Control Theory (035188)
lecture no. 11

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So far on state-space design

State feedback:

- pole placement rationale (Ackermann)
- optimization-based rationale (LQR)

State observer:

- pole placement rationale (Luenberger)
- optimization-based rationale (Kalman)

Output feedback:

- just combine state feedback with state observer

Common assumption:

- only initial conditions are uncertain
  (i.e. no disturbances / noised beyond impulses or other white signals)

Outline

Idea

Consider state reconstruction for

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), & x(0) = x_0, \\ y(t) = Cx(t), \end{cases}$$

If $d(t)$

- measurable, $\dot{\epsilon}(t) = A_L\epsilon(t)$ and hence $\epsilon(t) \to 0$
- unmeasurable, $\dot{\epsilon}(t) = A_L\epsilon(t) + Bd(t)$ and hence $\epsilon(t) \not\to 0$ in general

To overcome this problem, we may try to

- observe not only $x(t)$ but also $d(t)$,

which makes sense

- only if some information about $d(t)$ available

(like the observation of $x(t)$ makes sense only if a state model is available).

Disturbance generators

Possible model of (unmeasurable) $d(t)$:

$$\begin{cases} \dot{x}_d(t) = A_d x_d(t), \quad x_d(0) = x_{d,0}, \\ d(t) = C_d x_d(t), \end{cases}$$

for known A_d and C_d (reflect our knowledge about $d(t)$) and unknown $x_{d,0}$ (reflects uncertainty in $d(t)$). This system

- called disturbance generator

and typically A_d has all its eigenvalues on the jω-axis (persistent signals). This model describes the family of signals, whose Laplace transforms

$$D(s) = C_d(sI - A_d)^{-1} x_{d,0}$$

for some unknown $x_{d,0}$.

Examples of disturbance generators: step

Let

$$d(t) = d_0 \cdot \mathbb{1}(t)$$

for some unknown d_0. Laplace transform of this signal is

$$D(s) = \frac{d_0}{s},$$

which corresponds to the following signal generator:

$$\begin{cases} \dot{x}_d(t) = 0 \cdot x_d(t), \quad x_d(0) = d_0, \\ d(t) = 1 \cdot x_d(t), \end{cases}$$

i.e. with $A_d = 0$ and $C_d = 1$.

Examples of disturbance generators: ramp

Let

$$d(t) = (d_0 + d_r \cdot t) \cdot \mathbb{1}(t)$$

for some unknown d_0 and d_r. Laplace transform of this signal is

$$D(s) = \frac{d_0 s + d_r}{s^2},$$

which corresponds to the following signal generator:

$$\begin{cases} \dot{x}_d(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_d(t), \quad x_d(0) = \begin{bmatrix} d_0 \\ d_r \end{bmatrix}, \\ d(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(t), \end{cases}$$

i.e. with $A_d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Examples of disturbance generators: harmonic signal

Let

$$d(t) = a \sin(\omega t + \phi) \cdot \mathbb{1}(t)$$

for some known ω and unknown a and ϕ. Laplace transform of this signal is

$$D(s) = \frac{a \sin(\phi)\, s + a\omega \cos(\phi)}{s^2 + \omega^2},$$

which corresponds to the following signal generator[1]:

$$\begin{cases} \dot{x}_d(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x_d(t), \quad x_d(0) = \begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix} a, \\ d(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(t), \end{cases}$$

i.e. with $A_d = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$ and $C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

[1] Take the observer form and apply the similarity transformation with $T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right]$.

Combined system: plant + disturbance

Now we have two systems (assume minimality of both):

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)) \\ y(t) = Cx(t) \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_d(t) = A_d x_d(t) \\ d(t) = C_d x_d(t) \end{cases}$$

with corresponding initial conditions. This can be written as

$$P_a : \begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \xi(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad \xi(0) = \begin{bmatrix} x_0 \\ x_{d,0} \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \xi(t), \end{cases}$$

with $\xi := \left[\begin{smallmatrix} x \\ x_d \end{smallmatrix}\right]$, with uncontrollable modes of A_d. Important is that

- the combined system has no unmeasurable inputs,

only unknown initial conditions. Hence, a Luenberger observer can be built to asymptotically estimate both x and x_d, if the realization is detectable.

Combined system: observability

Consider the series

$$G(s) = C(sI - A)^{-1}B \cdot C_d(sI - A_d)^{-1}.$$

We know (Lecture 6) that its realization is

$$G : \begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \xi(t), \end{cases}$$

and (Lecture 7) that it remains observable iff

- no modes of A_d are canceled by zeros of $C(sI - A)^{-1}B$.

Because G and P_a have the same "A" and "C" parameters, we have that

- P_a is observable iff the plant has no zeros in $\operatorname{spec}(A_d)$,

which is reasonable.

Observer for combined system

Straightforward use of known formulae:

$$\begin{aligned} \dot{\hat{\xi}}(t) &= \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} \left(y(t) - \begin{bmatrix} C & 0 \end{bmatrix} \hat{\xi}(t)\right) \\ &= \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} y(t) \end{aligned}$$

with $\hat{\xi}(0) = \hat{\xi}_0$. In this case error $\epsilon(t) := \xi(t) - \hat{\xi}(t)$ satisfies

$$\dot{\epsilon}(t) = \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \epsilon(t), \quad \epsilon(0) = \xi_0 - \hat{\xi}_0,$$

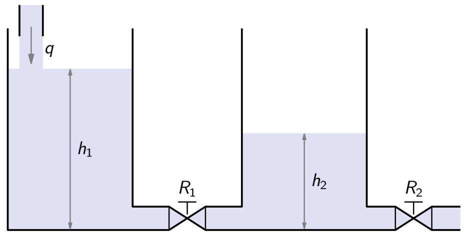
and asymptotically converges to zero if L and L_d are chosen properly.

Because $\xi = \left[\begin{smallmatrix} x \\ x_d \end{smallmatrix}\right]$,

- $\hat{\xi}$ reconstructs both x (plant state) and x_d (disturbance state).

Example

Return to the two-tank example from Lecture 8:



whose (linearized) model is

$$\begin{bmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q(t),$$

The goal is to

- reconstruct $h_2(t)$ from measured $h_1(t)$

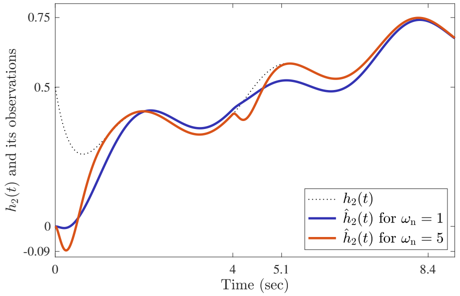
despite load disturbances of the form $d = d_0 \mathbb{1}$ for an unknown d_0.

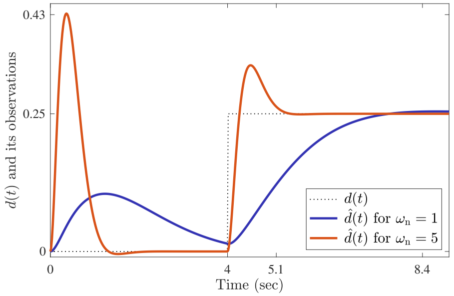
Example (contd)

With $q(t) = 0.5(\sin(2t) + 1)$, $d(t) = 0.25\mathbb{1}(t - 4)$, and

$$\hat{\chi}_{\rm cl}(s) = (s^2 + 2\zeta\omega_{\rm n}s + \omega_{\rm n}^2)(s + 7) \quad \text{for } \zeta = 0.8 \text{ and } \omega_{\rm n} = \{1, 5\}$$

as the observer characteristic polynomial, we end up with





Outline

Disturbance observers

Observer-based feedback with disturbance observers

Accounting for colored noise

Sampled-data systems

Idea

Consider controller design for

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), & x(0) = x_0, \\ y(t) = Cx(t), \end{cases}$$

If both x and d were measurable, we could use

$$u(t) = Kx(t) - d(t)$$

to stabilize the system and reject d (the reference signal can be handled by the 2DOF architecture, so its addition changes nothing).

We know what to do when

- x is not measurable $\implies$ observer-based feedback.

What if we use the same idea with a disturbance observer?

Controller

Naturally,

$$\begin{cases} \dot{\hat{\xi}}(t) = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} \left(y(t) - \begin{bmatrix} C & 0 \end{bmatrix} \hat{\xi}(t)\right) \\ u(t) = \begin{bmatrix} K & -C_d \end{bmatrix} \hat{\xi}(t) \end{cases}$$

The state relation reads

$$\begin{aligned} \dot{\hat{\xi}}(t) &= \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K & -C_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \hat{\xi}(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} y(t) \\ &= \begin{bmatrix} A + BK + LC & 0 \\ L_d C & A_d \end{bmatrix} \hat{\xi}(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} y(t) \end{aligned}$$

and we end up with the controller

$$C_y(s) = -\begin{bmatrix} K & -C_d \end{bmatrix} \left(sI - \begin{bmatrix} A + BK + LC & 0 \\ L_d C & A_d \end{bmatrix} \right)^{-1} \begin{bmatrix} L \\ L_d \end{bmatrix}$$

Closed-loop dynamics

Combining the plant of controller, the closed-loop state

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{x}}_d(t) \end{bmatrix} = \begin{bmatrix} A & BK & -BC_d \\ -LC & A+BK+LC & 0 \\ -L_dC & L_dC & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{x}_d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} d(t)$$

With the standard (by now) trick of replacing $\hat{x} \to \epsilon_x := x - \hat{x}$,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ -\dot{\hat{x}}_d(t) \end{bmatrix} = \begin{bmatrix} A+BK & -BK & BC_d \\ 0 & A+LC & BC_d \\ 0 & L_dC & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon_x(t) \\ -\hat{x}_d(t) \end{bmatrix} + \begin{bmatrix} B \\ B \\ 0 \end{bmatrix} d(t)$$

which are stable, provided $A+BK$ and

$$\begin{bmatrix} A+LC & BC_d \\ L_dC & A_d \end{bmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix}$$

are Hurwitz (state feedback and observer dynamics are separated yet again).

Disturbance response

If d is indeed generated by its model, then

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ -\dot{\hat{x}}_d(t) \\ \dot{x}_d(t) \end{bmatrix} = \begin{bmatrix} A+BK & -BK & BC_d & BC_d \\ 0 & A+LC & BC_d & BC_d \\ 0 & L_dC & A_d & 0 \\ 0 & 0 & 0 & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon_x(t) \\ -\hat{x}_d(t) \\ x_d(t) \end{bmatrix}$$

with some initial conditions. Introducing $\epsilon_d := x_d - \hat{x}_d$, these dynamics read

$$\left[\begin{array}{c} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ \dot{\epsilon}_d(t) \\ \hline \dot{x}_d(t) \end{array}\right] = \left[\begin{array}{ccc|c} A+BK & -BK & BC_d & 0 \\ 0 & A+LC & BC_d & 0 \\ 0 & L_dC & A_d & 0 \\ \hline 0 & 0 & 0 & A_d \end{array}\right] \left[\begin{array}{c} x(t) \\ \epsilon_x(t) \\ \epsilon_d(t) \\ \hline x_d(t) \end{array}\right], \quad \left[\begin{array}{c} x(0) \\ \epsilon_x(0) \\ \epsilon_d(0) \\ \hline x_d(0) \end{array}\right] = \dots$$

Therefore,

- x is decoupled from $x_d \implies y = Cx$ is decoupled from $d = C_dx_x$

meaning perfect asymptotic rejection of disturbances from a given class.

Controller structure

Returning to

$$C_y(s) = -\begin{bmatrix} K & -C_d \end{bmatrix} \left(sI - \begin{bmatrix} A+BK+LC & 0 \\ L_dC & A_d \end{bmatrix} \right)^{-1} \begin{bmatrix} L \\ L_d \end{bmatrix}$$

its "A" matrix has all eigenvalues of A_d as its eigenvalues. Moreover, it can be shown that

- eigenvalues of A_d are always poles of $C_y(s)$

(to this end we need to prove that all eigenvalues of A_d are both controllable and observable in the realization above, which is true).

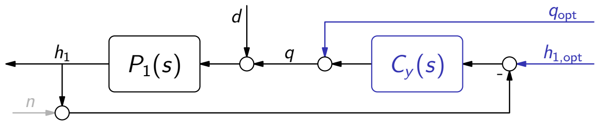
This is a version of the Internal Model Principle, roughly saying that

- disturbance model should be a part of the controller.

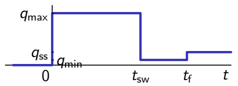
We are supposed to know it well for the case of $A_d = 0$ (integral action) . . .

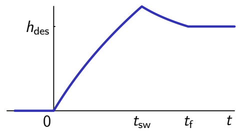
Example

For the two-tank example from Lecture 8 use the 2DOF control



for the time-optimal

$q_{opt}(t) =$ 

and $h_{1,opt}(t) =$ 

under given bounds q_{min} and q_{max}.

Assuming that $d = d_0\mathbb{1}$ for an unknown d_0, we design an

- observer-based $C_y(s)$ that contains an integral action.

Example (contd)

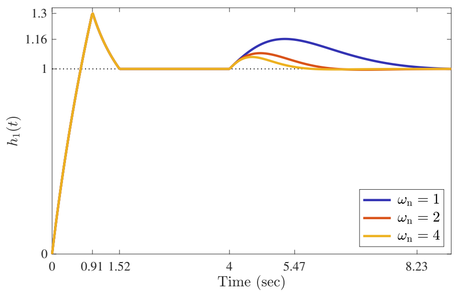
With $q_{\min} = 0.2$, $q_{\max} = 2$, $d(t) = 0.25\mathbb{1}(t-4)$,

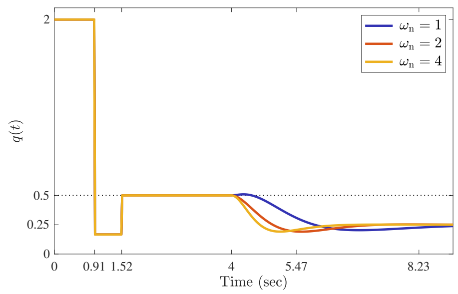
$$\chi_{\mathrm{cl}}(s) = s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2 \quad \text{for } \zeta = 0.8 \text{ and } \omega_{\mathrm{n}} = \{1, 2, 4\}$$

as the state-feedback characteristic polynomial (independent of W_d), and

$$\hat{\chi}_{\mathrm{cl}}(s) = (s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)(s+7) \quad \text{for } \zeta = 0.8 \text{ and } \omega_{\mathrm{n}} = 2$$

as the observer characteristic polynomial, we end up with





Outline

Kalman filtering setup

Simple version:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bd(t), & x(0) = 0, \\ y(t) = Cx(t) + n(t), \end{cases}$$

where $d(t)$ and $n(t)$ are white noise signals with intensities σ_d and σ_n.

More general formulation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), & x(0) = 0, \\ y(t) = Cx(t) + D_w w(t), \end{cases}$$

where each element of $w(t)$ is a white unit-intensity noise.

Is white noise assumption realistic?

The white noise assumption might not reflect our knowledge of properties of $d(t)$ and $n(t)$.

- Low-frequency harmonics typically dominate spectrum of $d(t)$. In this case, it might be more natural to assume that

$$|D(\omega)|^2 = \Phi_d(\omega)$$

  for some $\Phi_d(\omega) \geq 0$ (power spectral density), large at low frequencies and small at high frequencies (like $a^2/(\omega^2 + a^2)$, $a \neq 0$).
- High-frequency harmonics typically dominate spectrum of $n(t)$. In this case, it might be more natural to assume that

$$|N(\omega)|^2 = \Phi_n(\omega)$$

  for some $\Phi_n(\omega) \geq 0$ (power spectral density), small at low frequencies and large at high frequencies (like $\omega^2/(\omega^2 + a^2)$, $a \neq 0$).

Kalman filtering assumptions do not account for such situations explicitly.

Spectral factorization

Situation is not hopeless though. Fundamental is the following result:

Theorem (spectral factorization)

If $\Phi_x(\omega) \not\equiv 0$ is real-rational and such that $\Phi_x(\omega) \geq 0$, then there is $W_x(s)$ having no poles and zeros in $\operatorname{Re} s > 0$ and satisfying

$$\Phi_x(\omega) = W_x(-\mathrm{j}\omega)W_x(\mathrm{j}\omega) = |W_x(\mathrm{j}\omega)|^2.$$

If $0 < \Phi_x(\omega) < \infty$, then $W_x(s)$ is stable and minimum-phase and is called the spectral factor of $\Phi_x(\omega)$.

Examples:

- $\Phi_x(\omega) = \dfrac{a^2}{\omega^2 + a^2} \implies W_x(s) = \dfrac{a}{s+a}$ as $\Phi_x(\omega) = \dfrac{a^2}{-s^2+a^2}\Big|_{s=\mathrm{j}\omega}$
- $\Phi_x(\omega) = \dfrac{\omega^2}{\omega^2 + a^2} \implies W_x(s) = \dfrac{s}{s+a}$
- $\Phi_x(\omega) = \dfrac{1}{\omega^2} \implies W_x(s) = \dfrac{1}{s}$

Whitening signals / "colorifying" white noise

If x is such that $|X(\omega)|^2 = \Phi_x(\omega)$, then the signal

- $\tilde{x} := \dfrac{1}{W_x(s)}x$ is white with unit intensity.

Indeed,

$$|\tilde{X}(\omega)|^2 = \frac{|X(\omega)|^2}{|W_x(\mathrm{j}\omega)|^2} = \frac{\Phi_x(\omega)}{|W_x(\mathrm{j}\omega)|^2} \equiv 1$$

Hence, the term "whitening filter" for $1/W_x$. Important implication is that

- signal with any spectrum can be generated by white noise

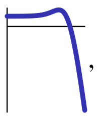
with the use of shaping filter (cf. signal generator for disturbance observer),

$$x = W_x(s)\tilde{x}$$

Example: Dryden gusts

A widely used spectral model of the lateral wind turbulence for an aircraft flying at a (constant) speed V through a frozen turbulence field is

$$\Phi_v(\omega) = \frac{\sigma_v^2 L_v V}{\pi}\frac{3L_v^2\omega^2 + V^2}{(L_v^2\omega^2 + V^2)^2} =$$



where σ_v and L_v represent turbulence intensity and scale length, resp. As

$$a_1^2\omega^2 + a_0^2 = -a_1^2s^2 + a_0^2|_{s=\mathrm{j}\omega} = (-a_1s + a_0)(a_1s + a_0)|_{s=\mathrm{j}\omega},$$

we end up with the following spectral factor:

$$W_v(s) = \sigma_v\sqrt{\frac{L_vV}{\pi}}\,\frac{\sqrt{3}L_vs + V}{(L_vs + V)^2},$$

which is an LPF. Similar models exist for longitudinal and vertical gusts.

Colorifying disturbances

Let shaping filter for d, $W_d(s)$, have (minimal) state space realization

$$\begin{cases} \dot{x}_d(t) = A_dx_d(t) + B_dw_d(t) \\ d(t) = C_dx_d(t) \end{cases}$$

where w_d is white. State-space realization of plant and filter is then

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_d(t) \end{bmatrix} = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix}\begin{bmatrix} x(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ B_d \end{bmatrix}w_d(t) \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ x_d(t) \end{bmatrix} + n(t) \end{cases}$$

This fits the general Kalman filtering formulation under

$$\left[\begin{array}{c:c} A & B_w \\ \hdashline C & D_w \end{array}\right] = \left[\begin{array}{cc:cc} A & BC_d & 0 & 0 \\ 0 & A_d & B_d\sqrt{\sigma_{w_d}} & 0 \\ \hdashline C & C_n & 0 & \sqrt{\sigma_n} \end{array}\right] \quad \text{and} \quad w(t) = \begin{bmatrix} \sqrt{1/\sigma_{w_d}}\,w_d(t) \\ \sqrt{1/\sigma_n}\,n(t) \end{bmatrix}$$

Colorifying measurement noise

Now, let shaping filter for n, $W_n(s)$, have (minimal) state space realization

$$\begin{cases} \dot{x}_n(t) = A_n x_n(t) + B_n w_n(t) \\ n(t) = C_n x_n(t) + D_n w_n(t) \end{cases}$$

where w_n is white. Joint state-space realization:

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_n \end{bmatrix} \begin{bmatrix} x(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} B & 0 \\ 0 & B_n \end{bmatrix} \begin{bmatrix} d(t) \\ w_n(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} C & C_n \end{bmatrix} \begin{bmatrix} x(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 & D_n \end{bmatrix} \begin{bmatrix} d(t) \\ w_n(t) \end{bmatrix} \end{cases}$$

This fits the general Kalman filtering formulation under

$$\left[\begin{array}{c|c} A & B_w \\ \hline C & D_w \end{array}\right] = \left[\begin{array}{cc|cc} A & 0 & B\sqrt{\sigma_d} & 0 \\ 0 & A_n & 0 & B_n\sqrt{\sigma_{w_n}} \\ \hline C & C_n & 0 & D_n\sqrt{\sigma_{w_n}} \end{array}\right] \quad \text{and} \quad w(t) = \begin{bmatrix} \sqrt{1/\sigma_d}\, d(t) \\ \sqrt{1/\sigma_{w_n}}\, w_n(t) \end{bmatrix}$$

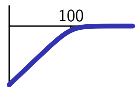
Example 1: high-frequency measurement noise

Consider again

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + n(t), \end{cases}$$

and now assume that n is a high-pass signal with the power spectral density

$$\Phi_n(\omega) = \frac{\omega^2}{\omega^2 + 10^4} =$$



$$= \frac{-j\omega}{-j\omega + 100} \frac{j\omega}{j\omega + 100},$$

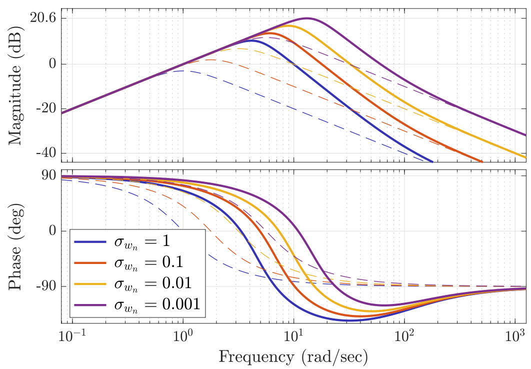
so that $W_n(s) = s/(s + 100)$ or, equivalently,

$$\begin{cases} \dot{x}_n(t) = -100 x_n(t) - 100 w_n(t) \\ n(t) = x_n(t) + w_n(t) \end{cases}$$

for a white $w_n(t)$.

Example 1: frequency responses

Bode plots of $G_{\hat{v}y}(s)$ for different choices of σ_{w_n}:



For each σ_{w_n} the filter is more aggressive than in the case of white n. Why?

Example 1: insight

Now our *assumed* measurement signal is

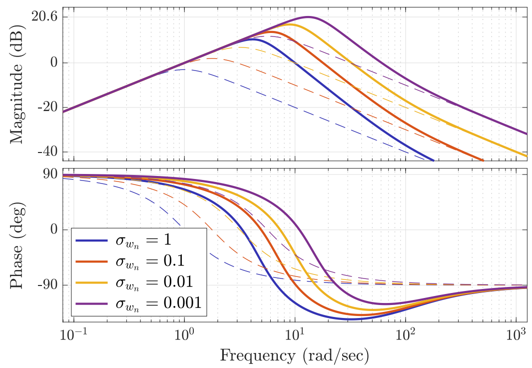
$$y = \frac{1}{s^2} d + \frac{s}{s + 100} w_n =: y_s + y_n.$$

Its signal to noise ratio

$$\text{SNR}(\omega) := \frac{|Y_s(j\omega)|}{|Y_n(j\omega)|} = \frac{\sqrt{\sigma_d}\sqrt{1 + 100^2/\omega^2}}{\sqrt{\sigma_{w_n}}\,\omega^2} > \frac{\sqrt{\sigma_d}}{\sqrt{\sigma_{w_n}}\,\omega^2}$$

especially for $\omega < 100$. But the quality of measurements returns to that of the white n with the intensity σ_{w_n} as ω goes beyond 100.

Example 1: insight (contd)



In our case, with $\sigma_d = 1$,

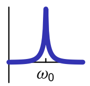
$$\mathrm{SNR}(\omega) = 1 \iff \omega = \{4.6433, 6.8182, 10.0167, 14.7306\}$$

vs. $\omega = \{1, 1.7783, 3.1623, 5.6234\}$ under white n.

Example 2: measurement noise at 10Hz

Assume now that $n(t)$ has dominant harmonic at $\omega_0 = 20\pi$. A way to model this is to assume the power spectral density

$$\Phi_n(\omega) = \frac{\omega^4 + \omega_0^4}{(\omega^2 - \omega_0^2)^2} =$$



$$= \frac{\omega^2 + \mathrm{j}\sqrt{2}\omega_0\omega - \omega_0^2}{\omega^2 - \omega_0^2}\,\frac{\omega^2 - \mathrm{j}\sqrt{2}\omega_0\omega - \omega_0^2}{\omega^2 - \omega_0^2},$$

so that

$$W_n(s) = \frac{s^2 + \sqrt{2}\omega_0 s + \omega_0^2}{s^2 + \omega_0^2}$$

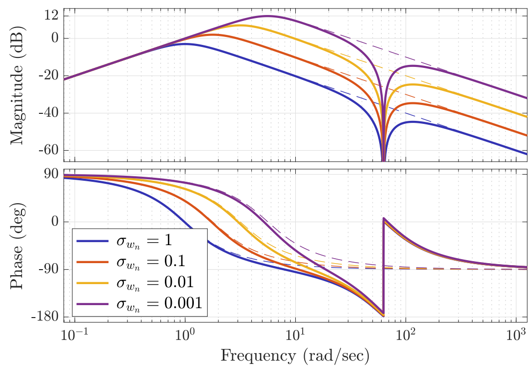
or, equivalently,

$$\begin{cases} \dot{x}_n(t) = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} x_n(t) + \begin{bmatrix} \sqrt{2}\omega_0 \\ 0 \end{bmatrix} w_n(t) \\ n(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_n(t) + w_n(t) \end{cases}$$

for a white $w_n(t)$.

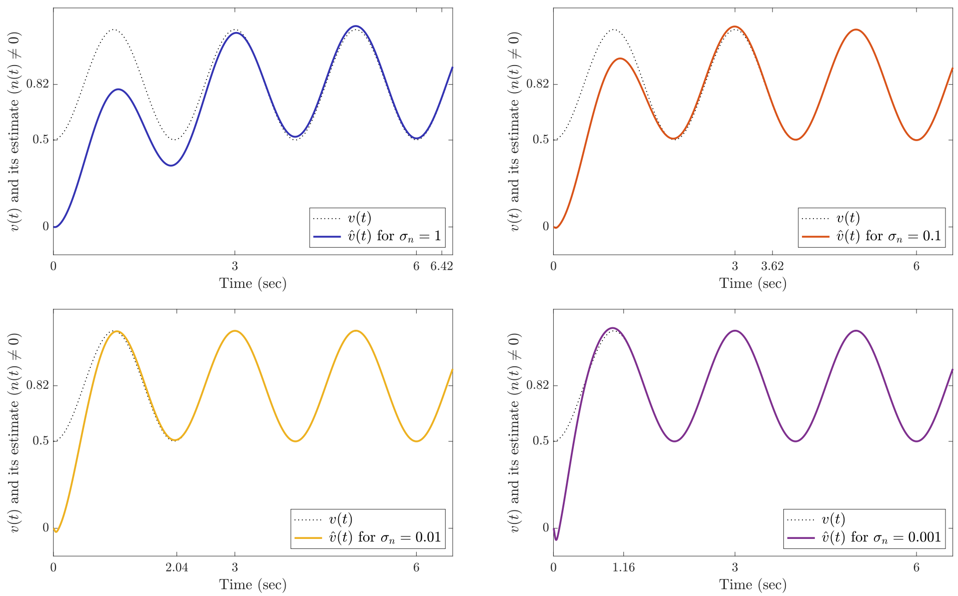
Example 2: measurement noise at 10Hz (contd)

The filters $G_{\hat{v}y}(s)$ for different choices of σ_{w_n},



have the notch property at $\omega = 20\pi\,[\mathrm{rad/sec}]$, imposed (implicitly) via the model of $n(t)$, because the assumed $\mathrm{SNR}(\omega) = 0$ at $\omega = \omega_0$.

Example 2: time responses with $n(t) = 0.2\sin(20\pi t)$



Outline

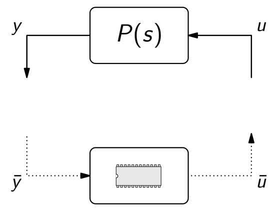
Disturbance observers

Observer-based feedback with disturbance observers

Accounting for colored noise

Sampled-data systems

Computer-controller systems

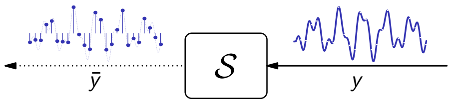


Control laws are often implemented on digital computers. Feedback loops in this case contain

- analog plant, having analog i/o signals
- digital controller, having digital i/o signals (digital sequences)

We, therefore, need to provide analog/digital and digital/analog interfaces.

A/D conversion (sampling)



By sampler (A/D converter) we understand

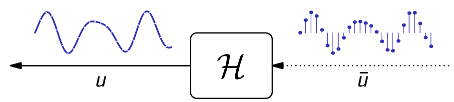
- any device transforming analog signal into digital sequence.

Constant $h > 0$ called the sampling period.

For example:

- ideal sampler: $\bar{y}[k] = y(kh), \quad k \in \mathbb{Z}^+$,
- averaging sampler: $\bar{y}[k] = \dfrac{1}{h}\displaystyle\int_{(k-1)h}^{kh} y(t)\mathrm{d}t, \quad k \in \mathbb{Z}^+$,
- ...

D/A conversion (hold)



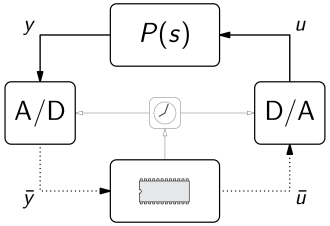
By hold device (D/A converter) we understand

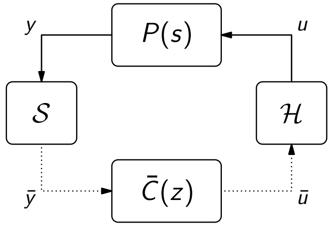
- any device transforming digital sequence into analog signal.

For example (formulae below are for all $k \in \mathbb{Z}^+$ and $\tau \in [0, h)$):

- zero-order hold: $u(kh + \tau) = \bar{u}[k]$,
- first-order hold: $u(kh + \tau) = \frac{\tau}{h}\bar{u}[k + 1] + \frac{h-\tau}{h}\bar{u}[k]$ (non-causal)
- delayed first-order hold: $u(kh + \tau) = \frac{\tau}{h}\bar{u}[k] + \frac{h-\tau}{h}\bar{u}[k - 1]$ (causal)
- predictive first-order hold: $u(kh + \tau) = \frac{\tau+h}{h}\bar{u}[k] - \frac{\tau}{h}\bar{u}[k - 1]$
- ...

Sampled-data systems systems





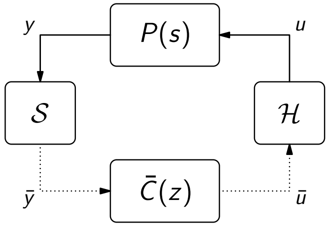
Thus, general sampled-data control system consists of

1. continuous-time plant $P(s)$,
2. digital part of the controller $\bar{C}(z)$,
3. A/D converter (sampler) $\mathcal{S}$,
4. D/A converter (hold) $\mathcal{H}$.

We assume that the

– last three devices are synchronized, with sampling period h.

Why sampled-data systems?



– Progress in computer technology

– Sampling due to measurements
(e.g. radar measurements, economic systems, visual systems, etc)

– Sampling due to pulsed operation
(e.g. internal-combustion engines, particle accelerator, etc)

Pros and cons of sampled-data control

Digital equipment is

- $\ddot\smile$ more flexible
- $\ddot\smile$ more reliable
- $\ddot\smile$ cheaper

than its analog counterpart. Consequently,

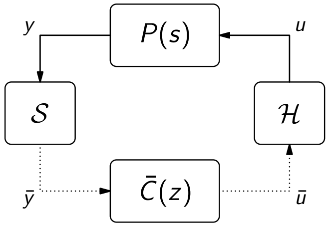
- $\ddot\smile$ more sophisticated control algorithms can be implemented.

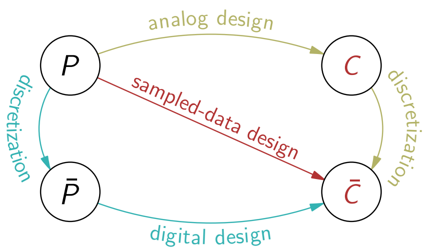
On the other hand,

- $\ddot\frown$ in intersample time control system is open loop;
- $\ddot\frown$ control signals are waveform limited (e.g. piecewise constant);
- $\ddot\frown$ rigorous analysis of sampled-data systems is more complicated

comparing with their continuous-time counterparts.

Three approaches to sampled-data control design





1. Digital redesign of analog controllers
(do your favorite analog design first, then discretize the resulting controller)
2. Discrete-time design
(discretize the problem first, then do your favorite discrete design)
3. Direct digital (sampled-data) design
(design discrete-time controller $\bar{C}(z)$ directly for analog specs)

In this course we address mainly the first approach . . .