Linear Control Systems (036012)
chapter 3

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Outline

Basics

Signals are mappings from $\mathbb{R}$ (time) to $\mathbb{F}^n$ for $n \in \mathbb{N}$, denoted $x : \mathbb{R} \to \mathbb{F}^n$. The value of x at a time instance t is denoted $x(t) \in \mathbb{F}^n$. We say that x is

- scalar-valued if $n = 1$
- vector-valued if $n > 1$

Set of signals is a vector space, with

addition: $x = y + z$ reads $x(t) = y(t) + z(t)$ for at t

multiplication by scalar: $x = \alpha y$ reads $x(t) = \alpha y(t)$ for all t

Another important operation is

shift: $\mathbb{S}_\tau x$ reads $(\mathbb{S}_\tau x)(t) = x(t + \tau)$ for all t

The set

$$\operatorname{supp}(x) := \{t \in \mathbb{R} \mid x(t) \neq 0\}$$

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Normed (Banach) spaces

Admissibility $\rightarrow$ signal spaces:

$$L_1^n(\mathbb{R}) := \left\{ x : \mathbb{R} \rightarrow \mathbb{F}^n \,\middle|\, \|x\|_1 := \int_{\mathbb{R}} \|x(t)\|_1 \mathrm{d}t < \infty \right\}$$

$$L_\infty^n(\mathbb{R}) := \left\{ x : \mathbb{R} \rightarrow \mathbb{F}^n \,\middle|\, \|x\|_\infty := \sup_{t \in \mathbb{R}} \|x(t)\|_\infty < \infty \right\}$$

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$$L_{2+}^n(\mathbb{R}) := \left\{ x \in L_2^n(\mathbb{R}) \mid x(t) = 0 \text{ for all } t < 0 \right\}$$

$$L_{2-}^n(\mathbb{R}) := \left\{ x \in L_2^n(\mathbb{R}) \mid x(t) = 0 \text{ for all } t > 0 \right\}$$

L_2 is Hilbert, with

$$\langle x, y \rangle_2 := \int_{\mathbb{R}} y'(t) x(t) \mathrm{d}t$$

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Fourier transform

Defined

$$\mathfrak{F}\{x\} = X(\mathrm{j}\omega) := \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t,$$

where $\omega \in \mathbb{R}$ called the (angular) frequency and measured in rad/sec. $\mathfrak{F}\{x\}$ is the frequency-domain representation or spectrum of x.

$$\mathfrak{F}^{-1}\{X\} = x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega,$$

i.e. x is a superposition, weighted by $X(\mathrm{j}\omega)$, of harmonic signals $\exp_{\mathrm{j}\omega}$ with frequencies ω.

Fourier transform (contd)

Existence:

- well defined (absolute convergence) for $x \in L_1$
- extendible (weaker convergence) for $x \in L_2$ by the Plancherel Theorem

$\mathfrak{F}\{\cdot\}$ is a unitary mapping $L_2(\mathbb{R}) \to L_2(\mathrm{j}\mathbb{R})$, i.e. it preserves sizes:

$$\underbrace{\int_{\mathbb{R}} \|x(t)\|^2 \mathrm{d}t}_{\|x\|_2^2} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \|X(\mathrm{j}\omega)\|^2 \mathrm{d}\omega}_{\|X\|_2^2}$$

It also preserves angles:

$$\underbrace{\int_{\mathbb{R}} [x_2(t)]' x_1(t) \mathrm{d}t}_{\langle x_1, x_2 \rangle} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} [X_2(\mathrm{j}\omega)]' X_1(\mathrm{j}\omega) \mathrm{d}\omega}_{\langle X_1, X_2 \rangle_2}$$

(Parseval's theorem, in engineering literature).

Laplace transform

The two-sided (bilateral) Laplace transform:

$$\mathfrak{L}\{x\} = F(s) := \int_{\mathbb{R}} x(t)\mathrm{e}^{-st}\mathrm{d}t$$

at all $s \in \mathbb{C}$ where the integral converges (region of convergence, aka RoC). We need $x \exp_{-s} \in L_1$ for absolute convergence.

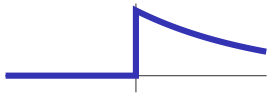
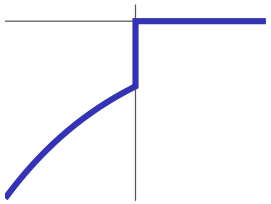
Normally used for signals supported on semi-axes:

- if $\mathrm{supp}(x) \subset \mathbb{R}_+$, then RoC is typically $\mathbb{C}_\alpha$ (may be $\bar{\mathbb{C}}_\alpha$)
- if $\mathrm{supp}(x) \subset \mathbb{R}_-$, then RoC is typically $\mathbb{C} \setminus \bar{\mathbb{C}}_\alpha$ (may be $\mathbb{C} \setminus \mathbb{C}_\alpha$)

for some $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$.

Laplace transform: mind RoC

Readily verifiable that

- $x(t) = x_+(t) := \mathrm{e}^{-t}\mathbb{1}(t) =$  $\implies X(s) = \dfrac{1}{s+1}$
- $x(t) = x_-(t) := -\mathrm{e}^{-t}\mathbb{1}(-t) =$  $\implies X(s) = \dfrac{1}{s+1}$

i.e.

- X alone does not define x unambiguously.

If the knowledge of X is complemented by its RoC:

- $X(s) = \dfrac{1}{s+1}$ and RoC $= \mathbb{C}_{-1}$ $\implies x = x_+$
- $X(s) = \dfrac{1}{s+1}$ and RoC $= \mathbb{C} \setminus \mathbb{C}_{-1}$ $\implies x = x_-$

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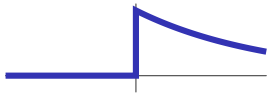
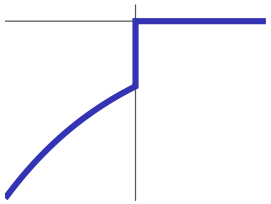
- $x(t) = x_+(t) := \mathrm{e}^{-t}\mathbb{1}(t) =$  $\implies X(s) = \dfrac{1}{s+1}$
- $x(t) = x_-(t) := -\mathrm{e}^{-t}\mathbb{1}(-t) =$  $\implies X(s) = \dfrac{1}{s+1}$

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Paley–Wiener and H_2 space

$\mathfrak{L}\{\cdot\}$ is a unitary mapping $L_{2+}(\mathbb{R}) \to H_2$, where

$$H_2^n := \left\{ X : \mathbb{C}_0 \to \mathbb{C}^n \;\middle|\; X(s) \text{ is holomorphic in } \mathbb{C}_0 \text{ and } \|X\|_2 := \sup_{\sigma>0} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|X(\sigma + \mathrm{j}\omega)\|^2 \, \mathrm{d}\omega \right)^{1/2} < \infty \right\}$$

Important:

- the boundary function $\tilde{X}(\mathrm{j}\omega) := \lim_{\sigma \downarrow 0} X(\sigma + \mathrm{j}\omega)$ exists for almost all ω and $\|\tilde{X}\|_2 = \|X\|_2$ (so $\tilde{X} \in L_2(\mathrm{j}\mathbb{R})$)
- H_2 functions are identified with boundary functions $\implies H_2 \subset L_2(\mathrm{j}\mathbb{R})$
- H_2-norm (provided $X \in H_2$):

$$\|X\|_2 = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|X(\mathrm{j}\omega)\|^2 \, \mathrm{d}\omega \right)^{1/2}$$

Outline

Continuous-time signals

Continuous-time dynamic systems in time domain

Continuous-time dynamic LTI systems in transformed domains

Coprime factorization of transfer functions over H_∞

Real-rational transfer functions

Poles, zeros, & C°

Linear systems on L_2

Linear operators

$$G : \mathfrak{D}_G \subset L_2^m \to L_2^p$$

for some domain $\mathfrak{D}_G$.

Sufficiently general class of $G : u \mapsto y$ satisfies

$$y(t) = \int_{\mathbb{R}} g(t,s)u(s)\mathrm{d}s$$

where $g : \mathbb{R}^2 \to \mathbb{R}^{p\times m}$ is the impulse response (kernel) of G, whose

- $g_{\bullet j}(t,s)$ is $y(t)$ under $u(t) = e_j\delta(t-s)$.

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Some systems $u \mapsto y$

G_{int} (integrator) $\dot{y}(t) = u(t)$

G_{dint} (discrete integrator)

$$y(t) = y(t - T) + u(t)$$

G_{fmint} (finite-memory integrator)

$$y(t) = \int_{t-T}^{t} u(s)\mathrm{d}s$$

D_τ (τ-delay operator) $y(t) = u(t - \tau)$

F_{ilp} (ideal low-pass filter)

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$$f_{\text{ilp}}(t,s) = \frac{\omega_{\text{b}}}{\pi} \operatorname{sinc}(\omega_{\text{b}}(t-s))$$

Causality

$G : u \mapsto y$ is causal if for every $t_c \in \mathbb{R}$

- $y(t) = 0$ for all $t \leq t_c$ whenever $u(t) = 0$ for all $t \leq t_c$.

Roughly, it says that

- $y(t)$ may only depend on past and present inputs u for all t.

Consequently,

- causal systems may be thought of as $\mathfrak{D}_G \subset L_{2+}^m \to L_{2+}^p$.

Criterion:

$$y(t) = \int_{t_c}^{\infty} g(t,s)u(s)ds = \int_{t_c}^{\infty} g(t,s)u(s)ds = 0, \quad \forall t < t_c, u \in \mathfrak{D}_G$$

whence

$$G \text{ is causal} \iff g(t,s) = 0 \text{ for all } s > t.$$

Note: G is an $\mathbb{R}$-linear operator. Consequently, when $\mathfrak{D}_G$ is a linear space, G is causal if and only if G is $\mathbb{R}$-linear.

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Remark: G is said to be anti-causal if y may only depend on future and present inputs u.
A linear G is anti-causal $\iff g(t,s) = 0$ for all $s < t$.

Time (shift) invariance

Linear $G : \mathfrak{D}_G \subset L_2^m \to L_2^p$ is time invariant (shift invariant) if

- $G\mathbb{S}_\tau = \mathbb{S}_\tau G$ for all $\tau \in \mathbb{R}$

If G LTI, its impulse response $g(t,s) = (G\mathbb{S}_{-s}\delta)(t)$ and then

$$y(t) = \int_{\mathbb{R}} (G\mathbb{S}_{-s}\delta)(t)u(s)\mathrm{d}s = \int_{\mathbb{R}} (\mathbb{S}_{-s}G\delta)(t)u(s)\mathrm{d}s$$

$$= \int_{\mathbb{R}} (G\delta)(t-s)u(s)\mathrm{d}s,$$

i.e. only the response of G to δ applied at $t = 0$ matters. We then treat

- $g : \mathbb{R} \to \mathbb{R}^{p\times m}$ (i.e. $g(t)$)
- can write the response as the convolution integral

$$y(t) = \int_{\mathbb{R}} g(t-s)u(s)\mathrm{d}s =: g * u$$

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Adjoint system

L_2 is Hilbert $\implies$ G has its adjoint G', defined via $\langle Gu, v\rangle = \langle u, G'v\rangle$. If

$$(Gu)(t) = \int_{\mathbb{R}} g(t,s)u(s)\mathrm{d}s,$$

then

$$\begin{aligned}
\langle Gu, v\rangle &= \int_{\mathbb{R}} v'(t)(Gu)(t)\mathrm{d}t = \int_{\mathbb{R}} v'(t)\int_{\mathbb{R}} g(t,s)u(s)\mathrm{d}s\mathrm{d}t \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \big(v'(t)g(t,s)\big)u(s)\mathrm{d}s\mathrm{d}t = \int_{\mathbb{R}}\int_{\mathbb{R}} \big(g'(t,s)v(t)\big)'u(s)\mathrm{d}t\mathrm{d}s \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} g'(t,s)v(t)\mathrm{d}t\right)'u(s)\mathrm{d}s = \int_{\mathbb{R}}\Bigg(\underbrace{\int_{\mathbb{R}} g'(s,t)v(s)\mathrm{d}s}_{(G'v)(t)}\Bigg)'u(t)\mathrm{d}t \\
&= \langle u, G'v\rangle.
\end{aligned}$$

Thus, the impulse response of G' is $[g(s,t)]'$, or $[g(-t)]'$ if G is LTI.

- G is causal $\implies$ G' is anti-causal.

Stability

Linear $G : \mathfrak{D}_G \subset L_2^m \to L_2^p$ is stable (L_2-stable)

- $\mathfrak{D}_G = L_2^m$ and
- $\|G\| := \sup_{\|u\|_2=1} \|Gu\|_2 < \infty$ (L_2-induced norm)

It is known (Young's convolution inequality) that

$$g \in L_1, u \in L_2 \implies g * u \in L_2 \text{ and } \|g * u\|_2 \leq \|g\|_1 \|u\|_2.$$

Hence, if G is LTI, then it is

- stable whenever $g \in L_1$.

But $g, u \in L_2$ might not imply that $g * u \in L_2$, so

- $g \in L_2$ does not necessarily imply the stability of G.

Unfortunately,

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Hence, if G is LTI, then it is

- stable whenever $g \in L_1$.

But $g, u \in L_2$ might not imply that $g * u \in L_2$, so

- $g \in L_2$ does not necessarily imply the stability of G.

Unfortunately,

☹ no N&S stability test in terms of the impulse response in general.

Outline

Basic property

Because

$$y = g * u \iff \mathfrak{L}\{y\} = \mathfrak{L}\{g\}\mathfrak{L}\{u\} \iff \mathfrak{F}\{y\} = \mathfrak{F}\{g\}\mathfrak{F}\{u\}$$

convolution representations become product in transformed domains, i.e. if G is LTI, then

$$y = Gu \iff Y(\mathrm{j}\omega) = G(\mathrm{j}\omega)U(\mathrm{j}\omega)$$

whenever both g and u are Fourier transformable and

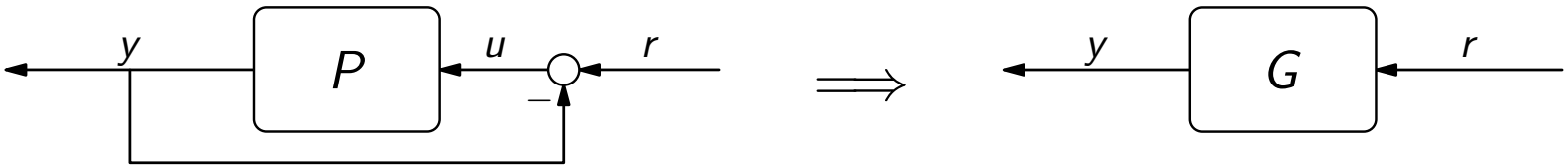
$$y = Gu \iff Y(s) = G(s)U(s)$$

for all $s \in \mathrm{RoC}(g) \cap \mathrm{RoC}(u)$. Here

- $G(s) = (\mathfrak{L}\{g\})(s)$ is the transfer function of G
- $G(\mathrm{j}\omega) = (\mathfrak{F}\{g\})(\mathrm{j}\omega)$ is the frequency response of G

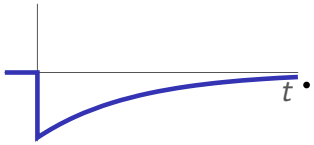
Beware of frequency response analysis: example

Consider



for

$$p(t) = -2\,\mathrm{e}^{-t}\mathbb{1}(t) =$$



.

Then

$$y(t) = -2\int_{-\infty}^{t} \mathrm{e}^{-(t-s)}u(s)\mathrm{d}s \iff \dot{y}(t) = -y(t) - 2u(t).$$

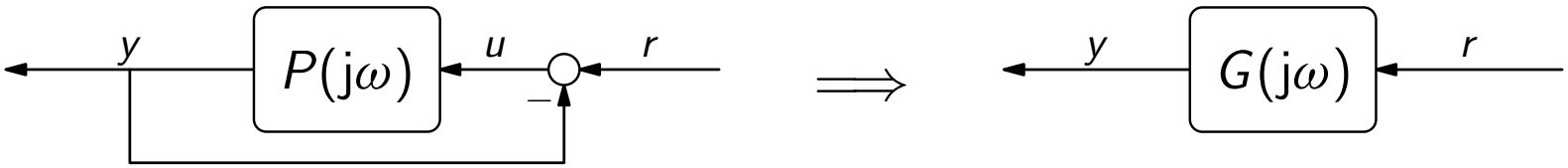
As $u(t) = r(t) - y(t)$,

$$\dot{y}(t) = y(t) - 2r(t) \iff y(t) = -2\int_{-\infty}^{t} \mathrm{e}^{t-s}u(s)\mathrm{d}s,$$

so G is causal, with $g(t) = -2\mathrm{e}^{t}\mathbb{1}(t)$, and unstable.

Beware of frequency response analysis: example (contd)

In the frequency domain,



with $Y(\mathrm{j}\omega) = P(\mathrm{j}\omega)(R(\mathrm{j}\omega) - Y(\mathrm{j}\omega))$. Hence

$$G(\mathrm{j}\omega) = \frac{P(\mathrm{j}\omega)}{1 + P(\mathrm{j}\omega)} = \frac{2}{1 - \mathrm{j}\omega} \quad \Longrightarrow \quad g(t) = 2\,\mathrm{e}^{t}\,\mathbb{1}(-t) =$$

This G is anti-causal and stable (for this $g \in L_1$), which makes no sense.

Hazards of analyzing systems in the Fourier domain:

- hard to cope with exponentially growing signals
- hard to trace causality

Beware of frequency response analysis: example (contd)

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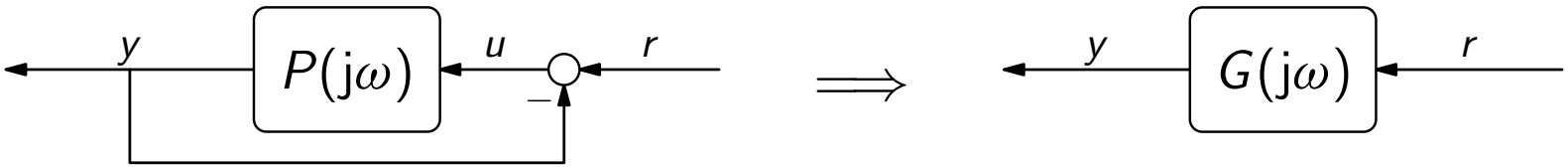

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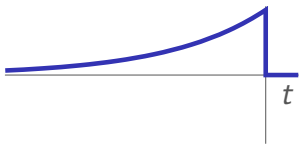

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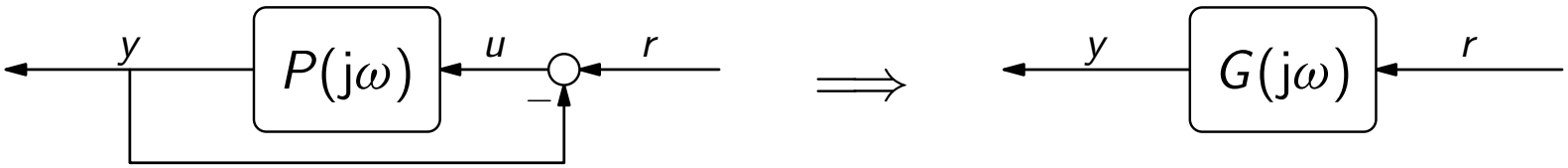

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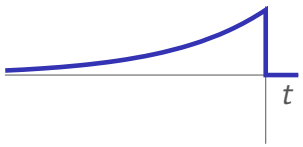

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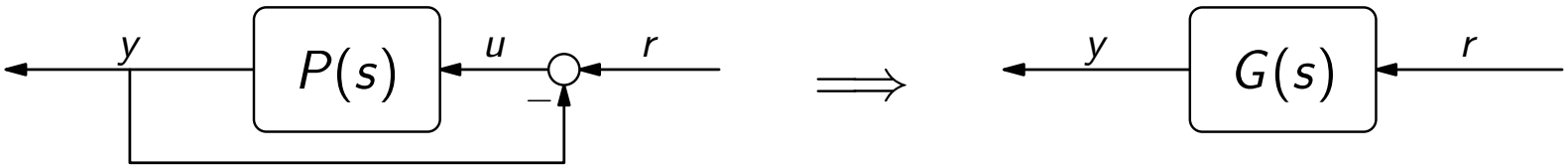
Feedback in Laplace domain: example

In the Laplace domain,



for

$$p(t) = -2\,\mathrm{e}^{-t}\mathbb{1}(t) \quad \Longrightarrow \quad P(s) = -\frac{2}{s+1}$$

whose RoC $= \mathbb{C}_{-1}$ (includes $\mathrm{j}\mathbb{R}$). Then, via $Y(s) = P(s)(R(s) - Y(s))$,

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It is not unreasonable to assume that

- causality is preserved under this feedback $\Longrightarrow$ RoC must remain a RHP

The correct impulse response can then be obtained immediately.

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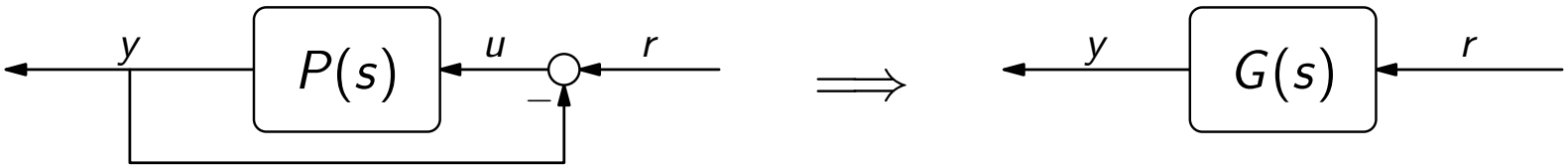

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Systems in Laplace domain

Typically,

- control applications are concerned with causal systems
- impulse responses are supported in $\mathbb{R}_+$
- signals are assumed to have support in $\mathbb{R}_+$ too
- RoC's are $\mathbb{C}_\alpha$ for some $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$
- causal LTI systems treated as operators $G : \mathfrak{D}_G \subset L_{2+}^m \to L_{2+}^p$

Outcomes:

- causal LTI systems $\rightleftharpoons$ transfer functions
- dynamical systems can be manipulated algebraically

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Transfer function: examples

Systems we already saw:

- $g_{\text{int}}(t) = \mathbb{1}(t)$
- $g_{\text{dint}}(t) = \sum \delta(t - iT)$
- $g_{\text{fmint}}(t) = \mathbb{1}_{[0,T]}(t)$
- $d_\tau(t) = \delta(t - \tau)$

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whose RoC's are $\mathbb{C}_0$, $\mathbb{C}_0$, $\mathbb{C}$, and $\mathbb{C}$, respectively.

Causality + stability in Laplace domain

An LTI G is causal and stable iff its transfer function $G \in H_\infty^{p\times m}$, where

$$H_\infty^{p\times m} := \left\{ G : \mathbb{C}_0 \to \mathbb{C}^{p\times m} \;\middle|\; G(s) \text{ is holomorphic and bounded in } \mathbb{C}_0 \right\}$$

Thus,

- $GH_2^m \subset H_2^p \iff G \in H_\infty^{p\times m}$
- if $p = m$ and $G, G^{-1} \in H_\infty^{m\times m}$, then $GH_2^m = H_2^m$

H_∞ is Banach, with $\|G\|_\infty := \sup_{s\in\mathbb{C}_0} \|G(s)\|$. Can be associated with its boundary function from

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Causality + stability and system poles

Let

$$G(s) = \frac{1}{1 + s + s\mathrm{e}^{-s}}.$$

Its poles are all in the OLHP $\mathbb{C} \setminus \bar{\mathbb{C}}_0$. To see this, let $s = \sigma + \mathrm{j}\omega$ be a pole, then $1 + \sigma + \mathrm{j}\omega + (\sigma + \mathrm{j}\omega)\mathrm{e}^{-\sigma - \mathrm{j}\omega} = 0$ reads

$$\mathrm{e}^{-\sigma}\mathrm{e}^{-\mathrm{j}\omega} = -1 - \frac{1}{\sigma + \mathrm{j}\omega} = -\left(1 + \frac{\sigma}{\sigma^2 + \omega^2}\right) + \mathrm{j}\frac{\omega}{\sigma^2 + \omega^2}.$$

Hence, σ must satisfy

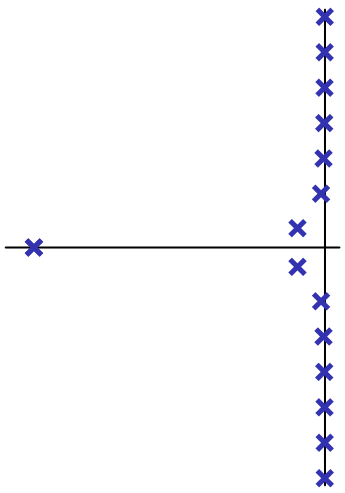
$$\mathrm{e}^{-\sigma} = \left|\left(1 + \frac{\sigma}{\sigma^2 + \omega^2}\right) - \mathrm{j}\frac{\omega}{\sigma^2 + \omega^2}\right| \geq \left|1 + \frac{\sigma}{\sigma^2 + \omega^2}\right| \geq 1.$$

which is a contradiction for all $\sigma > 0$. If $\sigma = 0$, we have $1 = 1 + \frac{1}{|\omega|}$, which also holds for none $\omega \in \mathbb{R}$.

Causality + stability and system poles (contd)

But $G \notin H_\infty$. To see this, let $\{s_k\} \in \mathbb{C} \setminus \bar{\mathbb{C}}_0$ be a sequence of poles of $G(s)$ satisfying

$$s_k + 1 + s_k \mathrm{e}^{-s_k} = 0, \quad \text{with } \lim_{k\to\pm\infty} |s_k| = \infty :$$



(known to exist). Then

$$G(-s_k) = \frac{1}{1 - s_k - s_k \mathrm{e}^{s_k}} = \frac{1}{1 - s_k + s_k^2/(1+s_k)} = 1 + s_k.$$

so there is a sequence $\{s_k\}$ in $\mathbb{C}_0$ such that $\lim_{k\to\infty} |G(-s_k)| = \infty$. Hence,

- G is not L_2-stable, despite having all poles in the OLHP

(curiously, $\frac{1}{s+1} G(s)$ is an H_∞ transfer function)

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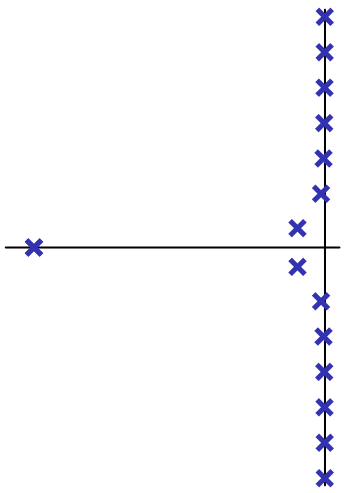

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$$G(-s_k) = \frac{1}{1 - s_k - s_k \mathrm{e}^{s_k}} = \frac{1}{1 - s_k + s_k^2/(1+s_k)} = 1 + s_k.$$

so there is a sequence $\{s_k\}$ in $\mathbb{C}_0$ such that $\lim_{k\to\infty} |G(-s_k)| = \infty$. Hence,

- G is not L_2-stable, despite having all poles in the OLHP

(curiously, $\frac{1}{s+1} G(s)$ is an H_∞ transfer function).

H_2 system space

Defined as

$$H_2^{p\times m} := \left\{ G : \mathbb{C}_0 \to \mathbb{C}^{p\times m} \,\middle|\, G(s) \text{ is holomorphic in } \mathbb{C}_0 \text{ and } \|G\|_2 := \sup_{\sigma>0} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(\sigma + \mathrm{j}\omega)\|_{\mathrm{F}}^2 \, \mathrm{d}\omega \right)^{1/2} < \infty \right\}$$

With the good ol' boundary function trick, $H_2 \subset L_2(\mathrm{j}\mathbb{R})$ and if $G \in H_2$,

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(\mathrm{j}\omega)\|_{\mathrm{F}}^2 \, \mathrm{d}\omega \right)^{1/2}$$

and H_2 inherits the inner product from $L_2(\mathrm{j}\mathbb{R})$.

Examples

- $G_{\text{int}} \notin H_2$ as $1/s$ is holomorphic in $\mathbb{C}_0$ but

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{d}\omega}{\sigma^2+\omega^2} = \frac{1}{2\sigma} \xrightarrow{\sigma\downarrow 0} \infty$$

(simpler, $\|\mathbb{1}\|_2 = \infty$)

- $G_{\text{dint}} \notin H_2$ for similar reasons
- $G_{\text{fmint}} \in H_2$ as $\left(1-\mathrm{e}^{-sT}\right)/s$ is holomorphic in $\mathbb{C}_0$ and

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1-2\mathrm{e}^{-\sigma T}\cos(\omega T)+\mathrm{e}^{-2\sigma T}}{\sigma^2+\omega^2}\,\mathrm{d}\omega = \frac{1-\mathrm{e}^{-2\sigma T}}{2\sigma} < T$$

(since $\|\mathbb{1}\|_2 = \sqrt{T/2}$)

- $D_\tau \notin H_2$ as $\mathrm{e}^{-s\tau}$ is holomorphic in $\mathbb{C}_0$ but

$$\frac{\mathrm{e}^{-2\sigma\tau}}{2\pi}\int_{\mathbb{R}} \mathrm{d}\omega = \infty$$

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(so $\|G_{\text{fmint}}\|_2 = \sqrt{T/(2\sigma)}$)

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H_2 system space (contd)

Is Hilbert, with

$$\langle G_1, G_2 \rangle_2 := \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{tr}\big([G_2(\mathrm{j}\omega)]' G_1(\mathrm{j}\omega)\big)\,\mathrm{d}\omega = \int_{\mathbb{R}} \mathrm{tr}\big([g_2(t)]' g_1(t)\big)\,\mathrm{d}t.$$

Usage:

- unrelated to stability ($D_\tau \in H_\infty$ but $D_\tau \notin H_2$, may be vice versa)
- popular performance measure (LQG, Kalman filtering)
  - $\|G\|_2^2$ equals the energy of $y = G\delta$
  - if u is Gaussian unit-intensity white, $\|G\|_2^2$ equals the variance of $y = Gu$

Properness

$G(s)$ is

- proper if $\exists \alpha \geq 0$ such that $\sup_{s \in \mathbb{C}_\alpha} \|G(s)\| < \infty$
- strictly proper if $\exists \alpha \geq 0$ such that $\lim_{|s| \to \infty, s \in \mathbb{C}_\alpha} \|G(s)\| = 0$

Examples:

- $G_{\text{int}}(s)$ is strictly proper (and thus proper)
- $G_{\text{dint}}(s)$ is proper but not strictly proper

$$\frac{1}{1+e^{-\sigma T}} \leq \frac{1}{|1-e^{-(\sigma+j\omega)T}|} \leq \frac{1}{1-e^{-\sigma T}}$$

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- $D_\tau(s)$ is proper but not strictly proper
  as $|e^{-(\sigma+j\omega)\tau}| = e^{-\sigma\tau} > 0$ for all finite $\sigma > 0$

Important:

- $G \in H_\infty \implies G(s)$ is proper $\implies$ stable causal G have proper t.f.'s
- $G \in H_2 \implies G(s)$ is strictly proper

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Conjugate transfer function

If G is LTI, its adjoint G' has impulse response $[g(-t)]'$ and

$$\mathfrak{L}\{g'\} = \int_{\mathbb{R}} [g(-t)]' \mathrm{e}^{-st} \mathrm{d}t = \left[\int_{\mathbb{R}} g(t) \mathrm{e}^{-(-\bar{s})t} \mathrm{d}t\right]' = [G(-\bar{s})]'.$$

with RoC in $\mathbb{C} \setminus \bar{\mathbb{C}}_{\alpha}$. Thus, the transfer function of G' is

$$G^{\sim}(s) := [G(-\bar{s})]',$$

known as conjugate transfer function and verifying $G^{\sim}(\mathrm{j}\omega) = [G(\mathrm{j}\omega)]'$.

Usage:

- mostly in analysis
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Inner and co-inner transfer functions

$G \in H_\infty^{p\times m}$ is

- inner if $G^\sim(s)G(s) = I_m$ (so $p \geq m$)
- co-inner if $G(s)G^\sim(s) = I_p$ (so $p \leq m$)

If $G(s)$ is inner, the system G is an isometry on $L_2(\mathbb{R})$:

$$\|Gu\|_2^2 = \|GU\|_2^2 = \langle GU, GU\rangle_2 = \langle G^\sim GU, U\rangle_2 = \langle U, U\rangle_2^2 = \|U\|_2^2 = \|u\|_2^2$$

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If $W_i(s)$ and $W_{ci}(s)$ are inner and co-inner, then

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Outline

Coprimeness over H_∞

$M \in H_\infty^{m\times m}$ and $N \in H_\infty^{p\times m}$ are (strongly) right coprime over H_∞ if there are Bézout factors $X \in H_\infty^{m\times m}$ and $Y \in H_\infty^{m\times p}$ satisfying

$$\begin{bmatrix} X(s) & Y(s) \end{bmatrix} \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = X(s)M(s) + Y(s)N(s) = I_m$$

(Bézout equality). Implies left invertibility of $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ over H_∞.

$\tilde{M} \in H_\infty^{p\times p}$ and $\tilde{N} \in H_\infty^{p\times m}$ are (strongly) left coprime over H_∞ if there are Bézout factors $\tilde{X} \in H_\infty^{p\times p}$ and $\tilde{Y} \in H_\infty^{m\times p}$ satisfying

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Coprime factorization

Effectively every stabilizable transfer function can be expressed as

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

for right / left coprime M, N / $\tilde{M}, \tilde{N} \in H_\infty$ and bi-proper $M(s)$ and $\tilde{M}(s)$.

Examples:

- $G_{\text{int}}(s) = \frac{1}{s} = \frac{1}{s+a} \cdot \left(\frac{s}{s+a}\right)^{-1}$, $a > 0$, with $X(s) = 1$ and $Y(s) = a$
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Constructing coprime factors:

- if $G \in H_\infty$, then $M(s) = I$, $N(s) = G(s)$, $X(s) = I$, and $Y(s) = 0$
- if $G(s) = \frac{p(s)}{q(s)}$, then $M(s) = \frac{q(s)}{\ldots}$

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- if $G \in H_\infty$, then $M(s) = I$, $N(s) = G(s)$, $X(s) = I$, and $Y(s) = 0$
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Coprime factorization

Effectively every stabilizable transfer function can be expressed as

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$$

for right / left coprime M, N / $\tilde{M}, \tilde{N} \in H_\infty$ and bi-proper $M(s)$ and $\tilde{M}(s)$.

Examples:

- $G_{\text{int}}(s) = \frac{1}{s} = \frac{1}{s+a} \cdot \left(\frac{s}{s+a}\right)^{-1}$, $a > 0$, with $X(s) = 1$ and $Y(s) = a$
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Two lemmas

Lemma

If $N_1 M_1^{-1} = N_2 M_2^{-1}$ *and* $\tilde{M}_1^{-1}\tilde{N}_1 = \tilde{M}_2^{-1}\tilde{N}_2$ *are rcf's and lcf's of some* G*, respectively, then*

$$\begin{bmatrix} M_2 \\ N_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} U \quad \text{and} \quad \begin{bmatrix} \tilde{M}_2 & \tilde{N}_2 \end{bmatrix} = \tilde{U} \begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix}$$

for some $U, U^{-1}, \tilde{U}, \tilde{U}^{-1} \in H_\infty$.

Implies that

- if $\det M_1(s_0) = 0$ for $s_0 \in \mathbb{C}_0$, then $\det M_2(s_0) = 0$ for any other *rcf*
- if $\det \tilde{M}_1(s_0) = 0$ for $s_0 \in \mathbb{C}_0$, then $\det \tilde{M}_2(s_0) = 0$ for any other *lcf*

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If $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ *are rcf and lcf, respectively, then*

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Domain of L_2 systems

If $G : \mathfrak{D}_G \subset L_2^m \to L_2^p$ is LTI and such that its transfer function admits a *rcf* over H_∞, $G(s) = N(s)M^{-1}(s)$, then

$$\mathfrak{D}_G = ML_2^m = \operatorname{Im} M = \{u \mid \exists v \in L_2^m \text{ such that } u = Mv\}.$$

Proof (outline).

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- $M \in H_\infty \implies ML_2^m \subset L_2^m$
- $GML_2^m = NL_2^m \subset L_2^p \implies ML_2^m \subset \mathfrak{D}_G$
- For any $u_0 \in \mathfrak{D}_G$, denote $v_0 := M^{-1}u_0$. We have:

$$L_2^{m+p} \ni \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} I \\ G \end{bmatrix} u_0 = \begin{bmatrix} M \\ N \end{bmatrix} v_0$$

Thus,

$$v_0 = Xu_0 + Yy_0 \in L_2^m \implies \mathfrak{D}_G \subset ML_2^m$$

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Doubly coprime factorization

Coprime factors of $G(s)$ and their Bézout can always be selected so that

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix},$$

i.e.

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M(s) & -\tilde{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix}$$

are invertible in H_∞.

Outline

Definition

We say that $G(s)$ is real-rational if

- $G_{ij}(s) = \frac{N_{ij}(s)}{M_{ij}(s)}$ for finite polynomials $N_{ij}(s)$ and $M_{ij}(s)$ with real coeff's.

Examples:

- $G_{\text{int}}(s) = \frac{1}{s}$
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Implications

Any real-rational $G(s)$

- is proper iff $\|G(\infty)\| < \infty$, i.e. $\deg(N_{ij}(s)) \leq \deg(M_{ij}(s))$, $\forall i, j$
- is strictly proper iff $\|G(\infty)\| = 0$, i.e. $\deg(N_{ij}(s)) < \deg(M_{ij}(s))$, $\forall i, j$
- $G \in H_\infty$ iff $G(s)$ is proper & has no poles in $\bar{\mathbb{C}}_0$
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By-products:

- stability $\iff$ proper + no poles in $\bar{\mathbb{C}}_0$
- $RH_2 \subset RH_\infty$
- always stabilizable by feedback

Outline

Diagonal case: poles, zeros, and . . .

Every

$$G(s) = \begin{bmatrix} G_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_m(s) \end{bmatrix} =: \operatorname{diag}\{G_i(s)\}$$

is effectively a union of m independent systems, so that

- poles and zeros of $G(s)$ are unions of poles and zeros of $G_i(s)$.

Consequences:

- may have uncancellable pole(s) and zero(s) at the same point
- $\det(G(s))$ might be a poor indicator of its dynamical properties
- mere location of poles and zeros is not sufficient

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Diagonal case: poles, zeros, and . . . (contd)

Poles and zeros of

$$G(s) = \begin{bmatrix} G_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_m(s) \end{bmatrix}$$

should be

- complemented by their association with subsystems
- complemented by their directions
  - if p_k (z_k) is a pole (zero) of $G_i(s)$, its direction is span(e_i)
  - if p_k (z_k) is a pole (zero) of $G_i(s)$ and $G_j(s)$, its direction is span(e_i, e_j)
  - pole direction of p_k: $\perp$ to any v for which $G(s)v$ has no pole at p_k
  - zero direction of z_k: span of all v for which $G(s)v|_{s \to z_k} = 0$
  - if p_k (z_k) is a pole (zero) of μ_k subsystems, its geometric multiplicity is μ_k
  - the multiplicity of p_k (z_k) in $G_i(s)$ is its ith partial multiplicity
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Diagonal case: poles, zeros, and . . . (contd)

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- ~~complemented by their association with subsystems~~
- complemented by their directions
  - if p_k (z_k) is a pole (zero) of $G_i(s)$, its direction is span(e_i)
  - if p_k (z_k) is a pole (zero) of $G_i(s)$ and $G_j(s)$, its direction is span(e_i, e_j)
  - pole direction of p_k: $\perp$ to any v for which $G(s)v$ has no pole at p_k
  - zero direction of z_k: span of all v for which $G(s)v|_{s\to z_k} = 0$
  - if p_k (z_k) is a pole (zero) of μ_k subsystems, its geometric multiplicity is μ_k
  - the multiplicity of p_k (z_k) in $G_i(s)$ is its ith partial multiplicity
  - the sum of all partial multiplicities of p_k is its algebraic multiplicity

General case: preliminaries

- normal rank: $\mathrm{nrank}(G(s)) := \max_{s\in\mathbb{C}} \mathrm{rank}(G(s))$
  if $G(s)$ is proper, then $\mathrm{rank}(G(s)) = \mathrm{nrank}(G(s))$ for all but a finitely many s
- unimodular polynomial matrix: square and $\det(U(s)) = \mathrm{const} \neq 0$
  $U^{-1}(s)$ is also a polynomial matrix
- polynomial $\beta(s)$ divides polynomial $\alpha(s)$ if $\frac{\alpha(s)}{\beta(s)}$ is a polynomial

Smith–McMillan form

Given a $p \times m$ transfer function $G(s)$ having $\operatorname{nrank}(G(s)) = r \leq \min\{p, m\}$, there are unimodular polynomial matrices $U(s)$ and $V(s)$ such that

$$U(s)G(s)V(s) = \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\alpha_i(s)$ divides $\alpha_{i+1}(s)$, $\beta_{i+1}(s)$ divides $\beta_i(s)$, and $\alpha_i(s)$ and $\beta_i(s)$ are coprime at every $i \in \mathbb{Z}_{1..r}$.

- roots of $\phi_p(s) := \prod_{i=1}^{r} \beta_i(s)$ are the poles of $G(s)$
- $n := \deg(\phi_p(s))$ is the McMillan degree (or degree) of $G(s)$
- roots of $\phi_z(s) := \prod_{i=1}^{r} \alpha_i(s)$ are the transmission zeros (or zeros) of $G(s)$

Smith–McMillan form & poles / degree / zeros

Given a $p \times m$ transfer function $G(s)$ having $\operatorname{nrank}(G(s)) = r \leq \min\{p, m\}$, there are unimodular polynomial matrices $U(s)$ and $V(s)$ such that

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where $\alpha_i(s)$ divides $\alpha_{i+1}(s)$, $\beta_{i+1}(s)$ divides $\beta_i(s)$, and $\alpha_i(s)$ and $\beta_i(s)$ are coprime at every $i \in \mathbb{Z}_{1..r}$.

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- roots of $\phi_{\mathrm{z}}(s) := \prod_{i=1}^{r} \alpha_i(s)$ are the transmission zeros (or zeros) of $G(s)$

Pole directions

Let $p_i \in \mathbb{C}$ be a pole of geometric multiplicity μ_i of

$$G(s) = U^{-1}(s) \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} V^{-1}(s)$$

input pole direction, $\mathrm{pdir_i}(G, p_i) \subset \mathbb{C}^m$:

$$\mathrm{pdir_i}(G, p_i) = \left(\mathrm{Im}\, V(p_i) \begin{bmatrix} e_{\mu_i+1} & \cdots & e_m \end{bmatrix}\right)^{\perp} = \ker \begin{bmatrix} e'_{\mu_i+1} \\ \vdots \\ e'_m \end{bmatrix} [V(p_i)]'$$

output pole direction, $\mathrm{pdir_o}(G, p_i) \subset \mathbb{C}^p$:

$$\mathrm{pdir_o}(G, p_i) = \ker \begin{bmatrix} \tilde{e}'_{\mu_i+1} \\ \vdots \\ \tilde{e}'_p \end{bmatrix} U(p_i) = \left(\mathrm{Im}[U(p_i)]' \begin{bmatrix} \tilde{e}_{\mu_i+1} & \cdots & \tilde{e}_p \end{bmatrix}\right)^{\perp}$$

Zero directions

Let $z_i \in \mathbb{C}$ be a pole of geometric multiplicity μ_i of

$$G(s) = U^{-1}(s) \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} V^{-1}(s)$$

input zero direction, $\mathrm{zdir_i}(G, p_i) \subset \mathbb{C}^m$:

$$\mathrm{zdir_i}(G, z_i) := \mathrm{Im}\, V(z_i) \begin{bmatrix} e_{r-\mu_i+1} & \cdots & e_m \end{bmatrix}$$

output zero direction, $\mathrm{zdir_o}(G, p_i) \subset \mathbb{C}^p$:

$$\mathrm{zdir_o}(G, p_i) := \mathrm{Im}[U(z_i)]' \begin{bmatrix} \tilde{e}_{r-\mu_i+1} & \cdots & \tilde{e}_p \end{bmatrix}$$

Example 1

Let

$$G(s) = \frac{1}{s}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \implies \overbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}^{U(s)} G(s) \overbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}^{V(s)} = \begin{bmatrix} 1/s & 0 \\ 0 & 0 \end{bmatrix}$$

One pole at $s = 0$, with

$$\mathrm{pdir}_i(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$\mathrm{pdir}_o(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

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Example 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \implies \overbrace{\begin{bmatrix} 1 & 0 \\ s & -1 \end{bmatrix}}^{U(s)} G(s) \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & -s \end{bmatrix}}^{V(s)} = \begin{bmatrix} 1/s & 0 \\ 0 & s \end{bmatrix}$$

One pole and one transmission zero at $s = 0$, with

$$\mathrm{pdir}_{\mathrm{i}}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\mathrm{pdir}_{\mathrm{o}}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\mathrm{zdir}_{\mathrm{i}}(G, 0) = \mathrm{Im}\, V(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\mathrm{zdir}_{\mathrm{o}}(G, 0) = \mathrm{Im}[U(0)]' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Example 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \implies \overbrace{\begin{bmatrix} 1 & 0 \\ s & -1 \end{bmatrix}}^{U(s)} G(s) \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & -s \end{bmatrix}}^{V(s)} = \begin{bmatrix} 1/s & 0 \\ 0 & s \end{bmatrix}$$

One pole and one transmission zero at $s = 0$, with

$$\mathrm{pdir_i}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\mathrm{pdir_o}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\mathrm{zdir_i}(G, 0) = \mathrm{Im}\, V(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\mathrm{zdir_o}(G, 0) = \mathrm{Im}\, [U(0)]' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Example 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \implies \overbrace{\begin{bmatrix} 1 & 0 \\ s & -1 \end{bmatrix}}^{U(s)} G(s) \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & -s \end{bmatrix}}^{V(s)} = \begin{bmatrix} 1/s & 0 \\ 0 & s \end{bmatrix}$$

One pole and one transmission zero at $s = 0$, with

$$\mathrm{pdir_i}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

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$$\mathrm{zdir_i}(G, 0) = \mathrm{Im}\, V(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\mathrm{zdir_o}(G, 0) = \mathrm{Im}[U(0)]' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Example 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \implies \overbrace{\begin{bmatrix} 1 & 0 \\ s & -1 \end{bmatrix}}^{U(s)} G(s) \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & -s \end{bmatrix}}^{V(s)} = \begin{bmatrix} 1/s & 0 \\ 0 & s \end{bmatrix}$$

One pole and one transmission zero at $s = 0$, with

$$\mathrm{pdir_i}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\mathrm{pdir_o}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\mathrm{zdir_i}(G, 0) = \mathrm{Im}\, V(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \perp \mathrm{pdir_i}(G, 0)$$

$$\mathrm{zdir_o}(G, 0) = \mathrm{Im}[U(0)]' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \perp \mathrm{pdir_o}(G, 0)$$

Example 3

Let

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

and define unimodular polynomials

$$U(s) = \frac{1}{6}\begin{bmatrix} 3 & 3 \\ s^3 - s^2 - 4s - 2 & s^3 - s^2 - 4s + 4 \end{bmatrix},$$

$$V(s) = \frac{1}{6}\begin{bmatrix} 2(s-2) & -6(s-1) & -3(s-1) \\ 4 & -24 & -6(s+2) \\ 0 & 6 & 3(s+2) \end{bmatrix}.$$

Then

$$U(s)G(s)V(s) = \begin{bmatrix} \frac{1}{(s^2-1)(s+2)} & 0 & 0 \\ 0 & \frac{s-1}{s+2} & 0 \end{bmatrix}.$$

Four poles, at $\{-2, -2, -1, 1\}$, and one transmission zero at $\{1\}$.

Example 3 (contd)

Pole directions:

$$\mathrm{pdir_i}(G,1) = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [V(1)]' = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

$$\mathrm{pdir_o}(G,1) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(1) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

$$\mathrm{pdir_i}(G,-1) = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [V(-1)]' = \mathrm{span}\left(\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right),$$

$$\mathrm{pdir_o}(G,-1) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(-1) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right),$$

$$\mathrm{pdir_i}(G,-2) = \ker \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} [V(-2)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right),$$

and $\mathrm{pdir_o}(G,-2) = \mathbb{C}^2$.

Example 3 (contd)

Zero directions:

$$\mathrm{zdir_i}(G,1) = \mathrm{Im}\, V(1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\mathrm{zdir_o}(G,1) = \mathrm{Im}[U(1)]' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Again,

$$\mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathrm{zdir_i}(G,1) \perp \mathrm{pdir_i}(G,1) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \mathrm{zdir_o}(G,1) \perp \mathrm{pdir_o}(G,1) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Example 3 (contd)

Zero directions:

$$\mathrm{zdir_i}(G,1) = \mathrm{Im}\, V(1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\mathrm{zdir_o}(G,1) = \mathrm{Im}[U(1)]' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Again,

$$\mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathrm{zdir_i}(G,1) \perp \mathrm{pdir_i}(G,1) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \mathrm{zdir_o}(G,1) \perp \mathrm{pdir_o}(G,1) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Example 4

Let

$$G(s) = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}.$$

Its Smith–McMillan form is

$$\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} G(s) \begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} 1/s^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Double pole at $s = 0$, with

$$\mathrm{pdir_i}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

and

$$\mathrm{pdir_o}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Although $e_1 \perp \mathrm{pdir_i}(G, 0)$, $G(s)e_1 = 1/s\, e_1$, i.e. it still has a pole at $s = 0$.

Example 4

Let

$$G(s) = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}.$$

Its Smith–McMillan form is

$$\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} G(s) \begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} 1/s^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Double pole at $s = 0$, with

$$\mathrm{pdir_i}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} [V(0)]' = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

and

$$\mathrm{pdir_o}(G, 0) = \ker \begin{bmatrix} 0 & 1 \end{bmatrix} U(0) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Although $e_1 \perp \mathrm{pdir_i}(G, 0)$, $G(s)e_1 = 1/s\, e_1$, i.e. it still has a pole at $s = 0$.

Simplifications

Let $\text{nrank}(G(s)) = r$. The following statements hold true:

1. $\phi_p(s)$ is the least common denominator of all nonzero minors of $G(s)$ of all orders provided all common poles and zeros in each of these minors were canceled.
2. $\phi_z(s)$ is the greatest common divisor of all the numerators of all r-order minors of $G(s)$ provided these minors have been adjusted to have $\phi_p(s)$ as their denominators.

Example 3 (contd)

For

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

nonzero minors of order 1 are

$$\frac{1}{s+1}, \quad \frac{s-1}{(s+1)(s+2)}, \quad -\frac{1}{s-1}, \quad \frac{1}{s+2}, \quad \text{and} \quad \frac{1}{s+2}$$

and the minors of order 2 are

$$-\frac{s-1}{(s+1)(s+2)^2}, \quad \frac{2}{(s+1)(s+2)}, \quad \text{and} \quad \frac{1}{(s+1)(s+2)}.$$

Hence,

$$\phi_{\mathrm{p}}(s) = (s+2)^2(s+1)(s-1) = (s+2)^2(s^2-1),$$

as before.

Example 3 (contd)

For

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

the minors of order 2 are:

$$-\frac{s-1}{(s+1)(s+2)^2}, \quad \frac{2}{(s+1)(s+2)}, \quad \text{and} \quad \frac{1}{(s+1)(s+2)}$$

or, equivalently, with $\phi_{\mathrm{p}}(s) = (s+2)^2(s+1)(s-1)$

$$-\frac{(s-1)^2}{\phi_{\mathrm{p}}(s)}, \quad \frac{2(s+2)(s-1)}{\phi_{\mathrm{p}}(s)}, \quad \text{and} \quad \frac{(s+2)(s-1)}{\phi_{\mathrm{p}}(s)}.$$

Hence,

$$\phi_{\mathrm{z}}(s) = s - 1,$$

as before.

Simplifications (contd)

Let $G(s)$ be a $p \times m$ real-rational proper transfer function.

1. If $z_i \in \mathbb{C}$ isn't a pole of $G(s)$, then it's a transmission zero of $G(s)$ iff $\operatorname{rank}\big(G(z_i)\big) < \operatorname{nrank}\big(G(s)\big)$ and $\operatorname{nrank}(G(s)) - \operatorname{rank}(G(z_i))$ equals the geometric multiplicity of the zero at z_i, with

$$\operatorname{zdir}_{\mathrm{i}}(G, z_i) = \ker G(z_i) \quad \text{and} \quad \operatorname{zdir}_{\mathrm{o}}(G, z_i) = \ker[G(z_i)]'.$$

2. If $p = m = \operatorname{nrank}(G(s))$ and $p_i \in \mathbb{C}$ isn't a transmission zero of $G(s)$, it's a pole of $G(s)$ iff $\det\big(G^{-1}(p_i)\big) = 0$ and $m - \operatorname{rank}(G^{-1}(p_i))$ equals the geometric multiplicity of the pole at p_i, with

$$\operatorname{pdir}_{\mathrm{i}}(G, p_i) = \ker[G^{-1}(p_i)]' \quad \text{and} \quad \operatorname{pdir}_{\mathrm{o}}(G, p_i) = \ker G^{-1}(p_i).$$