Linear Control Systems (036012)
lecture no. 1

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT

Outline

Course info

- Credit points: 3
- Prerequisite: Control Theory (035188)
- Grading policy: homework 100% (4 best out of 5 assignments)
  Homework solutions must be submitted electronically to *c036012@technion.ac.il*
- Course site: `http://leo.technion.ac.il/Courses/LCS/`
- Literature:
  1. My lecture notes (available at the course site)
  2. Skogestad, S. & I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design*, John Wiley & Sons, 1996.
  3. Doyle, J. C., B. A. Francis, & A. Tannenbaum. *Feedback Control Theory*, MacMillan, 1992 (available online).
  4. Zhou, K., J. C. Doyle, & K. Glover. *Robust and Optimal Control*, Prentice Hall, 1995.

Syllabus

1. Stand-alone systems
   - static MIMO systems
   - dynamic MIMO systems
     - basic notions (stability, causality, domain) in time and transformed domains
     - coprime factorization in H_∞
     - poles and zeros of rational transfer functions
     - state-space realizations and their structural and computational properties
     - model order reduction via balanced truncation

2. Interconnected systems
   - basic interconnections and their effects on dynamics, LFTs
   - stability and stabilization
     - internal stability
     - general stability results (Small Gain and Passivity)
     - all stabilizing controllers (Youla–Kučera)
   - optimization-based performance
     - weighted / mixed sensitivity problems, the standard problem
     - balanced sensitivity (H_∞ loop shaping)

Outline

Why and for whom

Course goals

1. MIMO literacy (badly lacking, especially in industry)
2. Power and limitations of optimization-based design methods

Background needed

- linear algebra (see Appendix A of Lecture Notes)
- SISO systems
- classical SISO control methods

Hazard (or opportunity, depends on preferences)

- analytic material (no choice, hand-waving hurts in MIMO more)

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A quiz

What in the order and what are poles and zeros of the transfer matrices:

1. $G(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$

2. $G(s) = \begin{bmatrix} 1/s & 1/s \\ 1/s & 1/s \end{bmatrix}$

3. $G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix}$

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1. $G(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$ $(n = 2$, poles: $\{0, 0\}$, zeros: $\varnothing)$

2. $G(s) = \begin{bmatrix} 1/s & 1/s \\ 1/s & 1/s \end{bmatrix}$

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3. $G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix}$ $\qquad$ ($n = 1$, pole: $\{0\}$, zero: $\{0\}$)

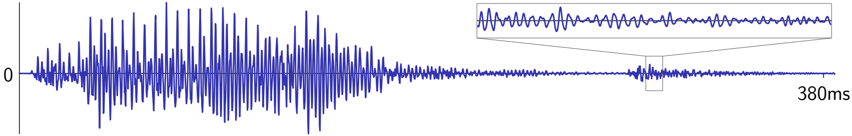
Nomenclature

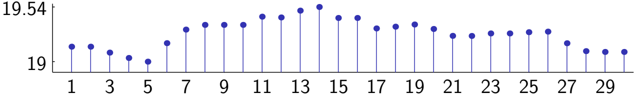
$\mathbb{R}$	the set of real numbers, $\mathbb{R} = (-\infty, \infty)$
$\mathbb{R}_+$	the set of nonnegative real numbers, $\mathbb{R}_+ = [0, \infty)$
$\mathbb{R}_-$	the set of nonpositive real numbers, $\mathbb{R}_- = (\infty, 0]$
$\mathrm{j}\mathbb{R}$	the set of pure imaginary numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_\alpha$	the half plain to the right of $\alpha \in \mathbb{R}$, i.e. $\mathbb{C}_\alpha : \{z \in \mathbb{C} \mid \operatorname{Re} z > \alpha\}$
$\bar{\mathbb{C}}_\alpha$	the closure of $\mathbb{C}_\alpha$, i.e. $\bar{\mathbb{C}}_\alpha : \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \alpha\}$
$\mathbb{T}$	the unit circle, $\mathbb{T} := \{z \in \mathbb{C} \mid \lvert z \rvert = 1\}$
$\mathbb{D}_\alpha$	the open α-disk, $\mathbb{D} := \{z \in \mathbb{C} \mid \lvert z \rvert < \alpha\}$
$\bar{\mathbb{D}}_\alpha$	the closed α-disk, $\bar{\mathbb{D}} := \{z \in \mathbb{C} \mid \lvert z \rvert \leq \alpha\} = \mathbb{D}_\alpha \cup (\alpha\mathbb{T})$
$\mathbb{F}$	alias of either $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of positive integers (natural numbers)
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}_-$	the set of nonpositive integers, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$
$\mathbb{Z}_{i_1..i_2}$	the interval $\{i_1, i_1 + 1, \ldots, i_2\}$

Outline

Signals

Represent evolving information:





Mathematically,

- functions of independent variables, $f(t)$ or $f[t]$

Signals

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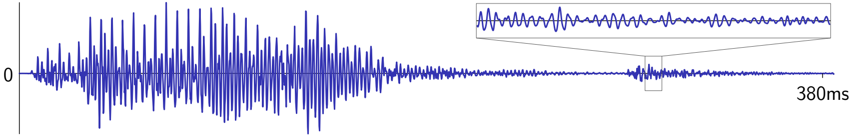
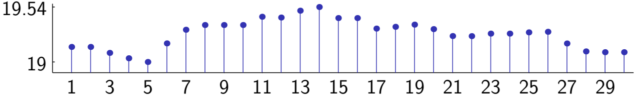


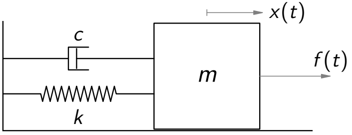


Mathematically,

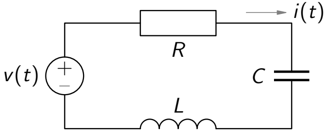
- functions of independent variables, $f(t)$ or $f[t]$

Systems

Constraints imposed on signals:



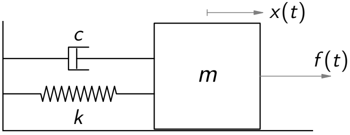
position $x(t)$ and force $f(t)$

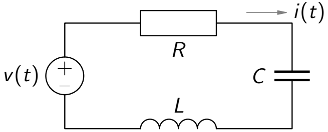


current $i(t)$ and voltage $v(t)$

Systems

Constraints imposed on signals:





position $x(t)$ and force $f(t)$

$x \mapsto f$ or $f \mapsto x$

current $i(t)$ and voltage $v(t)$

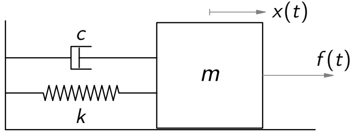
$v \mapsto i$ or $i \mapsto v$

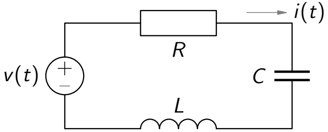
I/O view on systems:

- some signals act (inputs)
- some signals react (outputs)

Mathematical models

(Approximate) description in a mathematical language:





position and force linked as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

charge ($\dot{q} = i$) and voltage linked as

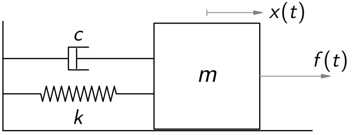
$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

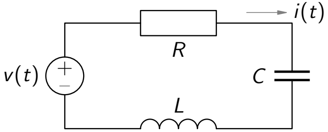
Abstract form:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k_{st}\omega_n^2 u(t)$$

Mathematical models

(Approximate) description in a mathematical language:





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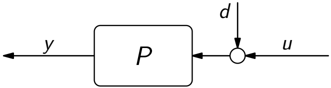
Outline

Prototype control problem



- y: regulated signal
- u: control signal (means)
- d: load disturbance
- P: plant

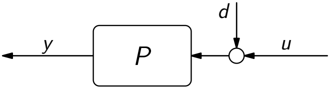
Goal:

$$u \quad \longrightarrow \quad y = r$$

where

- r: reference signal (goal)

Ultimate methodology: plant inversion



$$y = P(d + u) \;\wedge\; y = r$$

$$\Downarrow$$

$$r = P(d + u)$$

$$\Downarrow$$

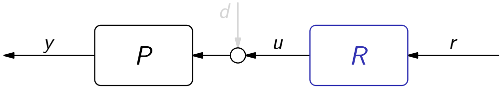
$$u = P^{-1}r - d$$

where

- P^{-1} is the inverse system

defined via $y = Pu \iff u = P^{-1}y$, with $P^{-1}(s) = \dfrac{1}{P(s)}$.

Open-loop plant inversion



with

$$R = P^{-1}$$

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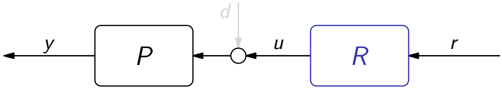

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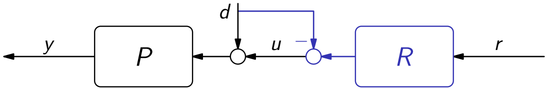

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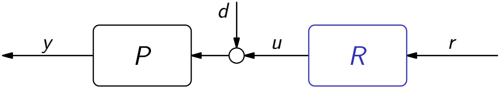
Limitations of open-loop plant inversion: stability



All signals,

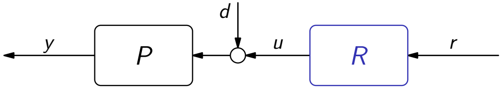
$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} PR & P \\ R & 0 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix},$$

bounded (internal stability).

Must have:

- P stable
- R stable, if $R = P^{-1} \implies P$ stably invertible

Limitations of open-loop plant inversion: stability



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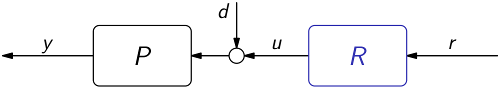

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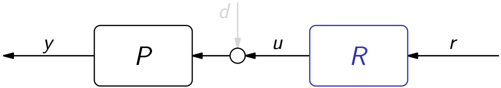
Approximate open-loop plant inversion



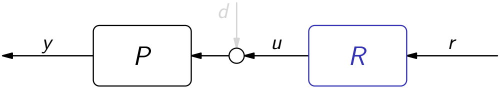
Pragmatic alternative:

$$R \approx P^{-1} r \quad \Longrightarrow \quad R = P^{-1} T_{\text{ref}}\, r$$

Reference model:

- T_{ref} stable
- $P^{-1} T_{\text{ref}}$ stable (proper, poles in $\operatorname{Re} s < 0$)
- $T_{\text{ref}} \approx 1$

Approximate open-loop plant inversion



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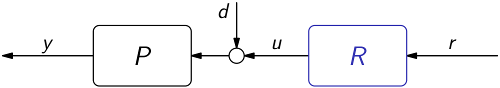
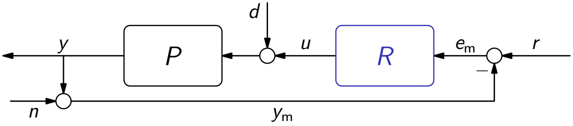
Limitations of open-loop plant inversion: other



- unmeasured d — nothing to do
- uncertain P — nothing to do
- limited u — bandwidth limitations

Closed-loop control



Gang of four:

$$\begin{bmatrix} S(s) & T_c(s) \\ T_d(s) & T(s) \end{bmatrix} := \frac{1}{1+P(s)R(s)} \begin{bmatrix} 1 & R(s) \\ P(s) & P(s)R(s) \end{bmatrix}$$

Signals:

$$\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} T & T_d & -T \\ T_c & -T & -T_c \\ S & -T_d & T \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix},$$

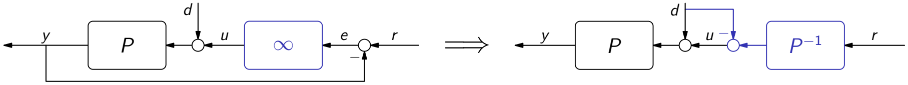
where $e := r - y = e_m + n$.

Closed-loop plant inversion

Because

$$T_{\mathrm{c}} = \frac{1}{1/R + P} \xrightarrow{R\to\infty} \frac{1}{P} \quad \text{and} \quad -T = -\frac{P}{1/R + P} \xrightarrow{R\to\infty} -1,$$

we have

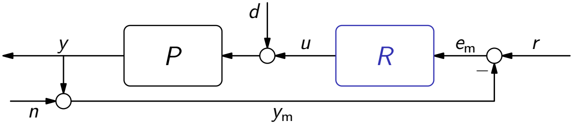


Thus,

$$T_{\mathrm{d}} = \frac{P}{1 + PR} \xrightarrow{R\to\infty} 0 \quad \text{and} \quad S = \frac{1}{1 + PR} \xrightarrow{R\to\infty} 0,$$

independently of the plant and w/o explicit measurements of d.

Limitations of closed-loop plant inversion



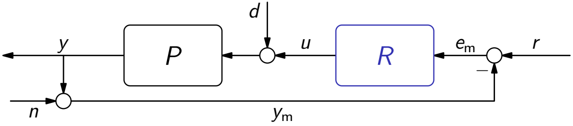
- closed-loop stability
- closed-loop stability
- closed-loop stability
- measurement noise sensitivity
- limited u
- ...

Hence,

- nontrivial tradeoffs

Outline

Problem



for

$$P(s) = \begin{bmatrix} 1+\alpha & 1-\alpha \\ -1+\alpha & -1-\alpha \end{bmatrix}, \quad \alpha \in [0,1]$$

Relations:

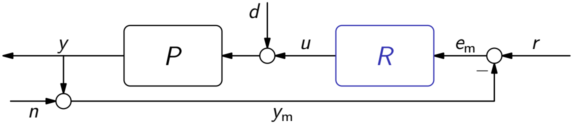
$$\begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} T_\mathrm{c} & -T_\mathrm{i} \\ S_\mathrm{o} & -T_\mathrm{d} \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix},$$

where

$$\begin{bmatrix} T_\mathrm{c}(s) & T_\mathrm{i}(s) \\ S_\mathrm{o}(s) & T_\mathrm{d}(s) \end{bmatrix} := \begin{bmatrix} R(s) \\ I \end{bmatrix} (I + P(s)R(s))^{-1} \begin{bmatrix} I & P(s) \end{bmatrix}.$$

($S_\mathrm{o} \neq I - T_\mathrm{i}$ in general).

Design 1



If

$$R(s) = k \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then

$$\begin{bmatrix} T_i & T_c \end{bmatrix} = \frac{k}{2k+1} \left[\begin{array}{cc:cc} \frac{4\alpha k-1-\alpha}{2\alpha k+1} & \frac{1-\alpha}{2\alpha k+1} & \frac{(1+\alpha)k+1}{2\alpha k+1} & \frac{(1-\alpha)k}{2\alpha k+1} \\ \frac{1-\alpha}{2\alpha k+1} & \frac{4\alpha k+1+\alpha}{2\alpha k+1} & -\frac{(1-\alpha)k}{2\alpha k+1} & -\frac{(1+\alpha)k+1}{2\alpha k+1} \end{array}\right]$$

and

$$\begin{bmatrix} T_d & S_o \end{bmatrix} = \frac{1}{2k+1} \left[\begin{array}{cc:cc} \frac{4\alpha k-1-\alpha}{2\alpha k+1} & \frac{1-\alpha}{2\alpha k+1} & \frac{(1+\alpha)k+1}{2\alpha k+1} & \frac{(1-\alpha)k}{2\alpha k+1} \\ -\frac{1-\alpha}{2\alpha k+1} & -\frac{4\alpha k+1+\alpha}{2\alpha k+1} & \frac{(1-\alpha)k}{2\alpha k+1} & \frac{(1+\alpha)k+1}{2\alpha k+1} \end{array}\right]$$

Design 1 (contd)

If $\alpha \neq 0$, then

$$\lim_{k\to\infty} -T_{\mathrm{i}} = -I \quad \text{and} \quad \lim_{k\to\infty} T_{\mathrm{c}} = \frac{1}{4\alpha}\begin{bmatrix} 1+\alpha & 1-\alpha \\ -1+\alpha & -1-\alpha \end{bmatrix} = P^{-1}$$

and

$$u \to P^{-1}r - d \quad \text{and} \quad e \to 0,$$

exactly as in the SISO case.

Design 1 (contd)

If $\alpha = 0$ ($\det P = 0$), then

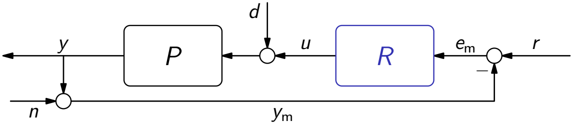
$$\lim_{k\to\infty} -T_{\mathrm{i}} = -\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \lim_{k\to\infty} T_{\mathrm{c}} = \frac{1}{2}\begin{bmatrix} k+1 & k \\ -k & -k-1 \end{bmatrix}\Bigg|_{k\to\infty}$$

and

$$u \to \frac{1}{2}\begin{bmatrix} k+1 & k \\ -k & -k-1 \end{bmatrix}\Bigg|_{k\to\infty} r - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} d \quad \text{and} \quad e \to \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} r,$$

different from the SISO case.

Design 2



If

$$R(s) = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then

$$\begin{bmatrix} T_i & T_c \end{bmatrix} = \frac{k}{4\alpha k^2 - 1} \left[\begin{array}{cc:cc} 4\alpha k - 1 - \alpha & -1 + \alpha & (1+\alpha)k - 1 & (1-\alpha)k \\ 1 - \alpha & 4\alpha k + 1 + \alpha & -(1-\alpha)k & -(1+\alpha)k - 1 \end{array} \right]$$

and

$$\begin{bmatrix} T_d & S_o \end{bmatrix} = \frac{1}{4\alpha k^2 - 1} \left[\begin{array}{cc:cc} 4\alpha k - 1 - \alpha & -1 + \alpha & (1+\alpha)k - 1 & (1-\alpha)k \\ 1 - \alpha & 4\alpha k + 1 + \alpha & -(1-\alpha)k & -(1+\alpha)k - 1 \end{array} \right]$$

Design 2 (contd)

If $\alpha \neq 0$, then

$$\lim_{k\to\infty} -T_{\mathrm{i}} = -I \quad \text{and} \quad \lim_{k\to\infty} T_{\mathrm{c}} = \frac{1}{4\alpha}\begin{bmatrix} 1+\alpha & 1-\alpha \\ -1+\alpha & -1-\alpha \end{bmatrix} = P^{-1}$$

and

$$u \to P^{-1}r - d \quad \text{and} \quad e \to 0,$$

exactly as in the SISO case.

Design 2 (contd)

If $\alpha = 0$ ($\det P = 0$), then

$$T_{\rm d} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad S_{\rm o} = \begin{bmatrix} -k+1 & -k \\ k & k+1 \end{bmatrix}, \quad T_{\rm i} = kT_{\rm d}, \quad T_{\rm c} = kS_{\rm o}$$

and

$$u \to \left(\begin{bmatrix} -k(k-1) & -k^2 \\ k^2 & k(k+1) \end{bmatrix} r - k \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} d \right) \Big|_{k\to\infty} \quad \text{and} \quad e \to \frac{1}{k} u$$

different from the SISO case.