Control Theory (035188)
lecture no. 12

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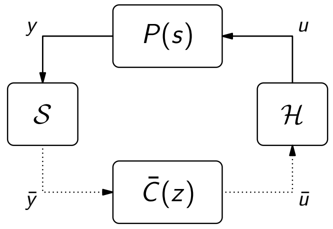
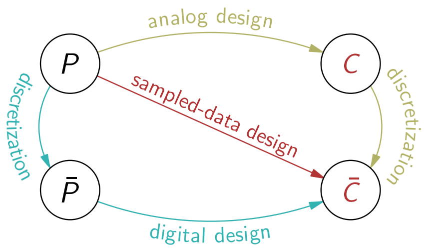
Three approaches to sampled-data control design

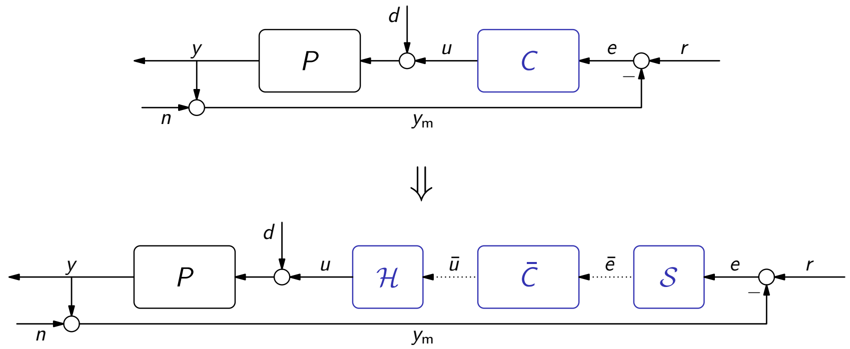




1. Digital redesign of analog controllers
   (do your favorite analog design first, then discretize the resulting controller)
2. Discrete-time design
   (discretize the problem first, then do your favorite discrete design)
3. Direct digital (sampled-data) design
   (design discrete-time controller $\bar{C}(z)$ directly for analog specs)

Outline

The redesign problem

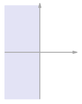
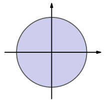


Starting point:
- "good" analog controller C (designed by whatever method)

Goal:
- find $\bar{C}$ such that $\mathcal{H}\bar{C}\mathcal{S} \approx C$

(we consider $\mathcal{S} = \mathcal{S}_{\text{idl}}$, $\mathcal{H} = \mathcal{H}_{\text{ZOH}}$, and periodic sampling with given $h > 0$).

Discrete transfer functions

Continuous-time systems	Discrete-time systems
Laplace transform	$\mathcal{Z}$-transform
s is the derivative in the time domain	z is the shift in the time domain
Left-half plane in the s-plane : 	 : Unit disk in the z-plane
$j\omega$-axis	Unit circle
Static gain is $G(s)\vert_{s=0} = G(0)$	Static gain is $G(z)\vert_{z=1} = G(1)$
Integral action: pole at $s = 0$	Integral action: pole at $z = 1$

Choice of $\bar{C}(z)$

Philosophy is to

– imitate $C(s)$ in low-frequency and crossover ranges.

Often based on numerical differentiation rules, like

forward Euler: $\dot{x}(ih) \approx \dfrac{x(ih+h) - x(ih)}{h} \implies s \approx \dfrac{z-1}{h}$

backward Euler: $\dot{x}(ih) \approx \dfrac{x(ih) - x(ih-h)}{h} \implies s \approx \dfrac{z-1}{hz}$

Tustin[1]: $\dfrac{\dot{x}(ih+h) + \dot{x}(ih)}{2} \approx \dfrac{x(ih+h) - x(ih)}{h} \implies s \approx \dfrac{2}{h}\dfrac{z-1}{z+1}$

making sense if h is "small enough."

Example

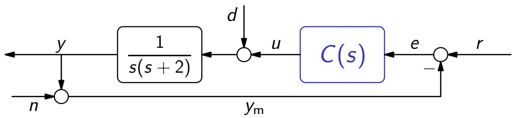
If $C(s) = 1/(s+1)$, then

$$\bar{C}(z) = C(s)\big|_{s=\frac{2}{h}\frac{z-1}{z+1}} = \frac{1}{2/h\cdot(z-1)/(z+1)+1} = \frac{h(z+1)}{(h+1)z+h-1}.$$

[1] MATLAB: `c2d(C,h,'tustin')`, where `C` is a continuous-time system.

Example: DC motor

A DC motor from Lecture 1, controlled in closed loop



Requirements:

– closed-loop stability (of course)
– zero steady-state error for a step in r — always holds
– zero steady-state error for a step in d — integrator in $C(s)$
– good stability margins
– $\omega_c \approx 2$ [rad/sec]

Design:

– LQG loop shaping, with a PI weight W (like in Lecture 10)

Example: analog design

Weight:

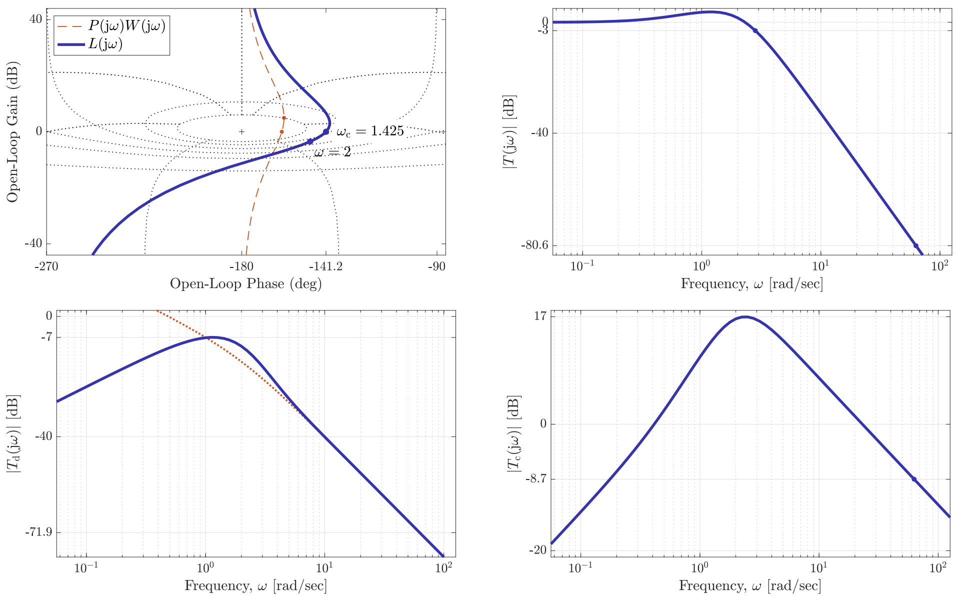
$$W(s) = 5.06\Big(1+\frac{1}{s}\Big)$$

Controller:

$$C(s) = W(s)C_a(s) = \frac{23.081(s+2.075)(s+0.5346)}{s(s^2+6.155s+17.44)}$$

(a pole of $C_a(s)$ cancels the zero of $W(s)$ at $s = 1$). The actual crossover is $\omega_c = 1.4248$ and the closed-loop bandwidth is $\omega_b = 2.8155$.

Example: analog design (contd)



Example: controller discretization

Using Tustin, the discretized controllers are

$$h = 0.01: \quad C(z) = \frac{0.11337(z+1)(z-0.9795)(z-0.9947)}{(z-1)(z^2-1.939z+0.9403)}$$

$$h = 0.1: \quad \bar{C}(z) = \frac{0.96777(z+1)(z-0.812)(z-0.9479)}{(z-1)(z^2-1.415z+0.5445)}$$

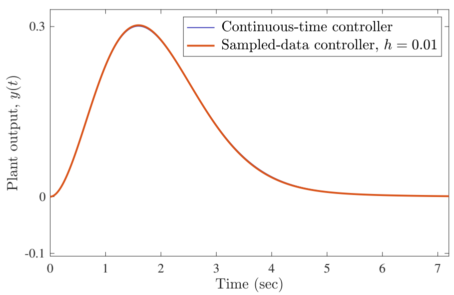
Both

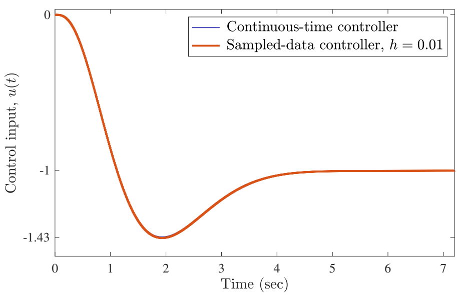
– preserve integral actions (pole at $s = 0 \to$ pole at $z = 1$)

which is a general property of the Tustin transformation.

Example: $d(t) = \mathbb{1}(t)$ and $n(t) = 0$

Responses with $h = 0.01$:





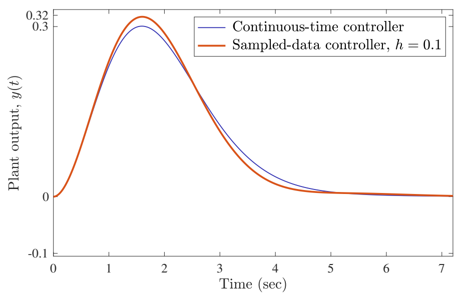
sampled-data response $\approx$ analog response

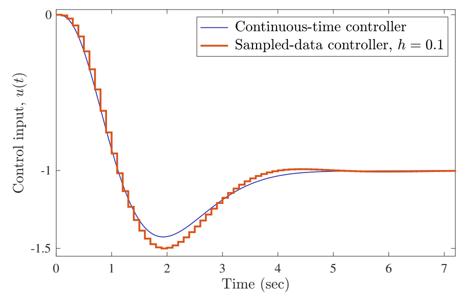
$\Downarrow$

adequate sampling rate

Example: $d(t) = \mathbb{1}(t)$ and $n(t) = 0$

Now the same with $h = 0.1$:





sampled-data response starts getting worse than analog response

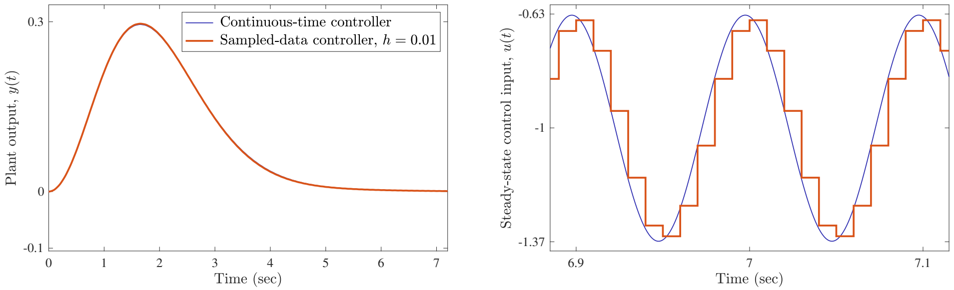
$\Downarrow$

sampling rate starts to become problematic

(further increase of h eventually results in an unstable closed-loop system).

Example: $d(t) = \mathbb{1}(t)$ and $n(t) = \sin(20\pi t + 0.1)$

Responses with $h = 0.01$:



sampled-data response $\approx$ analog response

$\Downarrow$

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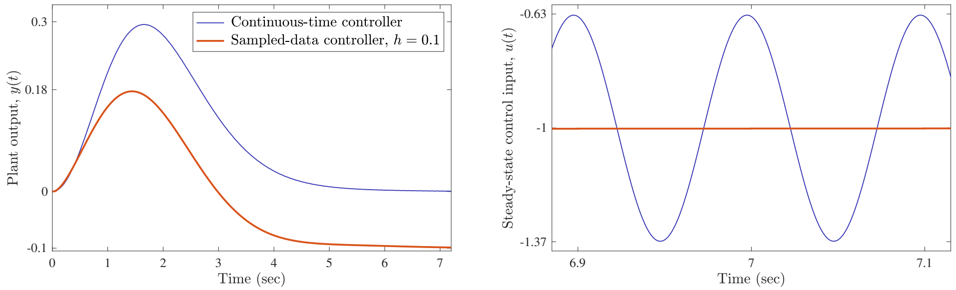
Example: $d(t) = \mathbb{1}(t)$ and $n(t) = \sin(20\pi t + 0.1)$

Now the same with $h = 0.1$:



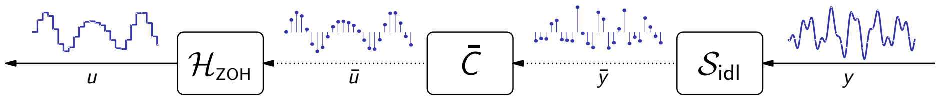
Oops,

– sampled-data response is qualitatively different from analog response
  (steady-state error is nonzero, the harmonic of measurement noise disappears)

Why? It seems that we lack understanding of what's going on here ...

Sampled-data controllers

Consider



(dubbed sample-and-hold circuit if $\bar{C} = 1$). Our goal below is to understand the relation between $Y(j\omega)$ and $U(j\omega)$. That might not be easy because

– sampled-data controllers are not time invariant

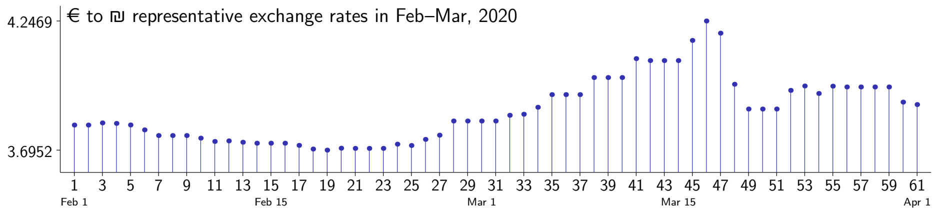
with all consequences of that:

– no convolution representation
– no transfer function / frequency response as multiplication
– harmonic inputs might not remain harmonic at the output

Outline

Signals

reflect evolving information:

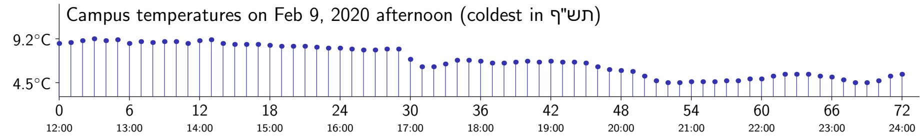


Mathematically, signals are

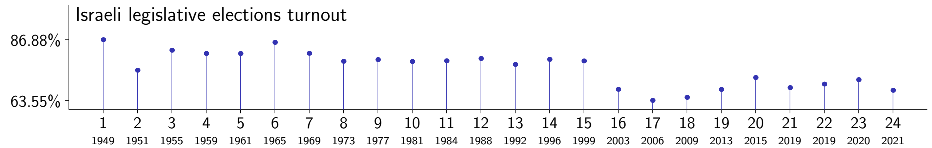
- functions of independent variables, like time
- discrete signals are functions $\mathbb{Z} \to \mathbb{R}^n$
- denoted as $f[t]$ (square brackets differentiate from analog signals)

Examples of discrete signals

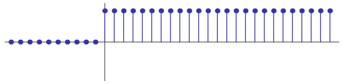
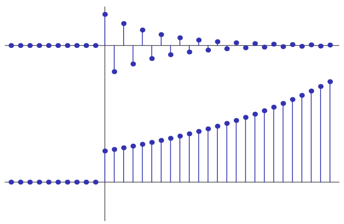
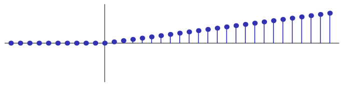
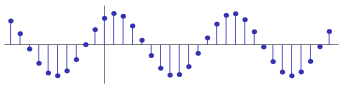
Some are discretized versions of analog signals:



Some are intrinsically discrete:



Basic discrete signals

- pulse: $f[t] = \delta[t] =$ 
- step: $f[t] = \mathbb{1}[t] =$ 
- exponential: $f[t] = \lambda^t\, \mathbb{1}[t] =$  if $|\lambda| < 1$; if $|\lambda| > 1$
- ramp: $f[t] = t\mathbb{1}[t] =$ 
- sinusoid[2]: $f[t] = \sin[\omega t + \phi] =$ 

[2] Periodic, not necessarily with $T = 2\pi/\omega$, only if $\omega = 2\pi\alpha$ for some $\alpha \in \mathbb{Q}$ (rational).

Energy and power

Energy of signal $f[t]$ is the quantity

$$E_f = \sum_{t=-\infty}^{\infty} |f[t]|^2$$

It can be viewed as a

- measure of size of f for a decaying f or f having a finite support

Power of signal $f[t]$ is defined as averaged energy per unit time:

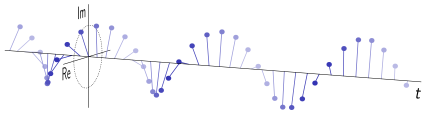
$$P_f = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{t=-M}^{M} |f[t]|^2$$

It can be viewed as a

- measure of size of f for a persistent f

Discrete-time harmonic signals

Signal

$$f[t] = \gamma e^{j\theta t} =$$ 

$$, \quad \gamma \in \mathbb{C}, \theta \in \mathbb{R}$$

where θ is the frequency, $|\gamma|$ is the amplitude, and $\phi = \arg \gamma$ is the initial phase, is called the discrete harmonic signal. By Euler's formula,

$$\mathrm{Re}(\gamma e^{j\theta t}) = |\gamma| \cos[\theta t + \phi] \quad \text{and} \quad \mathrm{Im}(\gamma e^{j\theta t}) = |\gamma| \sin[\theta t + \phi].$$

Hence, the discrete harmonic signal may be thought of as a plain sinusoid.

Two qualitative deviations from the continuous-time case:

- $\gamma e^{j\theta t}$ might not be periodic (if $2\pi/\theta$ is irrational);
- because

$$e^{j(\theta + 2\pi i)t} = e^{j\theta t} e^{j2\pi it} = e^{j\theta t}, \qquad \forall i \in \mathbb{Z}$$

  we may only consider $\theta \in [-\pi, \pi]$ and the highest frequency is $|\theta| = \pi$.

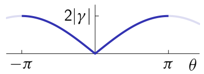
Pace of harmonic signals

The difference operator Δ, for which

$$(\Delta f)[t] := f[t+1] - f[t],$$

may be viewed as the discrete counterpart of the derivative. A size of Δf is then a measure of the pace of f.

If $f[t] = \gamma e^{j\theta t}$, then

$$|f[t+1] - f[t]| = |\gamma e^{j\theta t}(e^{j\theta} - 1)| = |\gamma| \sqrt{2 - 2\cos\theta} =$$ 

Because this is a strictly increasing function of $|\theta| \in [0, \pi]$, we conclude that

- $\gamma e^{j\theta_1 t}$ is faster (slower) than $\gamma e^{j\theta_2 t}$ if $|\theta_1| > |\theta_2|$ ($|\theta_1| < |\theta_2|$),

provided both θ_1 and θ_2 are in $[-\pi, \pi]$). Thus, the fastest discrete harmonic signal is

$$\gamma e^{\pm j\pi t} = \gamma(-1)^t.$$

Discrete-time Fourier transform (DTFT)

Given $f : \mathbb{Z} \to \mathbb{F}^n$, its discrete-time Fourier transform

$$\mathfrak{F}\{f\} = F(e^{j\theta}) := \sum_{t \in \mathbb{Z}} f[t] e^{-j\theta t},$$

for the angular frequency $\theta \in [-\pi, \pi]$ (in radians per step).

If the range of θ is extended to the whole $\mathbb{R}$, then $F(e^{j\theta})$ is 2π-periodic as a function of θ, $F(e^{j(\theta + 2\pi)}) = F(e^{j\theta})$.

Strictly speaking, $\mathfrak{F}\{f\}$ exists as a function of θ only if

- $\sum_t |f[t]| < \infty$ (or $E_f < \infty$, if a weaker convergence is used).

Inverse DTFT:

$$\mathfrak{F}^{-1}\{F\} = f[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\theta}) e^{j\theta t} d\theta.$$

DTFT: interpretation

It follows from

$$f[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\theta}) e^{j\theta t} d\theta.$$

that $f[t]$ is a superposition of elementary harmonic signals $e^{j\theta t}$. The signal

- $F(e^{j\theta})$ is the frequency-domain representation (or spectrum) of $f[t]$.

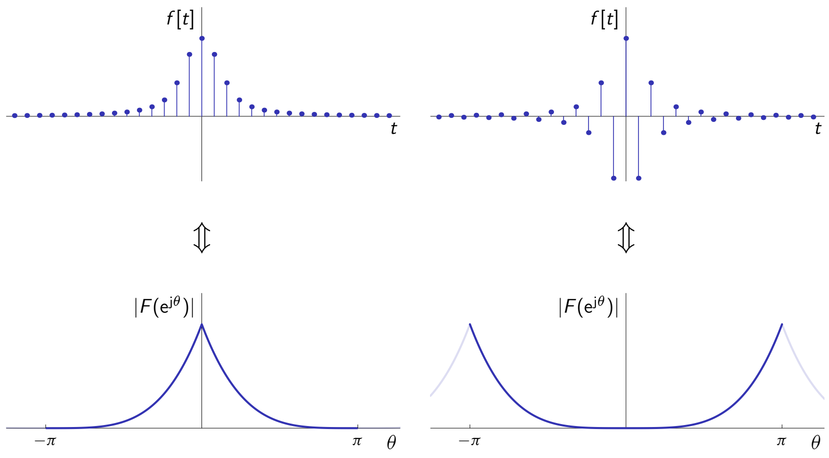
$F(e^{j\theta_0})$ quantifies the contribution of $e^{j\theta_0 t}$ to $f[t]$.

Hence, spectrum offers a

- viewpoint on f, where fast and slow components are separated.

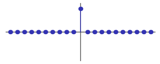
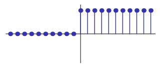
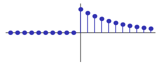
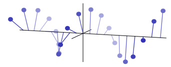
DTFT: interpretation (contd)

"slow" $f[t]$ ($E_{\Delta f} = 0.0188$): "fast" $f[t]$ ($E_{\Delta f} = 0.425$):



DTFTs of some discrete signals

Assuming $\theta \in [-\pi, \pi]$,

$f[t]$		$F(e^{j\theta})$	condition
$\delta[t] =$		1	
$\mathbb{1}[t] =$		$\dfrac{1}{1 - e^{-j\theta}} + \pi\delta(\theta)$	
$\lambda^t \mathbb{1}[t] =$		$\dfrac{1}{1 - \lambda e^{-j\theta}}$	$\lvert\lambda\rvert < 1$
$e^{j\theta_0 t} =$		$2\pi\delta(\theta - \theta_0)$	$\theta_0 \in [-\pi, \pi]$

Some properties of DTFT (assuming transforms exist)

Linearity: for all constants α_1 and α_2,

$$\mathfrak{F}\{\alpha_1 f_1 + \alpha_2 f_2\} = \alpha_1 \mathfrak{F}\{f_1\} + \alpha_2 \mathfrak{F}\{f_2\},$$

Time shift: if $qf[t] := f[t+1]$, then for every $\tau \in \mathbb{Z}$,

$$\mathfrak{F}\{q^\tau f\} = e^{j\theta\tau}\mathfrak{F}\{f\}$$

Time reversal: if $g[t] = f[-t]$, then

$$\mathfrak{F}\{g\}(e^{j\theta}) = \mathfrak{F}\{f\}(e^{-j\theta}).$$

Convolution: for all f and g,

$$\mathfrak{F}\{f * g\} = \mathfrak{F}\{f\}\,\mathfrak{F}\{g\}$$

where $(f * g)[t] := \sum_{s\in\mathbb{Z}} f[t-s]g[s] = \sum_{s\in\mathbb{Z}} f[s]g[t-s]$.

Parseval's theorem

If $f[t]$ is a finite energy signal, then

$$E_f = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(e^{j\theta})|^2 d\theta =: \frac{1}{2\pi} E_F$$

i.e. the energy of $f[t]$ equals that of its DTFT $F(e^{j\theta})$, modulo the factor $1/(2\pi)$, exactly like in the continuous-time case.

Implications:

- $|F(e^{j\theta})|$ shows the contribution of the harmonic $e^{j\theta t}$ to E_f
- reduction any parts of $|F(e^{j\theta})|$ reduces the power of $f[t]$
- harmonics with highest $|F(e^{j\theta})|$ dominate the behavior of $f[t]$

Discrete systems in time and frequency domains

Any LTI system $y = Gu$ can be described as

$$y[t] = (g * u)[t] = \sum_{s\in\mathbb{Z}} g[t-s]u[s] = \sum_{s\in\mathbb{Z}} g[s]u[t-s]$$

(convolution form), where $g(t)$ is the impulse response of G. Hence,

$$Y(\mathrm{e}^{\mathrm{j}\theta}) = G(\mathrm{e}^{\mathrm{j}\theta})U(\mathrm{e}^{\mathrm{j}\theta})$$

where

– $G(\mathrm{e}^{\mathrm{j}\theta}) = \mathfrak{F}\{g\}$ the frequency response of G

(and $G(\mathrm{e}^{\mathrm{j}\theta}) = G(z)|_{z=\mathrm{e}^{\mathrm{j}\theta}}$). Because

$$u[t] = \mathrm{e}^{\mathrm{j}\theta t} \implies (Gu)[t] = \sum_{s\in\mathbb{Z}} g[s]\mathrm{e}^{\mathrm{j}\theta(t-s)} = G(\mathrm{e}^{\mathrm{j}\theta})\mathrm{e}^{\mathrm{j}\theta t}$$

at each frequency $\theta \in [-\pi, \pi]$

– $G(\mathrm{e}^{\mathrm{j}\theta})$ characterizes how the harmonic $\mathrm{e}^{\mathrm{j}\theta t}$ processed by the system G.

Outline

Analog redesign: Part I

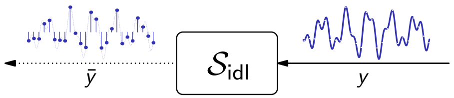
Discrete signals in time and frequency domains

A/D conversion in frequency domain

D/A conversion in frequency domain

The Sampling Theorem (Whittaker-Kotel'nikov-Shannon)

What do we lose by sampling analog signals ?



From the time-domain relation

$$\bar{y}[i] = y(ih)$$

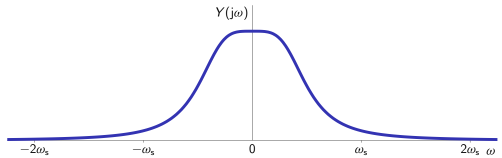
we know that all intersample information about $y(t)$ is lost. But

– to what extent is it important (if at all) ?

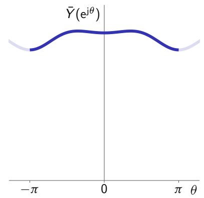
To answer this kind of questions, frequency-domain analysis is indispensable.

Key question

How can we squeeze the spectrum of a continuous-time signal $y(t)$,

$Y(\mathrm{j}\omega) =$ 

(with $\omega \in \mathbb{R}$) into the spectrum of its sampled version $\bar{y}[i] = y(ih)$,

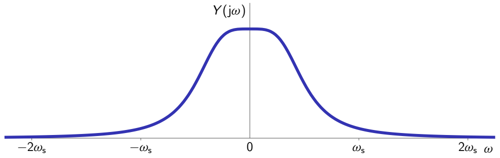
$\bar{Y}(\mathrm{e}^{\mathrm{j}\theta}) =$ 

(with $\theta \in [\pi, \pi]$) ?

A weird function

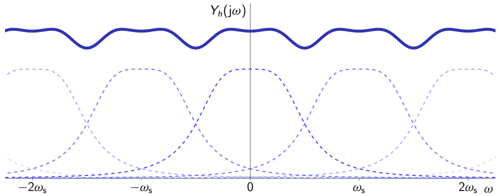
Consider an analog signal $y(t)$ with the spectrum $Y(\mathrm{j}\omega)$, e.g.

$$Y(\mathrm{j}\omega) =$$



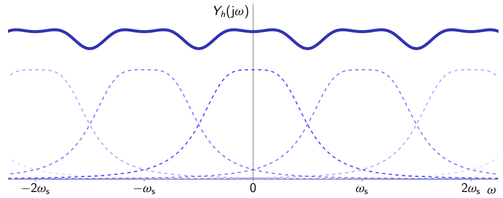
and define, for some $h > 0$, the function

$$Y_h(\mathrm{j}\omega) := \frac{1}{h}\sum_{i\in\mathbb{Z}} Y(\mathrm{j}(\omega + \omega_s i)) =$$



where $\omega_s := 2\pi/h$ (in rad/sec). Note that the mapping $Y \mapsto Y_h$ is linear.

A weird function: Fourier series



As $Y_h(\mathrm{j}\omega)$ is ω_s-periodic, we can bring in its Fourier series expansion

$$Y_h(\mathrm{j}\omega) = \sum_{i\in\mathbb{Z}} c_i\,\mathrm{e}^{\mathrm{j}\frac{2\pi}{\omega_s}i\omega} = \sum_{i\in\mathbb{Z}} c_i\,\mathrm{e}^{\mathrm{j}\omega h i},$$

where Fourier coefficients are calculated as

$$c_i = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} Y_h(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega h i}\mathrm{d}\omega.$$

A weird function: Fourier coefficients

With some extra efforts:

$$\begin{aligned}
c_i &= \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2}\frac{1}{h}\sum_{k\in\mathbb{Z}} Y(\mathrm{j}(\omega+\omega_s k))\,\mathrm{e}^{-\mathrm{j}\omega h i}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\omega_s/2}^{\omega_s/2} Y(\mathrm{j}(\omega+\omega_s k))\mathrm{e}^{-\mathrm{j}(\omega+\omega_s k)hi}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\omega_s/2+\omega_s k}^{\omega_s/2+\omega_s k} Y(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega h i}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} Y(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega h i}\mathrm{d}\omega \qquad \left(\text{remember, } y(t) = \frac{1}{2\pi}\int_{\mathbb{R}} Y(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega\right) \\
&= y(-ih).
\end{aligned}$$

A weird function: Fourier series (contd)

Thus, we end up with

$$Y_h(\mathrm{j}\omega) = \sum_{i\in\mathbb{Z}} y(-ih)\mathrm{e}^{\mathrm{j}\omega h i} = \sum_{i\in\mathbb{Z}} y(ih)\mathrm{e}^{-\mathrm{j}\omega h i}.$$

Compare it with the DTFT of $\bar{y} = \mathcal{S}_{\mathrm{idl}}y$,

$$\bar{Y}(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{i\in\mathbb{Z}} \bar{y}[i]\mathrm{e}^{-\mathrm{j}\theta i} = \sum_{i\in\mathbb{Z}} y(ih)\mathrm{e}^{-\mathrm{j}\theta i}.$$

We can therefore say that

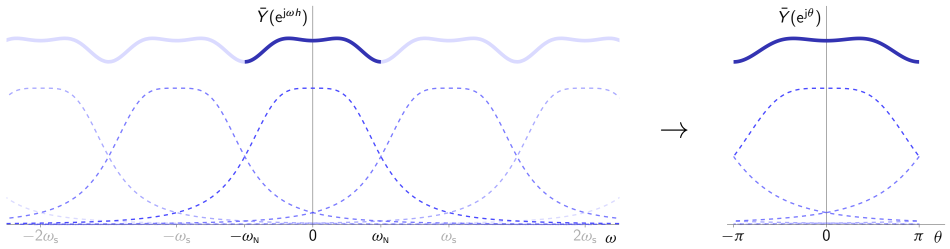
$$Y_h(\mathrm{j}\omega) = \frac{1}{h}\sum_{i\in\mathbb{Z}} Y(\mathrm{j}(\omega+\omega_s i)), \quad \text{where } \omega_s = \frac{2\pi}{h} \text{ (the sampling frequency)}$$

is the DTFT of the sampled signal $\bar{y}$ modulo scaling, $\theta = \omega h$, i.e.

$$\bar{Y}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{h}\sum_{i\in\mathbb{Z}} Y(\mathrm{j}(\theta/h + \omega_s i)) =: Y_h(\mathrm{j}\theta/h).$$

Spectrum of sampled signal

We thus end up with

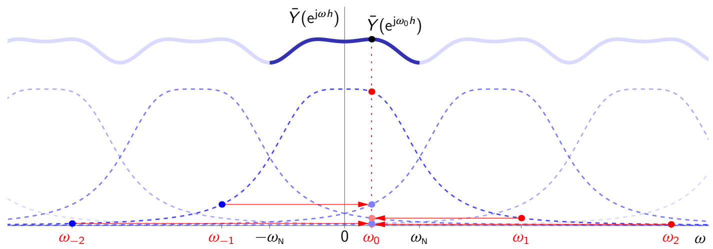


The frequency

$$\omega_N := \frac{\omega_s}{2} = \frac{\pi}{h}$$

is called the Nyquist frequency associated with the sampling period h (it is measured in rad/sec if h is in sec).

Spectrum of sampled signal: aliasing



Thus, the spectrum of $\bar{y}$ at each frequency $\theta_0 = \omega_0 h$ is a

– blend of analog frequency responses at $\omega_i := \omega_0 + \omega_s i$, $\forall i \in \mathbb{Z}$.

In other words,

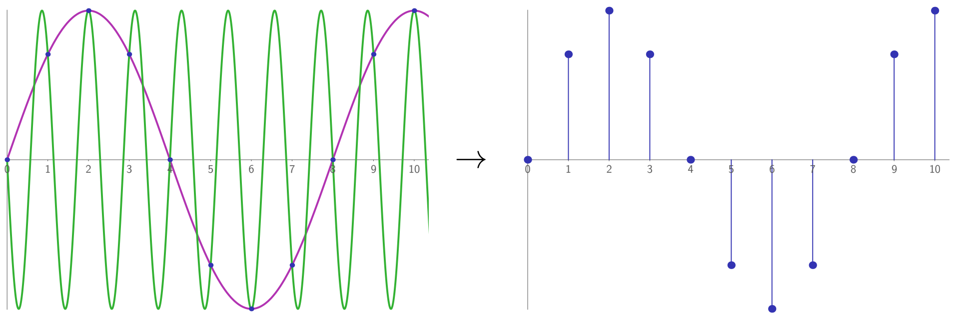
– every discrete frequency $\theta_0 \in [-\pi, \pi]$ is an alias of all ω_i, $i \in \mathbb{Z}$.

This phenomenon is dubbed aliasing, with respect to the base frequency ω_0.

Aliasing means information loss, we can no longer tell $Y(j\omega_i)$ from $Y(j\omega_j)$ in their effect on $\bar{Y}(e^{j\theta_0})$ (unless we know their dependencies).

Aliasing: example

Consider signals $y_1(t) = \sin\left(\frac{\pi}{4}t\right)$ and $y_2(t) = \sin\left(-\frac{7\pi}{4}t\right)$ sampled at $h = 1$:

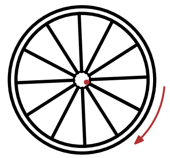
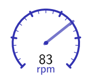


Sampling frequency is $\omega_s = 2\pi$, so that

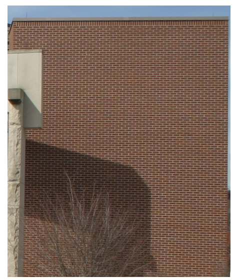
– both $\omega_0 = \dfrac{\pi}{4}$ and $\omega_{-1} = -\dfrac{7\pi}{4} = \omega_0 - \omega_s$ have aliases at $\theta_0 = \omega_0$

and, consequently, produce the same sampled signal.

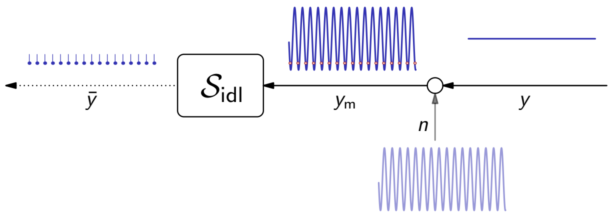
Aliasing: non-control examples

Wagon-wheel effect:  

(shot with 12 FPS frame rate)

Moiré pattern:  $\xrightarrow{\text{downsampling}}$ 

Aliasing: control implications



Let $y(t) = \text{const}$ be measured via a noisy sensor with $n(t) = \sin(2\omega_{\text{N}} t + \phi)$ (their spectra are well separated). But the sampled measured signal

$$\bar{y}[i] = y(ih) + \sin(\phi)$$

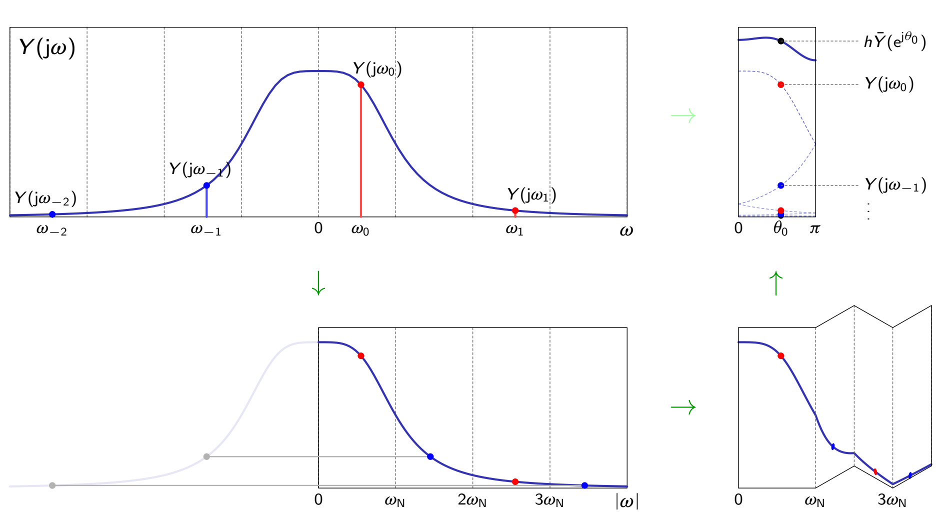
is offset, with no way to separate y from n (cf. Example in the first section). Such phenomena might have acute consequences on feedback designs that are hinged upon spectra separation between $r(t)/d(t)$ and $n(t)$. The

– spectrum of sampled n might interweave with those of r/d,

confusing the controller. And this cannot be corrected by a digital $\bar{C}(z)$.

Frequency folding

If $Y(\mathrm{j}\omega) = Y(-\mathrm{j}\omega) \in \mathbb{R}$, $\forall \omega$, the spectrum of its sampled version, $\bar{Y}(\mathrm{e}^{\mathrm{j}\theta})$, can be constructed also via the following folding procedure:



Instability of the ideal sampler

Let $y(t)$ be a signal with

$$Y(\mathrm{j}\omega) = \frac{\sqrt{2}}{\sqrt{\omega^2 + 1}}.$$

By Parseval,

$$E_y = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2}{\omega^2 + 1} \mathrm{d}\omega = 1,$$

so y is unit-energy (so, bounded) signal. Frequency response of $\bar{y} = \mathcal{S}_{\text{idl}} y$ is

$$\bar{Y}(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta/h + 2\pi/h\, i)^2 + 1}} = \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta + 2\pi\, i)^2 + h^2}} = \infty$$

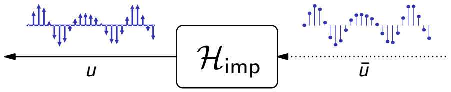
for every $\theta \in [-\pi, \pi]$. This means that $\bar{y}$ is unbounded, i.e. that

– the ideal sampler $\mathcal{S}_{\text{idl}}$ is unstable in the L_2 sense

Remark: $\mathcal{S}_{\text{idl}}$ does produce finite-energy discrete signals from analog inputs, whose spectra decay faster than $1/|\omega|$ at high frequencies.

Outline

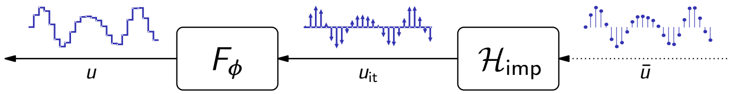
The impulsive hold



Acts as

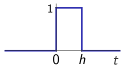
$$u(t) = \sum_{i \in \mathbb{Z}} \delta(t - ih)\bar{u}[i]$$

(known as the impulse train). Not quite practical by itself, but is the base for many other holds via the series with LTI filters:

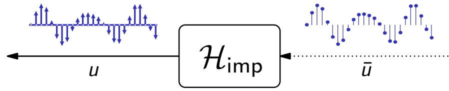


For example,

- $\mathcal{H}_{\text{ZOH}}$ corresponds to $F_\phi(s) = \dfrac{1 - \mathrm{e}^{-sh}}{s}$,

whose impulse response is $f_\phi(t) = \mathbb{1}(t) - \mathbb{1}(t-h) =$  (so it is FIR).

Spectrum of signals reconstructed by $\mathcal{H}_{\text{imp}}$



The Fourier transform of this $u(t)$ is

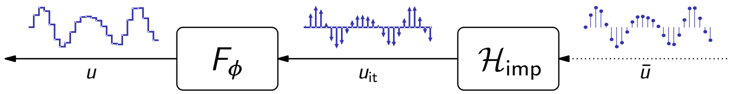
$$\begin{aligned} U(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} u(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \int_{-\infty}^{\infty} \sum_{i \in \mathbb{Z}} \delta(t - ih)\bar{u}[i]\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \\ &= \sum_{i \in \mathbb{Z}} \int_{-\infty}^{\infty} \delta(t - ih)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t\,\bar{u}[i] = \sum_{i \in \mathbb{Z}} \bar{u}[i]\mathrm{e}^{-\mathrm{j}(\omega h)i} \\ &= \bar{U}(\mathrm{e}^{\mathrm{j}\omega h}) = \bar{U}(\mathrm{e}^{\mathrm{j}\theta})|_{\theta = \omega h}. \end{aligned}$$

Because $\bar{U}(\mathrm{e}^{\mathrm{j}\omega h})$ is a $(2\pi/h)$-periodic function of ω,

- $\mathcal{H}_{\text{imp}}$ merely clones the spectrum of $\bar{u}$,

whose frequency θ-axis in $[-\pi, \pi]$ is scaled to fit the ω-axis in $[-\omega_{\text{N}}, \omega_{\text{N}}]$.

Spectrum of signals reconstructed by $\mathcal{H}_{\text{ZOH}}$

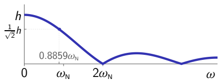


Just in two steps, again:

1. apply the analysis above to derive $U_{\text{it}}(\mathrm{j}\omega)$
2. filter u_{it} by the LTI F_ϕ to end up with

$$U(\mathrm{j}\omega) = F_\phi(\mathrm{j}\omega)U_{\text{it}}(\mathrm{j}\omega) = \frac{1 - \mathrm{e}^{-\mathrm{j}\omega h}}{\mathrm{j}\omega}\bar{U}(\mathrm{e}^{\mathrm{j}\omega h})$$

Note that

$|F_\phi(\mathrm{j}\omega)| =$ 

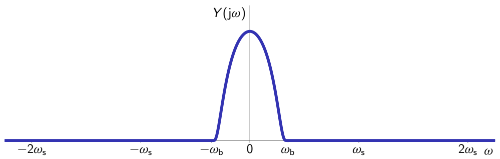
so this F_ϕ is a low-pass filter, whose (normalized) bandwidth $\omega_{\text{b}} \approx 0.886\omega_{\text{N}}$.

Outline

Spectrum of sampled bandlimited signal

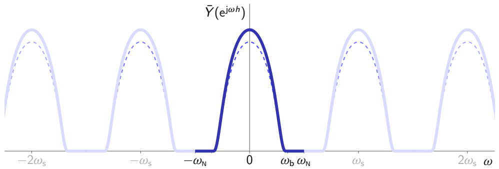
An analog signal $y(t)$ is said to be bandlimited if its spectrum, $Y(\mathrm{j}\omega)$, has support in $[-\omega_\mathrm{b}, \omega_\mathrm{b}]$ for some $\omega_\mathrm{b} > 0$ (bandwidth), like

$Y(\mathrm{j}\omega) =$



If $\omega_\mathrm{b} \le \omega_\mathrm{N}$, shifted $Y(\mathrm{j}(\omega + \omega_\mathrm{s} i))$ are mutually non-overlapping, so that

$\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h}) =$



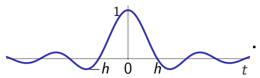
and there is no frequency blending in $\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h})$, i.e. no information is lost. In fact, $\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h}) = \frac{1}{h} Y(\mathrm{j}\omega)$ for every $\omega \in [-\omega_\mathrm{N}, \omega_\mathrm{N}]$ and we may expect that y is reconstructable from $\bar{y}$.

How to reconstruct bandlimited signal

Let $y(t)$ be bandlimited, with $\omega_\mathrm{b} \le \omega_\mathrm{N}$. Then $Y(\mathrm{j}\omega) = h\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h})$ and

$$
\begin{aligned}
y(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} Y(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \\
&= \frac{h}{2\pi}\int_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \bar{Y}(\mathrm{e}^{\mathrm{j}\omega h})\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega = \frac{h}{2\pi}\int_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \sum_{i\in\mathbb{Z}} y(ih)\mathrm{e}^{-\mathrm{j}\omega h i}\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \\
&= \sum_{i\in\mathbb{Z}} y(ih)\frac{1}{2\omega_\mathrm{N}}\int_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \mathrm{e}^{\mathrm{j}\omega(t-ih)}\mathrm{d}\omega = \sum_{i\in\mathbb{Z}} y(ih)\frac{1}{2\omega_\mathrm{N}}\frac{\mathrm{e}^{\mathrm{j}\omega(t-ih)}}{\mathrm{j}(t-ih)}\bigg|_{-\omega_\mathrm{N}}^{\omega_\mathrm{N}} \\
&= \sum_{i\in\mathbb{Z}} y(ih)\frac{\mathrm{e}^{\mathrm{j}\omega_\mathrm{N}(t-ih)} - \mathrm{e}^{-\mathrm{j}\omega_\mathrm{N}(t-ih)}}{2\mathrm{j}\,\omega_\mathrm{N}(t-ih)} = \sum_{i\in\mathbb{Z}} y(ih)\frac{\sin(\omega_\mathrm{N}(t-ih))}{\omega_\mathrm{N}(t-ih)} \\
&= \sum_{i\in\mathbb{Z}} \mathrm{sinc}_h(t-ih)y(ih)
\end{aligned}
$$

where $\mathrm{sinc}_h(t) := \dfrac{\sin(\omega_\mathrm{N} t)}{\omega_\mathrm{N} t} =$



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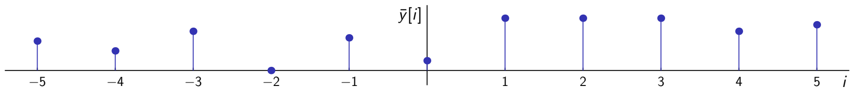
The Sampling Theorem

Theorem (Whittaker-Kotel'nikov-Shannon)

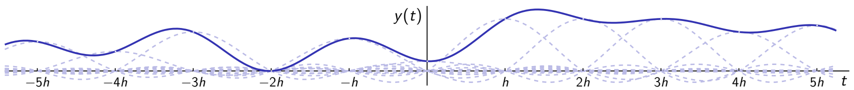
Let $y(t)$ be analog bandlimited signal with bandwidth ω_b. If $\omega_b \le \omega_N$, $y(t)$ can be perfectly reconstructed from its sampled measurements $\bar{y}[i] = y(ih)$ via the (non-causal) sinc*-interpolator*

$$y(t) = \sum_{i\in\mathbb{Z}} \mathrm{sinc}_h(t-ih)y(ih).$$

The sinc-interpolator acts as



$\downarrow$

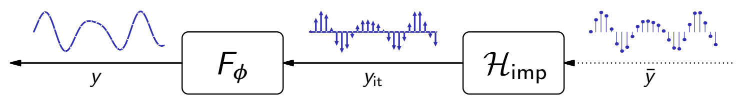


sinc-interpolator in the frequency domain

Readily seen that

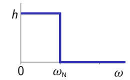
$$y(t) = \sum_{i\in\mathbb{Z}} \mathrm{sinc}_h(t-ih)y(ih) = \int_{-\infty}^{\infty} \mathrm{sinc}_h(t-s)y_\mathrm{it}(s)\mathrm{d}s$$

where the impulse train $y_\mathrm{it}(t) = \sum_{i\in\mathbb{Z}} \delta(t-ih)\bar{y}[i]$. Thus, we have



where

$F_\phi(\mathrm{j}\omega) = \mathfrak{F}\{\mathrm{sinc}_h\} = h(\mathbb{1}(\omega + \omega_\mathrm{N}) - \mathbb{1}(\omega - \omega_\mathrm{N})) =$



is the ideal low-pass filter with the bandwidth ω_N. This is intuitive (why?).