Linear Control Systems (036012)
chapter 2

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Static systems

is static (memoryless) if

- $y(t_1)$ depends only on $u(t)$ only at $t = t_1$ for all t_1

Hence,

- frozen-time analysis (no need in time dependence)

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Outline

Signals & systems

Signals are vectors:

$$u \in \mathbb{F}^m \quad \text{and} \quad y \in \mathbb{F}^p.$$

Systems are mappings:

$$G : \mathbb{F}^m \to \mathbb{F}^p.$$

If

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \sum_{i=1}^{m} u_i e_i \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = \sum_{i=1}^{p} y_i \tilde{e}_i$$

then

$$\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = \overbrace{\begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{p1} & \cdots & g_{pm} \end{bmatrix}}^{\text{system matrix (convenient)}} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

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Matrix notation and terminology

By $g_{\bullet j} \in \mathbb{F}^{p\times 1}$ and $g_{i\bullet} \in \mathbb{F}^{1\times m}$ we denote the jth column and the ith row of $G \in \mathbb{F}^{p\times m}$,

$$G = \begin{bmatrix} g_{\bullet 1} & \dots & g_{\bullet m} \end{bmatrix} = \begin{bmatrix} g_{1\bullet} \\ \vdots \\ g_{p\bullet} \end{bmatrix}.$$

We say that $G \in \mathbb{F}^{p\times m}$ is

- square if $m = p$, tall if $p > m$, and fat if $p < m$;
- upper (lower) triangular if its elements $g_{ij} = 0$ whenever $i > j$ ($i < j$);
- diagonal if $g_{ij} = 0$ whenever $i \neq j$.

If G is square, then

- its trace $\operatorname{tr}(G) := \sum_{i=1}^{m} g_{ii} \in \mathbb{F}$

Similarity transformations

If $\{v_1, \ldots, v_m\}$ be a (non-standard) basis on $\mathbb{F}^m$, then

$$x = \alpha_1 v_1 + \cdots + \alpha_m v_m = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} =: T x_\alpha$$

and $T \in \mathbb{F}^{m \times m}$ is invertible. The vector $x_\alpha = T^{-1} x$ represents the same signal, just from a different viewpoint. Now, defining $\tilde{u} := T_u^{-1} u$ and $\tilde{y} := T_y^{-1} y$, we end up with

$$y = Gu \iff \tilde{y} = T_y^{-1} G T_u \tilde{u},$$

where $T_y^{-1} G T_u$ is the matrix representation of G in these new coordinates.

If $p = m$, we may take $T_u = T_y = T$. Then $T^{-1} G T$ is called similar to G.

Similarity transformations: example

DFT basis

$$\phi_i := \frac{1}{n}\begin{bmatrix} 1 \\ (e^{j2\pi/n})^{i-1} \\ \vdots \\ (e^{j2\pi(n-1)/n})^{i-1} \end{bmatrix}, \quad i \in \mathbb{Z}_{1..n}.$$

Let

$$G = \frac{1}{3}\begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

and consider

$$T = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 2 & -1+j\sqrt{3} & -1-j\sqrt{3} \\ 2 & -1-j\sqrt{3} & -1+j\sqrt{3} \end{bmatrix} \left(T^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 2 & 2 \\ 2 & -1-j\sqrt{3} & -1+j\sqrt{3} \\ 2 & -1+j\sqrt{3} & -1-j\sqrt{3} \end{bmatrix} \right)$$

Then

$$T^{-1}GT = \operatorname{diag}\{1, 2, 2\}.$$

Signals: size matters

Hölder vector norms $\|x\|_q$, $q \geq 1$:

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|, \quad \|x\| = \|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad \|x\|_\infty := \max_{1\leq i\leq n} |x_i|.$$

x is unit vector in q-metric if $\|x\|_q = 1$.

All vector norms are equivalent, that is

$$\exists \gamma_2 > \gamma_1 > 0 \text{ such that } \gamma_1 \|x\|_b \leq \|x\|_a \leq \gamma_2 \|x\|_b, \quad \forall x.$$

e.g. if $x \in \mathbb{F}^n$, then

– $\|x\|_p \leq \|x\|_q \leq n^{1/q-1/p} \|x\|_p$ whenever $q < p$

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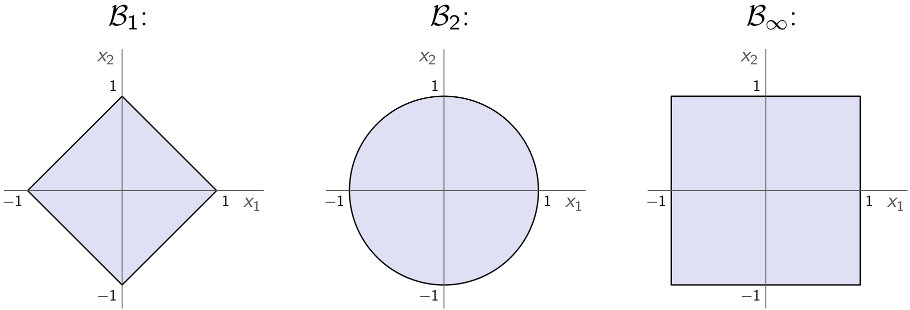
Signals: size matters (contd)

The unit ball:

$$\mathcal{B}_q := \{x : \|x\|_q \le 1\}.$$

e.g.:



Size of systems

Induced norms (largest gain):

$$\|A\|_q := \sup_{u\in\mathbb{F}^m, u\neq 0} \frac{\|Au\|_q}{\|u\|_q} = \sup_{\|u\|_q=1} \|Au\|_q = \sup_{\|u\|_q\in\mathcal{B}_q} \|Au\|_q.$$

e.g.

$$\|G\|_1 = \max_{1\leq j\leq m} \sum_{i=1}^{p} |a_{ij}|, \quad \|G\| = \|G\|_2 = \sqrt{\rho(A'A)}, \quad \|G\|_\infty = \max_{1\leq i\leq p} \sum_{j=1}^{m} |a_{ij}|$$

Frobenius norm (not induced)

$$\|G\|_\mathrm{F} := \sqrt{\mathrm{tr}(G'G)} = \left(\sum_{i=1}^{m} \|Ge_i\|^2\right)^{1/2} = \left(\sum_{i=1}^{p}\sum_{j=1}^{m} |g_{ij}|^2\right)^{1/2}$$

All matrix norms are also equivalent. For instance,

$$\|A\| \leq \|A\|_\mathrm{F} \leq \sqrt{\mathrm{rank}(A)}\,\|A\|.$$

Signals: direction matters too

The direction of $x \in \mathbb{F}^n$ is the

- 1-dimensional subspace $\text{span}(x) \subset \mathbb{F}^n$.

Relative direction may be quantified by the inner product notion,

$$\langle x, y \rangle := \sum_{i=1}^{n} \overline{y}_i x_i = y' x,$$

with

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

x and y are

orthogonal if $\langle x, y \rangle = 0$

co-directed if $|\langle x, y \rangle| = \|x\| \|y\|$ (i.e. $\exists \alpha \in \mathbb{F}$ such that $x = \alpha y$)

System gain(s)

Norms are a rough tool. For example, let

$$P = \begin{bmatrix} 1+\alpha & 1-\alpha \\ -1+\alpha & -1-\alpha \end{bmatrix}$$

for $\alpha \in [0,1]$. Then $\|P\|_2 = 2$, regardless α.

If $u = \left[\begin{smallmatrix} u_0 \\ u_0 \end{smallmatrix}\right]$, then $\|u\| = \sqrt{2}|u_0|$ and

$$y = P\begin{bmatrix} u_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 2u_0 \\ 2u_0 \end{bmatrix} \implies \|y\| = \sqrt{8}|u_0| = 2\|u\|.$$

But if $u = \left[\begin{smallmatrix} u_0 \\ -u_0 \end{smallmatrix}\right]$, then $\|u\| = \sqrt{2}|u_0|$ and

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might be significantly smaller if $|\alpha| \ll 1$.

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Structural properties

Given a $G : \mathbb{F}^m \to \mathbb{F}^p$, its kernel and image are

$$\ker G := \{u \in \mathbb{F}^m \mid Gu = 0\} \;\&\; \operatorname{Im} G := \{y \in \mathbb{F}^p \mid \exists u \in \mathbb{F}^m \text{ s.t. } y = Gu\}$$

and the rank, defined as $\operatorname{rank} G := \dim(\operatorname{Im} G)$, quantifies the richness of G.

Kernel describes the freedom of choice for the input.

$$y = Gu \iff u = u_0 + u_n \text{ for some } u_n \in \ker G$$

for any u_0 such that $y_0 = Gu_0$ and

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$$y = Gu \iff u = u_0 + u_\mathrm{n} \text{ for some } u_\mathrm{n} \in \ker G$$

for any u_0 such that $y_0 = Gu_0$ and

Outline

Unitary matrices

A matrix (system) $G \in \mathbb{F}^{m\times m}$ is said to be unitary if

$$\|Gu\| = \|u\|, \quad \forall u \in \mathbb{F}^m$$

i.e. unitary systems have unit norm in every direction. It can be shown that

- G is unitary iff $G'G = I$ or, equivalently, $G^{-1} = G'$.

Examples of unitary matrices on $\mathbb{F}^{2\times 2}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underbrace{R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{plain rotation}} \qquad \underbrace{Q_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}}_{\text{plain reflection}}$$

In real case, either rotations (if $\det G = 1$) or reflections (if $\det G = -1$).

Unitary matrices (contd)

Let

$$G = \begin{bmatrix} g_{\bullet 1} & g_{\bullet 2} & \cdots & g_{\bullet m} \end{bmatrix}, \quad g_{\bullet i} \in \mathbb{F}^m$$

be unitary. Then

$$G'G = \begin{bmatrix} g'_{\bullet 1} \\ \vdots \\ g'_{\bullet m} \end{bmatrix} \begin{bmatrix} g_{\bullet 1} & \cdots & g_{\bullet m} \end{bmatrix} = \begin{bmatrix} g'_{\bullet 1} g_{\bullet 1} & \cdots & g'_{\bullet 1} g_{\bullet m} \\ \vdots & \ddots & \vdots \\ g'_{\bullet m} g_{\bullet 1} & \cdots & g'_{\bullet m} g_{\bullet m} \end{bmatrix} = I,$$

i.e. the columns of unitary matrices are mutually orthogonal and unit. That means that any unitary matrix defines an orthonormal basis of $\mathbb{F}^m$. E.g.

- the identity matrix I corresponds to the standard basis $\left\{ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \right\}$
- plain rotation matrix corresponds to $\left\{ \left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix} \right], \left[\begin{smallmatrix} -\sin\theta \\ \cos\theta \end{smallmatrix} \right] \right\}$

Singular value decomposition

For every $G \in \mathbb{F}^{p\times m}$ there are[1] *unitary* matrices

$$U = \begin{bmatrix} u_1 & u_2 & \dots & u_p \end{bmatrix} \in \mathbb{F}^{p\times p} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 & \dots & v_m \end{bmatrix} \in \mathbb{F}^{m\times m}$$

such that

$$G = U\Sigma V',$$

where $\Sigma \in \mathbb{R}^{p\times m}$ is of the form

$$\Sigma = \begin{cases} \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} & \text{if } p \le m \\ \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} & \text{if } p \ge m \end{cases}$$

and $\Sigma_1 = \operatorname{diag}\{\sigma_1, \sigma_2, \dots, \sigma_{\min\{p,m\}}\}$ with $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_{\min\{p,m\}} \ge 0$.

[1] Matlab command is `svd(G)`

Singular value decomposition (contd)

- real numbers σ_i are called the singular values of G
  (it can be shown that σ_i^2 are the eigenvalues of both $G'G$ and GG')
- vectors u_i are called the left singular vectors of G
- vectors v_i are called the right singular vectors of G
- alternative expression of SVD:

$$G = \sum_{i=1}^{\min\{p,m\}} \sigma_i \, u_i v_i'$$

System action via SVD

Let $p = m$ (for simplicity) and consider $G : w \mapsto y$ such that

$$G = \sum_{i=1}^{m} \sigma_i \, u_i v_i'$$

Let

$$w = \sum_{i=1}^{m} w_i \, v_i, \quad \text{where } w_i \text{ are coordinates of } w \text{ in } \{v_i\}$$

By orthonormality of $\{v_i\}$,

$$y = \left(\sum_{i=1}^{m} \sigma_i \, u_i v_i' \right) \left(\sum_{i=1}^{m} w_i \, v_i \right) = \sum_{i=1}^{m} (\sigma_i w_i) u_i$$

meaning

– $y_i = \sigma_i w_i$ are the coordinates of y in the (orthonormal) basis $\{u_i\}$.

System gains via SVD

We know that $\|x\|^2 = \sum_{i=1}^{m} |x_i|^2$, where x_i are the coordinates of $x \in \mathbb{F}^m$ in any orthonormal basis of $\mathbb{F}^m$. Thus,

$$\|Gw\|^2 = \|y\|^2 = \sum_{i=1}^{m} \sigma_i^2 |w_i|^2$$

Hence,

- $\|y\| \le \sigma_1 \|w\|$
  (because $\|y\|^2 = \sum_{i=1}^{m} \sigma_i^2 |w_i|^2 \le \sum_{i=1}^{m} \sigma_1^2 |w_i|^2 = \sigma_1^2 \sum_{i=1}^{m} |w_i|^2 = \sigma_1^2 \|w\|^2$)
- $\|y\| \ge \sigma_m \|w\|$
  (because $\|y\|^2 = \sum_{i=1}^{m} \sigma_i^2 |w_i|^2 \ge \sum_{i=1}^{m} \sigma_m^2 |w_i|^2 = \sigma_m^2 \sum_{i=1}^{m} |w_i|^2 = \sigma_m^2 \|w\|^2$)

and the singular values show the maximal and minimal amplifications of G.

Singular vectors

If the input

$$w = v_j,$$

then the output

$$y = \left(\sum_{i=1}^{m} \sigma_i \, u_i v_i' \right) v_j = \sigma_j u_j \quad \Longrightarrow \quad \|y\| = \sigma_j$$

Hence,

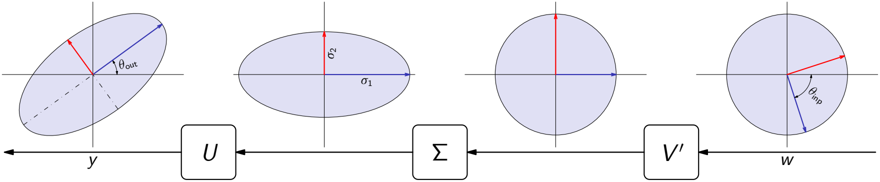
- span(v_j) is the input direction corresponding to gain σ_j
- span(u_j) is the output direction corresponding to gain σ_j

In particular

- span(v_1) is the input direction corresponding to the maximal gain
- span(u_1) is the output direction corresponding to the maximal gain
- span(v_m) is the input direction corresponding to the minimal gain
- span(u_m) is the output direction corresponding to the minimal gain

Interpretation

Any system G acts as the cascade of a rotation (reflection), scaling along "physical" directions, and another rotation (reflection):



Structural properties via SVD

Lemma

If

$$\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_{\min\{p,m\}} = 0$$

for some $r \leq \min\{p, m\}$*, then*

- $\operatorname{Im} G = \operatorname{span}(u_1, \ldots, u_r)$*,*
- $\operatorname{rank}(G) = r$*,*
- $\ker G = \operatorname{span}(v_{r+1}, \ldots, v_m)$*.*

Proof.

Follows from

$$G = \sum_{i=1}^{\min\{p,m\}} \sigma_i \, u_i v_i' = \sum_{i=1}^{r} \sigma_i u_i v_i' = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1' \\ \vdots \\ v_r' \end{bmatrix}$$

□

Low-rank approximation

Given a $G \in \mathbb{F}^{p\times m}$ with $\operatorname{rank} G = r$, then for every $l \le r$

$$\min_{\operatorname{rank}(H)\le l} \|G - H\| = \sigma_{l+1} \quad \text{and} \quad \min_{\operatorname{rank}(H)\le l} \|G - H\|_{\mathrm{F}} = \left(\sum_{i=l+1}^{r} \sigma_i^2\right)^{1/2}$$

and the minimizing

$$H = G_l := \sum_{i=1}^{l} \sigma_i u_i v_i'$$

in both cases.

Rank decomposition

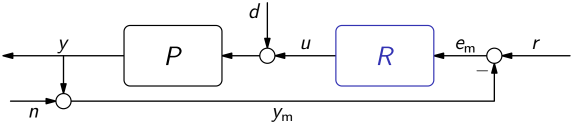
Given a $G \in \mathbb{F}^{p\times m}$ such that $\operatorname{rank}(G) = r \leq \min\{m, p\}$, there are full row rank $G_{\text{inp}} \in \mathbb{F}^{r\times m}$ and full column rank $G_{\text{out}} \in \mathbb{F}^{p\times r}$ such that

$$G = G_{\text{out}} G_{\text{inp}}.$$

In this case $\ker G = \ker G_{\text{inp}}$ and $\operatorname{Im} G = \operatorname{Im} G_{\text{out}}$. This follows directly from the SVD of G (as $\sigma_i \neq 0 \iff i \leq r$):

$$G = \sum_{i=1}^{r} \sigma_i u_i v_i' = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1' \\ \vdots \\ v_r' \end{bmatrix}.$$

Example: designs in Lect. 1



If

$$P = \begin{bmatrix} 1+\alpha & 1-\alpha \\ -1+\alpha & -1-\alpha \end{bmatrix}, \quad \text{with } R = \overbrace{k\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}^{R_1} \text{ or } R = \overbrace{k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{R_2},$$

then

$$R_1\colon \quad T_d \xrightarrow{\alpha\downarrow 0} \frac{1}{2k+1}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2k+1}P$$

$$R_2\colon \quad T_d \xrightarrow{\alpha\downarrow 0} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = P$$

– What makes them so different?
– Why R_2 doesn't affect disturbance response?

Example: plant geometry

Plant's SVD ($0 \leq \alpha \leq 1$):

$$P = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \right) \begin{bmatrix} 2 & 0 \\ 0 & 2\alpha \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right).$$

If $\alpha = 0$, then

- only inputs co-directed with $\text{span}\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right)$ can affect the response
- only outputs co-directed with $\text{span}\left(\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]\right)$ can be affected

and these directions are orthogonal. In other words,

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But

- $R_2 = kI$ doesn't change direction of measurements!

Hence, the system effectively works in open loop with this controller.

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Outline

Generating subspaces

Lemma

A set $\mathcal{S} \subset \mathbb{F}^n$ is a subspace iff either of the following conditions holds:

- $\exists\, d \le n$ *and a full-rank matrix* $S_i \in \mathbb{F}^{n\times d}$ *such that* $\mathcal{S} = \operatorname{Im} S_i$
- $\exists\, d \le n$ *and a full-rank matrix* $S_k \in \mathbb{F}^{(n-d)\times n}$ *such that* $\mathcal{S} = \ker S_k$

Moreover, this d is the dimension of $\mathcal{S}$.

Proof.

Because both image and kernel are subspaces, the "if" part in both cases is immediate as well as the fact that $\dim(\mathcal{S}) = d$. To show the "only if" part, let $\mathcal{S}$ be a d-dimensional subspace and $\{s_1, \ldots, s_d\}$ be its basis. By the very definition, $\mathcal{S} = \operatorname{Im} S_{\mathrm{i}}$ for $S_{\mathrm{i}} = \begin{bmatrix} s_1 & \cdots & s_d \end{bmatrix}$. Likewise, if $\{s_{d+1}, \ldots, s_n\}$ is a basis of the $(n-d)$-dimensional space $\mathcal{S}^\perp$, then $S_{\mathrm{k}} = \begin{bmatrix} s_{d+1} & \cdots & s_n \end{bmatrix}'$ is what we need. □

Example: freedom of choice

Let $y = Gu$ for $G = \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}$. Find all u such that $y = 1$. Remember that $y = Gu \iff u = u_0 + u_n$ for $u_n \in \ker G$. A particular solution is

$$u_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

To characterize the kernel, bring in SVD

$$G = 1 \cdot \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0 & 0.8 \\ -0.8 & 0 & 0.6 \\ 0 & 1 & 0 \end{bmatrix},$$

so

$$\ker G = \operatorname{span}\left(\begin{bmatrix} -0.8 \\ 0 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \operatorname{Im} \begin{bmatrix} -0.8 & 0 \\ 0 & 1 \\ 0.6 & 0 \end{bmatrix}$$

Hence,

$$u = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.8 & 0 \\ 0 & 1 \\ 0.6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for arbitrary v_1 and v_2.

Example: freedom of choice

Let $y = Gu$ for $G = \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}$. Find all u such that $y = 1$. Remember that $y = Gu \iff u = u_0 + u_n$ for $u_n \in \ker G$. A particular solution is

$$u_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

To characterize the kernel, bring in SVD

$$G = 1 \cdot \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0 & 0.8 \\ -0.8 & 0 & 0.6 \\ 0 & 1 & 0 \end{bmatrix},$$

so

$$\ker G = \operatorname{span}\left(\begin{bmatrix} -0.8 \\ 0 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \operatorname{Im} \begin{bmatrix} -0.8 & 0 \\ 0 & 1 \\ 0.6 & 0 \end{bmatrix}$$

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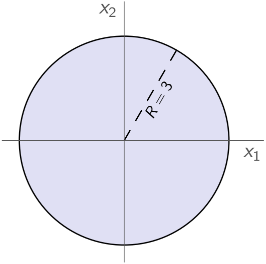
Exotic metrics

There may be reasons to define unorthodox metrics, like a metric in which

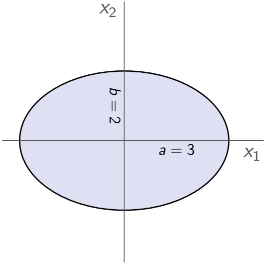
- x is "small" if $\|x\| \le \gamma$ for some $\gamma > 0$ (uniform scaling)
- x_i is more important than x_j (non-uniform scaling)
- important are various linear combinations of x_i

$\mathcal{B}_a$:



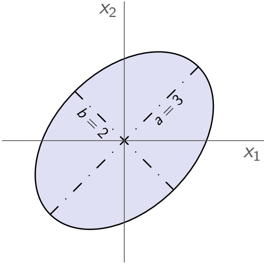
$$\sqrt{\left(\tfrac{x_1}{3}\right)^2 + \left(\tfrac{x_2}{3}\right)^2}$$

$\mathcal{B}_b$:



$$\sqrt{\left(\tfrac{x_1}{3}\right)^2 + \left(\tfrac{x_2}{2}\right)^2}$$

$\mathcal{B}_c$:



$$\sqrt{\left(\tfrac{x_1+x_2}{3\sqrt{2}}\right)^2 + \left(\tfrac{x_1-x_2}{2\sqrt{2}}\right)^2}$$

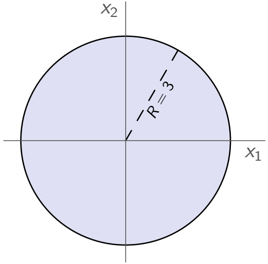
Exotic metrics modeled via standard metrics

Rather than dreaming up new metrics (unhandy), we can recycle existing. For example, we can define scaled unit balls as

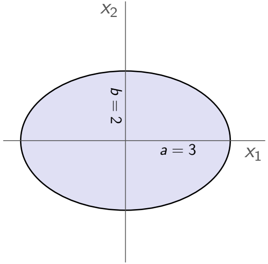
$$G\mathcal{B}_q := \left\{ x \mid x = Gu, \|u\|_q \leq 1 \right\},$$

which would result in

$\mathcal{B}_a = 3\mathcal{B}_2$:



$\mathcal{B}_b = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\mathcal{B}_2$:



$\mathcal{B}_c = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}\mathcal{B}_2$: