function $G \in H_\infty^{p \times m}$, where

$$H_\infty^{p \times m} := \left\{ G : \mathbb{C}_0 \to \mathbb{C}^{p \times m} \mid G(s) \text{ is holomorphic and bounded in } \mathbb{C}_0 \right\}$$

Thus,

- $G H_2^m \subset H_2^p \iff G \in H_\infty^{p \times m}$
- if $p = m$ and $G, G^{-1} \in H_\infty^{m \times m}$, then $G H_2^m = H_2^m$

H_∞ is Banach, with $\|G\|_\infty := \sup_{s \in \mathbb{C}_0} \|G(s)\|$. Can be associated with its boundary function from

$$L_\infty^{p \times m}(j\mathbb{R}) := \left\{ G : j\mathbb{R} \to \mathbb{C}^{p \times m} \mid \|G\|_\infty := \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \|G(j\omega)\| < \infty \right\}$$

and $H_\infty \subset L_\infty(j\mathbb{R})$. Then, provided $G \in H_\infty$,

$$\|G\|_\infty = \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \|G(j\omega)\|.$$

Examples

- $G_{\text{int}} \notin H_\infty$ as $1/s$ is holomorphic but not bounded in $\mathbb{C}_0$
- $G_{\text{dint}} \notin H_\infty$ as $1/(1 - e^{-sT})$ is holomorphic but not bounded in $\mathbb{C}_0$
- $G_{\text{fmint}} \in H_\infty$ as $(1 - e^{-sT})/s$ is holomorphic and bounded in $\mathbb{C}_0$
  for every $s = (\sigma + j\omega)/T$,

$$
\begin{aligned}
|G_{\text{fmint}}(s)|^2 &= T^2 \left| \frac{1 - e^{-(\sigma + j\omega)}}{\sigma + j\omega} \right|^2 = T^2 \frac{1 - 2e^{-\sigma}\cos\omega + e^{-2\sigma}}{\sigma^2 + \omega^2} \\
&= T^2 \left(\frac{1 - e^{-\sigma}}{\sigma} \right)^2 - T^2 \frac{4\omega^2 e^{-\sigma}}{\sigma^2(\sigma^2 + \omega^2)} \left(\sinh^2\left(\frac{\sigma}{2}\right) - 2\frac{1 - \cos\omega}{\omega^2}\left(\frac{\sigma}{2}\right)^2 \right) \\
&\leq T^2 \left(\frac{1 - e^{-\sigma}}{\sigma} \right)^2 - T^2 \frac{4\omega^2 e^{-\sigma}(\sinh^2(\sigma/2) - (\sigma/2)^2)}{\sigma^2(\sigma^2 + \omega^2)} \\
&\leq T^2 \left(\frac{1 - e^{-\sigma}}{\sigma} \right)^2 < T^2
\end{aligned}
$$

  where $2(1 - \cos\omega)/\omega^2 \leq 1$ and $\sinh^2 x > x^2$ for all $x \neq 0$ were used

- $D_\tau \in H_\infty$ as $e^{-s\tau}$ is holomorphic and bounded in $\mathbb{C}_0$
  for every $s = \sigma + j\omega$, $|e^{-(\sigma + j\omega)\tau}| = e^{-\sigma\tau} \leq 1$

Causality + stability and system poles

Let

$$
G(s) = \frac{1}{1 + s + se^{-s}}.
$$

Its poles are all in the OLHP $\mathbb{C} \setminus \bar{\mathbb{C}}_0$. To see this, let $s = \sigma + j\omega$ be a pole, then $1 + \sigma + j\omega + (\sigma + j\omega)e^{-\sigma - j\omega} = 0$ reads

$$
e^{-\sigma}e^{-j\omega} = -1 - \frac{1}{\sigma + j\omega} = -\left(1 + \frac{\sigma}{\sigma^2 + \omega^2}\right) + j\frac{\omega}{\sigma^2 + \omega^2}.
$$

Hence, σ must satisfy

$$
e^{-\sigma} = \left| \left(1 + \frac{\sigma}{\sigma^2 + \omega^2}\right) - j\frac{\omega}{\sigma^2 + \omega^2} \right| \geq \left| 1 + \frac{\sigma}{\sigma^2 + \omega^2} \right| \geq 1.
$$

which is a contradiction for all $\sigma > 0$. If $\sigma = 0$, we have $1 = 1 + \frac{1}{|\omega|}$, which also holds for none $\omega \in \mathbb{R}$.

Causality + stability and system poles (contd)

But $G \notin H_\infty$. To see this, let $\{s_k\} \in \mathbb{C} \setminus \bar{\mathbb{C}}_0$ be a sequence of poles of $G(s)$ satisfying

$$
s_k + 1 + s_k e^{-s_k} = 0, \quad \text{with } \lim_{k \to \pm\infty} |s_k| = \infty :
$$

(known to exist). Then

$$
G(-s_k) = \frac{1}{1 - s_k - s_k e^{s_k}} = \frac{1}{1 - s_k + s_k^2/(1 + s_k)} = 1 + s_k.
$$

so there is a sequence $\{s_k\}$ in $\mathbb{C}_0$ such that $\lim_{k \to \infty} |G(-s_k)| = \infty$. Hence,

- G is not L_2-stable, despite having all poles in the OLHP

(curiously, $\frac{1}{s+1}G(s)$ is an H_∞ transfer function).

H_2 system space

Defined as

$$
H_2^{p \times m} := \left\{ G : \mathbb{C}_0 \to \mathbb{C}^{p \times m} \,\middle|\, G(s) \text{ is holomorphic in } \mathbb{C}_0 \text{ and} \right.
$$

$$
\left. \|G\|_2 := \sup_{\sigma > 0} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(\sigma + j\omega)\|_F^2 \, d\omega \right)^{1/2} < \infty \right\}
$$

With the good ol' boundary function trick, $H_2 \subset L_2(j\mathbb{R})$ and if $G \in H_2$,

$$
\|G\|_2 = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(j\omega)\|_F^2 \, d\omega \right)^{1/2}
$$

and H_2 inherits the inner product from $L_2(j\mathbb{R})$.

Examples

- $G_{\text{int}} \notin H_2$ as $1/s$ is holomorphic in $\mathbb{C}_0$ but

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{\sigma^2 + \omega^2} = \frac{1}{2\sigma} \xrightarrow{\sigma \downarrow 0} \infty$$

  (simpler, $\|\mathbb{1}\|_2 = \infty$)

- $G_{\text{dint}} \notin H_2$ for similar reasons
- $G_{\text{fmint}} \in H_2$ as $(1 - e^{-sT})/s$ is holomorphic in $\mathbb{C}_0$ and

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 - 2e^{-\sigma T}\cos(\omega T) + e^{-2\sigma T}}{\sigma^2 + \omega^2}\,\mathrm{d}\omega = \frac{1 - e^{-2\sigma T}}{2\sigma} < T$$

  (simpler, $\|\mathbb{1}_{[0,T]}\|_2 = \sqrt{T} < \infty$)

- $D_\tau \notin H_2$ as $e^{-s\tau}$ is holomorphic in $\mathbb{C}_0$ but

$$\frac{e^{-2\sigma\tau}}{2\pi} \int_{\mathbb{R}} \mathrm{d}\omega = \infty$$

H_2 system space (contd)

Is Hilbert, with

$$\langle G_1, G_2 \rangle_2 := \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{tr}\big([G_2(\mathrm{j}\omega)]' G_1(\mathrm{j}\omega)\big)\,\mathrm{d}\omega = \int_{\mathbb{R}} \mathrm{tr}\big([g_2(t)]' g_1(t)\big)\,\mathrm{d}t.$$

Usage:

- unrelated to stability ($D_\tau \in H_\infty$ but $D_\tau \notin H_2$, may be vice versa)
- popular performance measure (LQG, Kalman filtering)
  - $\|G\|_2^2$ equals the energy of $y = G\delta$
  - if u is Gaussian unit-intensity white, $\|G\|_2^2$ equals the variance of $y = Gu$

Properness

$G(s)$ is

- proper if $\exists \alpha \geq 0$ such that $\sup_{s \in \mathbb{C}_\alpha} \|G(s)\| < \infty$
- strictly proper if $\exists \alpha \geq 0$ such that $\lim_{|s| \to \infty, s \in \mathbb{C}_\alpha} \|G(s)\| = 0$

Examples:

- $G_{\text{int}}(s)$ is strictly proper (and thus proper)
- $G_{\text{dint}}(s)$ is proper but not strictly proper

$$\frac{1}{1 + e^{-\sigma T}} \leq \frac{1}{|1 - e^{-(\sigma + \mathrm{j}\omega)T}|} \leq \frac{1}{1 - e^{-\sigma T}}$$

- $G_{\text{fmint}}(s)$ is strictly proper (and thus proper)
- $D_\tau(s)$ is proper but not strictly proper
  as $|e^{-(\sigma + \mathrm{j}\omega)\tau}| = e^{-\sigma\tau} > 0$ for all finite $\sigma > 0$

Important:

- $G \in H_\infty \implies G(s)$ is proper $\implies$ stable causal G have proper t.f.'s
- $G \in H_2 \implies G(s)$ is strictly proper

Conjugate transfer function

If G is LTI, its adjoint G' has impulse response $[g(-t)]'$ and

$$\mathcal{L}\{g'\} = \int_{\mathbb{R}} [g(-t)]' e^{-st}\,\mathrm{d}t = \left[\int_{\mathbb{R}} g(t) e^{-(-\bar{s})t}\,\mathrm{d}t\right]' = [G(-\bar{s})]'.$$

with RoC in $\mathbb{C} \setminus \bar{\mathbb{C}}_\alpha$. Thus, the transfer function of G' is

$$G^\sim(s) := [G(-\bar{s})]',$$

known as conjugate transfer function and verifying $G^\sim(\mathrm{j}\omega) = [G(\mathrm{j}\omega)]'$.

Usage:

- mostly in analysis
- limited to systems operating over the whole $\mathbb{R}$
  convolution theorem doesn't hold for non-causal systems if considered on $L_2(\mathbb{R}_+)$

Inner and co-inner transfer functions

$G \in H_\infty^{p \times m}$ is
- inner if $G^\sim(s)G(s) = I_m$ (so $p \geq m$)
- co-inner if $G(s)G^\sim(s) = I_p$ (so $p \leq m$)

If $G(s)$ is inner, the system G is an isometry on $L_2(\mathbb{R})$:

$$\|Gu\|_2^2 = \|GU\|_2^2 = \langle GU, GU \rangle_2 = \langle G^\sim GU, U \rangle_2 = \langle U, U \rangle_2^2 = \|U\|_2^2 = \|u\|_2^2$$

and if $G(s)$ is co-inner, the system G' is an isometry on $L_2(\mathbb{R})$.

If $W_{\mathrm{i}}(s)$ and $W_{\mathrm{ci}}(s)$ are inner and co-inner, then
- $\|G\|_\infty = \|W_{\mathrm{i}} G W_{\mathrm{ci}}\|_\infty$ for all $G \in H_\infty$
- $\|G\|_2 = \|W_{\mathrm{i}} G W_{\mathrm{ci}}\|_2$ for all $G \in H_2$

Outline

Coprimeness over H_∞

$M \in H_\infty^{m \times m}$ and $N \in H_\infty^{p \times m}$ are (strongly) right coprime over H_∞ if there are Bézout factors $X \in H_\infty^{m \times m}$ and $Y \in H_\infty^{m \times p}$ satisfying

$$\begin{bmatrix} X(s) & Y(s) \end{bmatrix} \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = X(s)M(s) + Y(s)N(s) = I_m$$

(Bézout equality). Implies left invertibility of $\begin{bmatrix} M \\ N \end{bmatrix}$ over H_∞.

$\tilde{M} \in H_\infty^{p \times p}$ and $\tilde{N} \in H_\infty^{p \times m}$ are (strongly) left coprime over H_∞ if there are Bézout factors $\tilde{X} \in H_\infty^{p \times p}$ and $\tilde{Y} \in H_\infty^{m \times p}$ satisfying

$$\begin{bmatrix} \tilde{M}(s) & \tilde{N}(s) \end{bmatrix} \begin{bmatrix} \tilde{X}(s) \\ \tilde{Y}(s) \end{bmatrix} = \tilde{M}(s)\tilde{X}(s) + \tilde{N}(s)\tilde{Y}(s) = I_p$$

(Bézout equality). Implies right invertibility of $\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}$ over H_∞.

If $p = m = 1$, then "left coprime" $\iff$ "right coprime" (so simply coprime).

Corona theorem

$M \in H_\infty^{m \times m}$ and $N \in H_\infty^{p \times m}$ are (strongly) right coprime over H_∞ iff

$$\inf_{s \in \mathbb{C}_0} \underline{\sigma} \left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} \right) > 0.$$

$\tilde{M} \in H_\infty^{p \times p}$ and $\tilde{N} \in H_\infty^{p \times m}$ are (strongly) left coprime over H_∞ iff

$$\inf_{s \in \mathbb{C}_0} \underline{\sigma} \left(\begin{bmatrix} \tilde{M}(s) & \tilde{N}(s) \end{bmatrix} \right) > 0.$$

Thus,
- $M(s) = \dfrac{1}{s+1}$ and $N(s) = \dfrac{se^{-s}}{s+1}$ are not coprime
- $M(s) = \dfrac{e^{-s}}{s+1}$ and $N(s) = \dfrac{s}{s+1}$ are coprime

Coprime factorization

Effectively every stabilizable transfer function can be expressed as

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

for right / left coprime M, N / $\tilde{M}, \tilde{N} \in H_\infty$ and bi-proper $M(s)$ and $\tilde{M}(s)$.

Examples:

- $G_{\text{int}}(s) = \frac{1}{s} = \frac{1}{s+a} \cdot \left(\frac{s}{s+a}\right)^{-1}$, $a > 0$, with $X(s) = 1$ and $Y(s) = a$
- $G_{\text{dint}}(s) = \frac{1}{1-e^{-sT}} = 1 \cdot (1 - e^{-sT})^{-1}$, with $X(s) = 1$ and $Y(s) = e^{-sT}$
- $G_{\text{fmint}}(s) = \frac{1-e^{-sT}}{s} = \frac{1-e^{-sT}}{s} \cdot 1^{-1}$, with $X(s) = 1$ and $Y(s) = 0$
- $D_\tau(s) = e^{-s\tau} = e^{-s\tau} \cdot 1^{-1}$, with $X(s) = 1$ and $Y(s) = 0$

Constructing coprime factors:

- if $G \in H_\infty$, then $M(s) = I$, $N(s) = G(s)$, $X(s) = I$, and $Y(s) = 0$
- if $G \notin H_\infty$, wait for state space

Two lemmas

Lemma
If $N_1 M_1^{-1} = N_2 M_2^{-1}$ and $\tilde{M}_1^{-1}\tilde{N}_1 = \tilde{M}_2^{-1}\tilde{N}_2$ are rcf's and lcf's of some G, respectively, then

$$\begin{bmatrix} M_2 \\ N_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} U \quad \text{and} \quad \begin{bmatrix} \tilde{M}_2 \ \tilde{N}_2 \end{bmatrix} = \tilde{U} \begin{bmatrix} \tilde{M}_1 \ \tilde{N}_1 \end{bmatrix}$$

for some $U, U^{-1}, \tilde{U}, \tilde{U}^{-1} \in H_\infty$.

Implies that

- if $\det M_1(s_0) = 0$ for $s_0 \in \mathbb{C}_0$, then $\det M_2(s_0) = 0$ for any other *rcf*
- if $\det \tilde{M}_1(s_0) = 0$ for $s_0 \in \mathbb{C}_0$, then $\det \tilde{M}_2(s_0) = 0$ for any other *lcf*

Lemma
If $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ are rcf and lcf, respectively, then

$$G \in H_\infty \iff M^{-1} \in H_\infty \iff \tilde{M}^{-1} \in H_\infty.$$

Domain of L_2 systems

If $G : \mathfrak{D}_G \subset L_2^m \to L_2^p$ is LTI and such that its transfer function admits a *rcf* over H_∞, $G(s) = N(s)M^{-1}(s)$, then

$$\mathfrak{D}_G = ML_2^m = \text{Im } M = \{u \mid \exists v \in L_2^m \text{ such that } u = Mv\}.$$

Proof (outline).

- $M \in H_\infty \implies ML_2^m \subset L_2^m$
- $G ML_2^m = NL_2^m \subset L_2^p \implies ML_2^m \subset \mathfrak{D}_G$
- For any $u_0 \in \mathfrak{D}_G$, denote $v_0 := M^{-1}u_0$. We have:

$$L_2^{m+p} \ni \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} I \\ G \end{bmatrix} u_0 = \begin{bmatrix} M \\ N \end{bmatrix} v_0$$

Thus,

$$v_0 = Xu_0 + Yy_0 \in L_2^m \implies \mathfrak{D}_G \subset ML_2^m \qquad \square$$

Doubly coprime factorization

Coprime factors of $G(s)$ and their Bézout can always be selected so that

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix},$$

i.e.

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix}$$

are invertible in H_∞.

Outline

Definition

We say that $G(s)$ is real-rational if

- $G_{ij}(s) = \frac{N_{ij}(s)}{M_{ij}(s)}$ for finite polynomials $N_{ij}(s)$ and $M_{ij}(s)$ with real coeff's.

Examples:

- $G_{\text{int}}(s) = \frac{1}{s}$ is real-rational
- $G_{\text{dint}}(s) = \frac{1}{1 - e^{-sT}}$ is not real-rational
- $G_{\text{fmint}}(s) = \frac{1 - e^{-sT}}{s}$ is not real-rational
- $D_\tau(s) = e^{-s\tau}$ is not real-rational

Implications

Any real-rational $G(s)$

- is proper iff $\|G(\infty)\| < \infty$, i.e. $\deg(N_{ij}(s)) \le \deg(M_{ij}(s))$, $\forall i, j$
- is strictly proper iff $\|G(\infty)\| = 0$, i.e. $\deg(N_{ij}(s)) < \deg(M_{ij}(s))$, $\forall i, j$
- $G \in H_\infty$ iff $G(s)$ is proper & has no poles in $\bar{\mathbb{C}}_0$ called RH_∞
- $G \in H_2$ iff $G(s)$ is strictly proper & has no poles in $\bar{\mathbb{C}}_0$ called RH_2
- admits doubly coprime factorizations over RH_∞

By-products:

- stability $\iff$ proper + no poles in $\bar{\mathbb{C}}_0$
- $RH_2 \subset RH_\infty$
- always stabilizable by feedback

Outline

Diagonal case: poles, zeros, and ...

Every

$$
G(s) = \begin{bmatrix} G_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_m(s) \end{bmatrix} =: \mathrm{diag}\{G_i(s)\}
$$

is effectively a union of m independent systems, so that

- poles and zeros of $G(s)$ are unions of poles and zeros of $G_i(s)$.

Consequences:

- may have uncancellable pole(s) and zero(s) at the same point
- $\det(G(s))$ might be a poor indicator of its dynamical properties
- mere location of poles and zeros is not sufficient

Diagonal case: poles, zeros, and ... (contd)

Poles and zeros of

$$
G(s) = \begin{bmatrix} G_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_m(s) \end{bmatrix}
$$

should be

- ~~complemented by their association with subsystems~~
- complemented by their directions
    - if p_k (z_k) is a pole (zero) of $G_i(s)$, its direction is $\mathrm{span}(e_i)$
    - if p_k (z_k) is a pole (zero) of $G_i(s)$ and $G_j(s)$, its direction is $\mathrm{span}(e_i, e_j)$
    - pole direction of p_k: $\perp$ to any v for which $G(s)v$ has no pole at p_k
    - zero direction of z_k: span of all v for which $G(s)v|_{s \to z_k} = 0$
    - if p_k (z_k) is a pole (zero) of μ_k subsystems, its geometric multiplicity is μ_k
    - the multiplicity of p_k (z_k) in $G_i(s)$ is its ith partial multiplicity
    - the sum of all partial multiplicities of p_k is its algebraic multiplicity

General case: preliminaries

- normal rank: $\mathrm{nrank}(G(s)) := \max_{s \in \mathbb{C}} \mathrm{rank}(G(s))$
  if $G(s)$ is proper, then $\mathrm{rank}(G(s)) = \mathrm{nrank}(G(s))$ for all but a finitely many s

- unimodular polynomial matrix: square and $\det(U(s)) = \mathrm{const} \neq 0$
  $U^{-1}(s)$ is also a polynomial matrix

- polynomial $\beta(s)$ divides polynomial $\alpha(s)$ if $\frac{\alpha(s)}{\beta(s)}$ is a polynomial

Smith–McMillan form & poles / degree / zeros

Given a $p \times m$ transfer function $G(s)$ having $\mathrm{nrank}(G(s)) = r \leq \min\{p, m\}$, there are unimodular polynomial matrices $U(s)$ and $V(s)$ such that

$$
U(s)G(s)V(s) = \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},
$$

where $\alpha_i(s)$ divides $\alpha_{i+1}(s)$, $\beta_{i+1}(s)$ divides $\beta_i(s)$, and $\alpha_i(s)$ and $\beta_i(s)$ are coprime at every $i \in \mathbb{Z}_{1..r}$.

- roots of $\phi_{\mathrm{p}}(s) := \prod_{i=1}^{r} \beta_i(s)$ are the poles of $G(s)$

- $n := \deg(\phi_{\mathrm{p}}(s))$ is the McMillan degree (or degree) of $G(s)$

- roots of $\phi_{\mathrm{z}}(s) := \prod_{i=1}^{r} \alpha_i(s)$ are the transmission zeros (or zeros) of $G(s)$

Pole directions

Let $p_i \in \mathbb{C}$ be a pole of geometric multiplicity μ_i of

$$G(s) = U^{-1}(s) \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} V^{-1}(s)$$

input pole direction, $\mathrm{pdir}_i(G, p_i) \subset \mathbb{C}^m$:

$$\mathrm{pdir}_i(G, p_i) = \left(\mathrm{Im}\, V(p_i) \begin{bmatrix} e_{\mu_i+1} & \cdots & e_m \end{bmatrix} \right)^{\perp} = \ker \begin{bmatrix} e'_{\mu_i+1} \\ \vdots \\ e'_m \end{bmatrix} [V(p_i)]'$$

output pole direction, $\mathrm{pdir}_o(G, p_i) \subset \mathbb{C}^p$:

$$\mathrm{pdir}_o(G, p_i) = \ker \begin{bmatrix} \tilde{e}'_{\mu_i+1} \\ \vdots \\ \tilde{e}'_p \end{bmatrix} U(p_i) = \left(\mathrm{Im}[U(p_i)]' \begin{bmatrix} \tilde{e}_{\mu_i+1} & \cdots & \tilde{e}_p \end{bmatrix} \right)^{\perp}$$

Zero directions

Let $z_i \in \mathbb{C}$ be a pole of geometric multiplicity μ_i of

$$G(s) = U^{-1}(s) \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} V^{-1}(s)$$

input zero direction, $\mathrm{zdir}_i(G, p_i) \subset \mathbb{C}^m$:

$$\mathrm{zdir}_i(G, z_i) := \mathrm{Im}\, V(z_i) \begin{bmatrix} e_{r-\mu_i+1} & \cdots & e_m \end{bmatrix}$$

output zero direction, $\mathrm{zdir}_o(G, p_i) \subset \mathbb{C}^p$:

$$\mathrm{zdir}_o(G, p_i) := \mathrm{Im}[U(z_i)]' \begin{bmatrix} \tilde{e}_{r-\mu_i+1} & \cdots & \tilde{e}_p \end{bmatrix}$$

Example 1

Let

$$G(s) = \frac{1}{s} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \implies \overbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}^{U(s)} G(s) \overbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}^{V(s)} = \begin{bmatrix} 1/s & 0 \\ 0 & 0 \end{bmatrix}$$

One pole at $s = 0$, with

$$\mathrm{pdir}_i(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$\mathrm{pdir}_o(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Example 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \implies \overbrace{\begin{bmatrix} 1 & 0 \\ s & -1 \end{bmatrix}}^{U(s)} G(s) \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & -s \end{bmatrix}}^{V(s)} = \begin{bmatrix} 1/s & 0 \\ 0 & s \end{bmatrix}$$

One pole and one transmission zero at $s = 0$, with

$$\mathrm{pdir}_i(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\mathrm{pdir}_o(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\mathrm{zdir}_i(G, 0) = \mathrm{Im}\, V(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \perp \mathrm{pdir}_i(G, 0)$$

$$\mathrm{zdir}_o(G, 0) = \mathrm{Im}[U(0)]' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \perp \mathrm{pdir}_o(G, 0)$$

Example 3

Let
$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

and define unimodular polynomials

$$U(s) = \frac{1}{6}\begin{bmatrix} 3 & 3 \\ s^3 - s^2 - 4s - 2 & s^3 - s^2 - 4s + 4 \end{bmatrix},$$

$$V(s) = \frac{1}{6}\begin{bmatrix} 2(s-2) & -6(s-1) & -3(s-1) \\ 4 & -24 & -6(s+2) \\ 0 & 6 & 3(s+2) \end{bmatrix}.$$

Then
$$U(s)G(s)V(s) = \begin{bmatrix} \frac{1}{(s^2-1)(s+2)} & 0 & 0 \\ 0 & \frac{s-1}{s+2} & 0 \end{bmatrix}.$$

Four poles, at $\{-2, -2, -1, 1\}$, and one transmission zero at $\{1\}$.

Example 3 (contd)

Pole directions:

$$\text{pdir}_i(G, 1) = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}[V(1)]' = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right),$$

$$\text{pdir}_o(G, 1) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix}U(1) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right),$$

$$\text{pdir}_i(G, -1) = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}[V(-1)]' = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right),$$

$$\text{pdir}_o(G, -1) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix}U(-1) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right),$$

$$\text{pdir}_i(G, -2) = \ker \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}[V(-2)]' = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right),$$

and $\text{pdir}_o(G, -2) = \mathbb{C}^2$.

Example 3 (contd)

Zero directions:

$$\text{zdir}_i(G, 1) = \text{Im } V(1)\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\text{zdir}_o(G, 1) = \text{Im}[U(1)]'\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

Again,

$$\text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \text{zdir}_i(G, 1) \perp \text{pdir}_i(G, 1) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \text{zdir}_o(G, 1) \perp \text{pdir}_o(G, 1) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Example 4

Let
$$G(s) = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}.$$

Its Smith–McMillan form is

$$\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} G(s) \begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} 1/s^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Double pole at $s = 0$, with

$$\text{pdir}_i(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix}[V(0)]' = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

and

$$\text{pdir}_o(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix}U(0) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

Although $e_1 \perp \text{pdir}_i(G, 0)$, $G(s)e_1 = 1/s\, e_1$, i.e. it still has a pole at $s = 0$.

Simplifications

Let $\text{nrank}(G(s)) = r$. The following statements hold true:

1. $\phi_p(s)$ is the least common denominator of all nonzero minors of $G(s)$ of all orders provided all common poles and zeros in each of these minors were canceled.

2. $\phi_z(s)$ is the greatest common divisor of all the numerators of all r-order minors of $G(s)$ provided these minors have been adjusted to have $\phi_p(s)$ as their denominators.

Example 3 (contd)

For
$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

nonzero minors of order 1 are
$$\frac{1}{s+1}, \quad \frac{s-1}{(s+1)(s+2)}, \quad -\frac{1}{s-1}, \quad \frac{1}{s+2}, \quad \text{and} \quad \frac{1}{s+2}$$

and the minors of order 2 are
$$-\frac{s-1}{(s+1)(s+2)^2}, \quad \frac{2}{(s+1)(s+2)}, \quad \text{and} \quad \frac{1}{(s+1)(s+2)}.$$

Hence,
$$\phi_p(s) = (s+2)^2(s+1)(s-1) = (s+2)^2(s^2-1),$$

as before.

Example 3 (contd)

For
$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

the minors of order 2 are:
$$-\frac{s-1}{(s+1)(s+2)^2}, \quad \frac{2}{(s+1)(s+2)}, \quad \text{and} \quad \frac{1}{(s+1)(s+2)}$$

or, equivalently, with $\phi_p(s) = (s+2)^2(s+1)(s-1)$
$$-\frac{(s-1)^2}{\phi_p(s)}, \quad \frac{2(s+2)(s-1)}{\phi_p(s)}, \quad \text{and} \quad \frac{(s+2)(s-1)}{\phi_p(s)}.$$

Hence,
$$\phi_z(s) = s-1,$$

as before.

Simplifications (contd)

Let $G(s)$ be a $p \times m$ real-rational proper transfer function.

1. If $z_i \in \mathbb{C}$ isn't a pole of $G(s)$, then it's a transmission zero of $G(s)$ iff $\text{rank}(G(z_i)) < \text{nrank}(G(s))$ and $\text{nrank}(G(s)) - \text{rank}(G(z_i))$ equals the geometric multiplicity of the zero at z_i, with
$$\text{zdir}_i(G, z_i) = \ker G(z_i) \quad \text{and} \quad \text{zdir}_o(G, z_i) = \ker[G(z_i)]'.$$

2. If $p = m = \text{nrank}(G(s))$ and $p_i \in \mathbb{C}$ isn't a transmission zero of $G(s)$, it's a pole of $G(s)$ iff $\det(G^{-1}(p_i)) = 0$ and $m - \text{rank}(G^{-1}(p_i))$ equals the geometric multiplicity of the pole at p_i, with
$$\text{pdir}_i(G, p_i) = \ker[G^{-1}(p_i)]' \quad \text{and} \quad \text{pdir}_o(G, p_i) = \ker G^{-1}(p_i).$$