Control Theory (035188)
lecture no. 10

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Outline

When is state feedback optimal ?

Theorem

If $m = 1$, (A, B) is controllable, and $K \in \mathbb{R}^{1\times n}$ satisfies

1. *$A + BK$ is stable,*
2. *$|1 - K(\mathrm{j}\omega I - A)^{-1}B| \ge 1$ for all $\omega \in \mathbb{R}$,*

then there is $0 \le Q = Q' \in \mathbb{R}^{n\times n}$ such that this K is LQR optimal for

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t$$

If (A, B) is not controllable, then there is Q for which $K(sI - A)^{-1}B$ is the optimal loop.

This result shows that LQR optimization is

- sufficiently rich, so that we won't miss any "good" controller

by using Q and r as design parameter under $S = 0$.

When is state feedback optimal ? (contd)

The condition $S = 0$ is essential. If S can also be chosen freely, then

- all stabilizing state-feedback gains are LQR-optimal,

regardless the second condition. For example, the cost

$$\mathcal{J} = \int_0^\infty (u(t) - K_\star x(t))^2 \mathrm{d}t$$

which corresponds to

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K_\star' \\ I \end{bmatrix} \begin{bmatrix} -K_\star & I \end{bmatrix} = \begin{bmatrix} K_\star' K_\star & -K_\star' \\ -K_\star & I \end{bmatrix},$$

produces the optimal $u(t) = K_\star x(t)$. in this case $\bar{X} = 0$ and $\mathcal{J}_{\mathrm{opt}} = 0$

Closed-loop poles ($m = 1$ and $S = 0$)

Because $1 + L_{sf}(s) = \chi_{cl}(s)/\chi_{ol}(s)$,

$$(1 + L_{sf}(-s))(1 + L_{sf}(s)) = 1 + \frac{1}{r} B'(-sI - A')^{-1} Q (sI - A)^{-1} B$$

reads

$$\frac{\chi_{cl}(-s)\chi_{cl}(s)}{\chi_{ol}(-s)\chi_{ol}(s)} = 1 + \frac{1}{r} G_*(s),$$

where

$$G_*(s) := B'(-sI - A')^{-1} Q (sI - A)^{-1} B$$

is such that $G_*(-s) = G_*(s)$, so that its poles / zeros are symmetric about the imaginary axis. If $x'Qx = y^2$, then $Q = C'C$ and $G_*(s) = P(-s)P(s)$. The solutions of

$$1 + \frac{1}{r} G_*(s) = 0 \iff -r = G_*(s),$$

which is a root-locus form under $k = 1/r$, determine then both the roots of $\chi_{cl}(s)$ (closed-loop LQR poles) and those of $\chi_{cl}(-s)$ (their reflection about the $j\omega$-axis).

Root locus with respect to $k = 1/r$

- has $2n$ loci symmetric about both real and imaginary axes
- starts for $r = \infty$ at roots of $\chi_{ol}(-s)\chi_{ol}(s)$
- ends as $r \downarrow 0$ at zeros of $G_*(s)$ or ∞ (depending on the cost)
- has an even number of asymptotes, whose centroid is at $\sigma_c = 0$
- if $G_*(s)$ is strictly proper (even pole excess), then asymptotes angles

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8
2	0	π						
4	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$				
6	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$		
8	$\frac{\pi}{8}$	$\frac{3\pi}{8}$	$\frac{5\pi}{8}$	$\frac{7\pi}{8}$	$\frac{9\pi}{8}$	$\frac{11\pi}{8}$	$\frac{13\pi}{8}$	$\frac{15\pi}{8}$

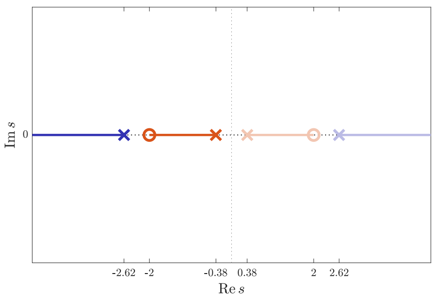
  half of them are in the LHP (and none on the imaginary axis).
- there are exactly n roots in the open LHP at each $r \in (0, \infty)$ (because $G_*(j\omega) \geq 0$, $-r = G_*(j\omega)$ cannot be solved in $r > 0$)

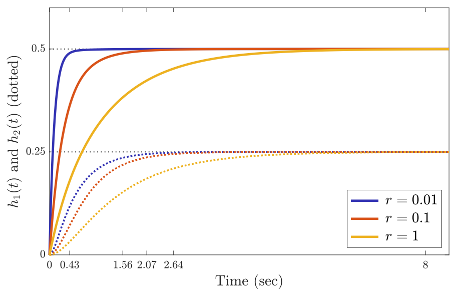
Example: design 1

LQR for $\mathcal{J} = \gamma_1 + r\gamma_u$ has $S = 0$ and

$$G_*(s) = -\frac{(s+2)(s-2)}{(s+2.618)(s+0.382)(s-0.382)(s-2.618)} = P_1(-s)P_1(s)$$

The root locus with respect to $1/r$:





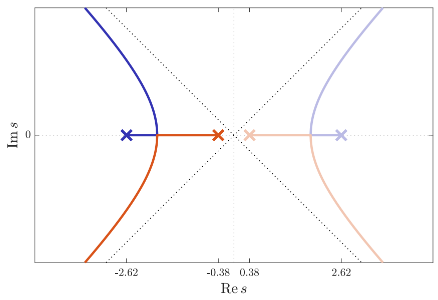
Closed-loop poles were at $\{-10.15, -1.97\}$, $\{-3.75, -1.71\}$, and $\{-2.7, -0.83\}$.

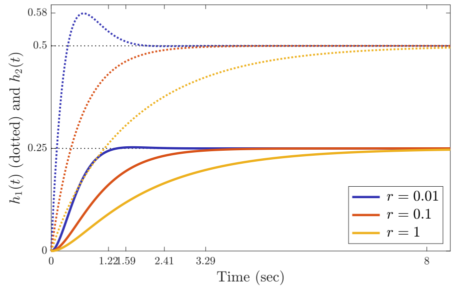
Example: design 2

LQR for $\mathcal{J} = \gamma_2 + r\gamma_u$ has $S = 0$ and

$$G_*(s) = \frac{1}{(s+2.618)(s+0.382)(s-0.382)(s-2.618)} = P_2(-s)P_2(s)$$

The root locus with respect to $1/r$:





Closed-loop poles were at $\{-2.6 \pm j1.81\}$, $\{-2.15, -1.54\}$, and $\{-2.59, -0.55\}$.

Limiting behavior: cheap control

The LQR problem with the cost

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t, \quad r \downarrow 0$$

is known as the cheap control LQR problem. By root-locus arguments, the roots of $\chi_{\text{cl}}(s)$ for $1/r \to \infty$

- either approach the stable zeros of $G_*(s)$
- or go to $\operatorname{Re} s = -\infty$ along the LHP asymptotes
  (it the pole excess of $G_*(s)$ is larger than 2, then the damping ratio of the "worst" asymptote is at most $1/\sqrt{2}$)

Limiting behavior: expensive control

We already know that the solution to the LQR for

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t$$

and

$$\mathcal{J}_r = \int_0^\infty \begin{bmatrix} x'(t) & u(t) \end{bmatrix} \begin{bmatrix} Q/r & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t = \frac{1}{r}\mathcal{J}$$

coincide. Thus, the LQR problem for the limiting case of $r \to \infty$ effectively corresponds to

$$\mathcal{J}_\infty = \int_0^\infty u^2(t)\mathrm{d}t \quad \text{or} \quad \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is the problem of

- stabilizing with the lowest control energy.

Limiting behavior: expensive control (contd)

Assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

$\mathcal{A}_2$: $R > 0$ and $Q - SR^{-1}S' \geq 0$
for $Q = 0$, $S = 0$, and $R = 1$ obviously holds

$\mathcal{A}_3$: $(A - BR^{-1}S', Q - SR^{-1}S')$ has no pure imaginary unobservable modes
for $Q = 0$, $S = 0$, and $R = 1$ equivalent to $\operatorname{spec}(A) \cap \mathrm{j}\mathbb{R} = \varnothing$

By root-locus arguments, the roots of $\chi_{\text{cl}}(s)$ for $1/r = 0$ are the stable roots of $\chi_{\text{ol}}(-s)\chi_{\text{ol}}(s)$. In other words,

- if λ is an OLHP root of $\chi_{\text{ol}}(s)$, then it is also a root of $\chi_{\text{cl}}(s)$,
- if λ is an ORHP root of $\chi_{\text{ol}}(s)$, then $-\lambda$ is a root of $\chi_{\text{cl}}(s)$.

Thus, the optimal controller for this problem

- reflects unstable poles about the $\mathrm{j}\omega$-axis and
- keeps stable poles untouched.

Outline

Setup

Consider system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bd(t), & x(0) = 0, \\ y(t) = Cx(t) + n(t), \end{cases}$$

where $u(t)$ is known input, $d(t)$ is (unknown) plant disturbance, and $n(t)$ is measurement noise.

Our purpose here is to reconstruct $x(t)$ so that reconstruction error,

$$\epsilon(t) = x(t) - \hat{x}(t),$$

is "small."

Clearly, solution of such state estimation problem depends on

1. what our assumptions about $d(t)$ and $n(t)$ are,
2. what measure of the estimation error $\epsilon(t)$ "smallness" is taken.

(Sloppy) introduction to white noise signals

By white noise signal we understand

– signal $f(t)$, whose frequency spectrum satisfies $|F(\omega)|^2 \equiv \sigma_f = \text{const}$

with the quantity $\sigma_f \geq 0$ called the intensity of $f(t)$. In other words, white noise signal

– has all harmonics equally represented in it.

We may interpret white noise signal as

– impulse response of any all-pass system—deterministic interpretation[1].

It may be safe to say that white noise doesn't exist in nature. It, however, is a convenient abstraction used as a building block for many realistic signals.

[1] Conventionally, white noise is defined and interpreted from a stochastic point of view (white random process). Yet such approach goes far beyond the scope of this course.

Linear-quadratic state estimation problem

In this formulation we assume that in the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bd(t), & x(0) = 0, \\ y(t) = Cx(t) + n(t), \end{cases}$$

inputs $d(t)$ and $n(t)$ are white noise signals with intensities σ_d and σ_n. We also assume that

$\mathcal{A}_4$: (C, A) is detectable,

$\mathcal{A}_5$: $\sigma_n > 0$,

$\mathcal{A}_6$: $(A, B\sigma_d)$ has no uncontrollable jω-axis modes.

Linear-quadratic estimation is formulated as reconstruction of x minimizing

$$\mathcal{J} := \int_0^\infty (x(t) - \hat{x}(t))'(x(t) - \hat{x}(t))\,\mathrm{d}t,$$

where $\hat{x}(t)$ is the reconstruction of $x(t)$.

Interpretation of d and n

Plant disturbance,

– $d(t)$, reflects inaccuracies of our model of $x(t)$.

Thus, high intensity of $d(t)$ means that we should correct our model-based estimation more aggressively.

Measurement noise,

– $n(t)$, reflects inaccuracies of our measurements.

As $y(t)$ is the only available information about deviations of $x(t)$ from its "predicted" values, high intensity of $n(t)$ means that we should rely more upon model.

Solution (steady-state Kalman filter)

Theorem

If $\mathcal{A}_{4\text{–}6}$ hold, then the unique optimal linear-quadratic estimator is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \sigma_n^{-1}\bar{Y}C'\big(y(t) - C\hat{x}(t)\big), \quad \hat{x}(0) = 0,$$

where $\bar{Y} = \bar{Y}' \geq 0$ is (unique) stabilizing solution of CARE

$$\bar{Y}A' + A\bar{Y} + \sigma_d BB' - \sigma_n^{-1}\bar{Y}C'C\bar{Y} = 0$$

such that $A_L = A + LC$ is Hurwitz, where $L = -\sigma_n^{-1}\bar{Y}C'$.

Filtering CARE

The CARE

$$\bar{Y}A' + A\bar{Y} + \sigma_d BB' - \sigma_n^{-1}\bar{Y}C'C\bar{Y} = 0$$

is a special case of

$$\bar{A}'\bar{X} + \bar{X}\bar{A} + \bar{Q} - (\bar{S} + \bar{X}\bar{B})\bar{R}^{-1}(\bar{S}' + \bar{B}'\bar{X}) = 0$$

with $\bar{A} = A'$, $\bar{B} = C'$, $\bar{Q} = \sigma_d BB'$, $\bar{R} = \sigma_u$, and $\bar{S} = 0$. Here

- (C, A) is detectable $\implies$ $(\bar{A}, \bar{B}) = (A', C')$ is stabilizable
- $(A, B\sigma_d)$ has no uncontrollable jω-axis modes and $\sigma_u > 0$ $\implies$

$$\begin{bmatrix} \bar{A} - \mathrm{j}\omega I & \bar{B} \\ \bar{Q} & \bar{S} \\ \bar{S}' & \bar{R} \end{bmatrix} = \left[\begin{array}{c:c} A' - \mathrm{j}\omega I & C' \\ \sigma_d BB' & 0 \\ \hdashline 0 & \sigma_u \end{array}\right]$$

has full column rank $\forall\omega \in \mathbb{R}$.

As such, it can be solved by `icare(A',C',sigmad*B*B',sigmau)`.

Kalman filter transfer functions

The Kalman filter has two inputs, u and y, and one vector output, $\hat{x}$. The transfer function from u to $\hat{x}$ and from y to $\hat{x}$ are

$$G_{\hat{x}u}(s) = (sI - A_L)^{-1}B \quad \text{and} \quad G_{\hat{x}y}(s) = -(sI - A_L)^{-1}L,$$

respectively, where $L = -\sigma_n^{-1}\bar{Y}C'$ and $A_L = A + LC$. Both result from the following form of the estimator equation:

$$\dot{\hat{x}}(t) = (A + LC)\hat{x}(t) + Bu(t) - Ly(t), \quad \hat{x}(0) = 0$$

which is a Luenberger observer under an optimization-based choice of L.

Estimating incomplete state

In some situations we do not need to reconstruct all state vector but rather only its part $v(t) = C_v x(t)$. Performance measure is then

$$\mathcal{J} := \int_0^\infty \big(C_v x(t) - \hat{v}(t)\big)'\big(C_v x(t) - \hat{v}(t)\big)\,\mathrm{d}t.$$

Remarkable property of Kalman filter is that in this case

$$\hat{v}(t) = C_v\hat{x}(t),$$

where $\hat{x}(t)$ is the optimal estimate of $x(t)$. In other words, the whole $x(t)$ should be estimated anyway. The transfer functions of the filter are then

$$G_{\hat{v}u}(s) = C_v(sI - A_L)^{-1}B \quad \text{and} \quad G_{\hat{v}y}(s) = -C_v(sI - A_L)^{-1}L$$

Example

Consider the process

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + n(t), \end{cases}$$

with $x = \left[\begin{smallmatrix} y \\ v \end{smallmatrix}\right]$, which is double integrator with noisy output measurements. Our purpose is

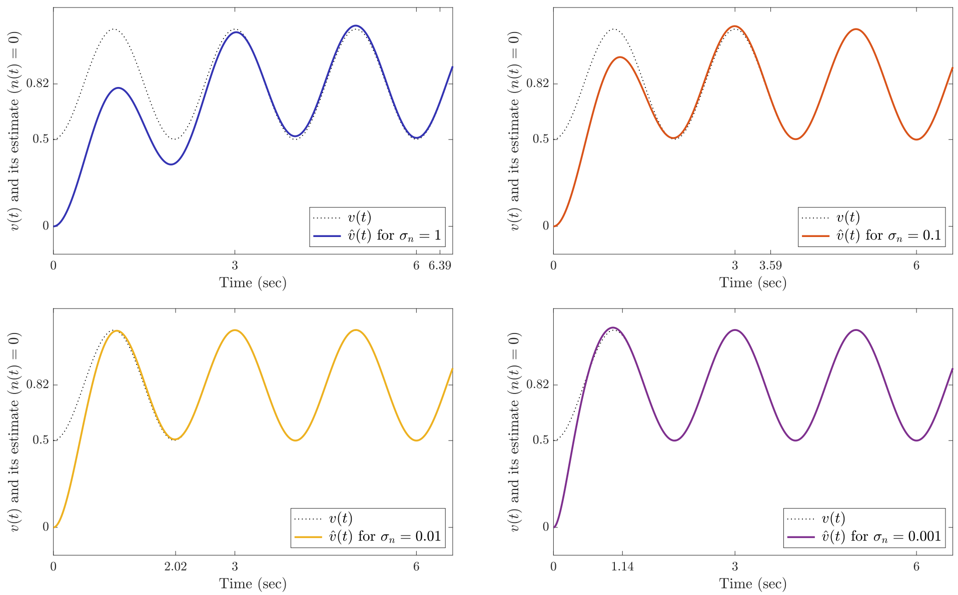
- to reconstruct $v(t)$ from $y(t)$ (i.e. $C_v = \begin{bmatrix} 0 & 1 \end{bmatrix}$).

We assume that

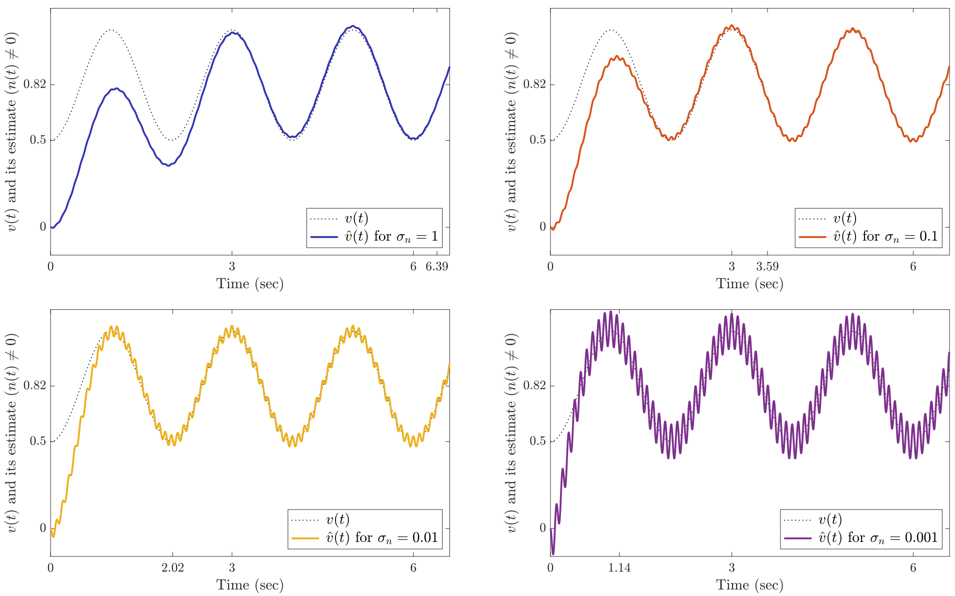
- $d(t) = \delta(t)$ (i.e. that $y(0) = 0$ and $v(0) = 1$) $\qquad \sigma_d = 1$
- $n(t) = 0.2\sin(20\pi t)$ $\qquad \sigma_n$ is tuned
- $u(t) = \sin(\pi t)$.

We'll be playing with σ_n, which is fictitious parameter.

Example: results with $d(t) = \delta(t)$ & $n(t) = 0$



Example: results with $d(t) = \delta(t)$ & $n(t) = 0.2\sin(20\pi t)$

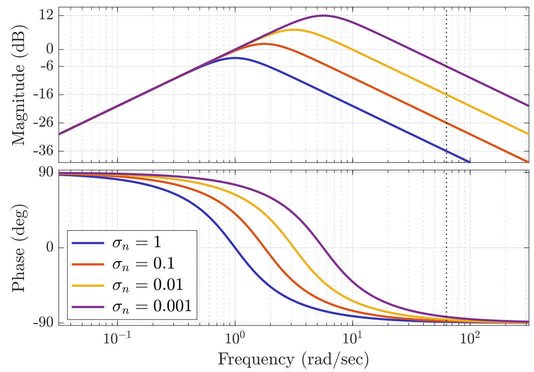


Example: frequency responses

Bode plots of the filter

$$G_{\hat{v}y}(s) = -C_v(sI - A_L)^{-1}L$$

for different choices of σ_n:



It starts as a differentiator (logical), but then cuts it off.

Example: insight

Our *assumed* measurement signal,

$$y = \frac{1}{s^2} d + n =: y_s + y_n,$$

can be thought of as comprised

- the "signal" y_s, which contains information about v (to be estimated),
- the "noise" y_n, which contains no information about x.

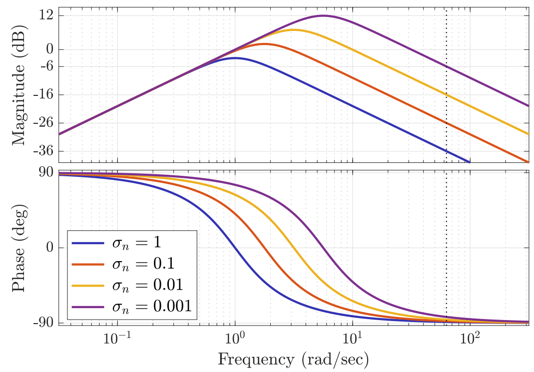
Convenient measure of the "usefulness" of y is the signal to noise ratio:

$$\mathrm{SNR}(\omega) := \frac{|Y_s(j\omega)|}{|Y_n(j\omega)|} = \frac{\sqrt{\sigma_d}}{\sqrt{\sigma_n}\,\omega^2}$$

The estimator then

- imitates the differentiator at frequencies where $\mathrm{SNR}(\omega) \gg 1$
- does nothing at frequencies where $\mathrm{SNR}(\omega) \ll 1$

Example: insight (contd)



In our case, with $\sigma_d = 1$,

$$\mathrm{SNR}(\omega) = 1 \iff \omega = \frac{1}{\sqrt[4]{\sigma_n}} = \{1, 1.7783, 3.1623, 5.6234\}$$

These frequencies agree with the breaks of tendency of $|G_{\hat{v}y}(j\omega)|$.

More general formulation

Let

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), \quad x(0) = 0, \\ y(t) = Cx(t) + D_w w(t), \end{cases}$$

for an unmeasured w, each element of which is a white unit-intensity signal independent of other elements of w, and under

$\mathcal{A}_4$: (C, A) is detectable,

$\mathcal{A}_5$: D_w has full row rank,

$\mathcal{A}_6$: $\begin{bmatrix} A - j\omega I & B_w \\ C & D_w \end{bmatrix}$ has full row rank $\forall \omega \in \mathbb{R}$

The previous formulation is its special case with

- $w = \begin{bmatrix} d \\ n \end{bmatrix}$, $B_w = \begin{bmatrix} B\sqrt{\sigma_d} & 0 \end{bmatrix}$, and $D_w = \begin{bmatrix} 0 & \sqrt{\sigma_n} \end{bmatrix}$.

More general formulation: solution

Theorem

If $\mathcal{A}_{4\text{–}6}$ hold, then the unique optimal linear-quadratic estimator is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + (D_w D_w')^{-1}(D_w B_w' + \bar{Y}C')(y(t) - C\hat{x}(t))$$

with $\hat{x}(0) = 0$, where $\bar{Y} = \bar{Y}' \geq 0$ is (unique) stabilizing solution of CARE

$$\bar{Y}A' + A\bar{Y} + B_w B_w' - (B_w D_w' + \bar{Y}C')(D_w D_w')^{-1}(D_w B_w' + C\bar{Y}) = 0$$

such that $A_L = A - (D_w D_w')^{-1}(D_w B_w' + \bar{Y}C')C$ is Hurwitz.

This corresponds to

- Luenberger observer with $L = -(D_w D_w')^{-1}(D_w B_w' + \bar{Y}C')$.
- CARE solution with $\bar{A} = A'$, $\bar{B} = C'$, $\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}' & \bar{R} \end{bmatrix} = \begin{bmatrix} B_w \\ D_w \end{bmatrix} \begin{bmatrix} B_w' & D_w' \end{bmatrix}$

Outline

LQR: solution properties (contd)

Linear-quadratic state observer (Kalman filter)

Linear Quadratic Gaussian problem

LQG controller design (optional self-study)

Problem formulation

Given system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bd(t), \quad x(0) = 0, \\ y(t) = Cx(t) + n(t), \end{cases}$$

where

- $u(t) \in \mathbb{R}$ is the control input
- $d(t) \in \mathbb{R}$ is the load disturbance (white, with intensity $\sigma_d \geq 0$)
- $n(t) \in \mathbb{R}$ is the measurement noise (white, with intensity $\sigma_n > 0$)

The LQG (Linear Quadratic Gaussian) problem is

- design a stabilizing $u(t)$, which is a function of $y(t)$, minimizing

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t$$

for some $R > 0$ and $Q - SR^{-1}S' \geq 0$.

Solution structure

Because

- LQR optimal control is state feedback,
- Kalman filter optimal state estimator is a Luenberger observer,

it is obvious that

- LQR + Kalman results in a stabilizing observer-based controller, with

$$\chi_{\mathrm{cl}}(s) = \det(sI - A_F)\det(sI - A_L)$$

Less obvious is that

- LQR + Kalman results in the LQG-optimal stabilizing controller

(the separation principle).

Problem solution

Theorem

If $\mathcal{A}_{1\text{–}6}$ hold, then the unique LQG controller is given by controller

$$\begin{cases} \dot{\hat{x}}(t) = (A + BK + LC)\hat{x}(t) - Ly(t), \quad \hat{x}(0) = 0, \\ u(t) = K\hat{x}(t), \end{cases}$$

where K and L are the LQR and Kalman filter gains. The optimal cost

$$\mathcal{J}_{opt} = \mathrm{tr}(\sigma_d \bar{X}BB' + \bar{Y}Q + \bar{X}A\bar{Y} + \bar{Y}A'\bar{X}),$$

where $\bar{X} = \bar{X}' \geq 0$ and $\bar{Y} = \bar{Y}' \geq 0$ are the stabilizing solutions of the LQR and filtering CAREs.

Properties:

- LQG controller is stabilizing, but not necessarily stable itself
- LQG controller always has a strictly proper transfer function
- stability margins of LQR are not achieved by LQG

Outline

LQR: solution properties (contd)

Linear-quadratic state observer (Kalman filter)

Linear Quadratic Gaussian problem

LQG controller design (optional self-study)

Think loop shaping, act LQG

Loop shaping:

- easy on the magnitude shaping
- might be intricate in the phase shaping around crossover

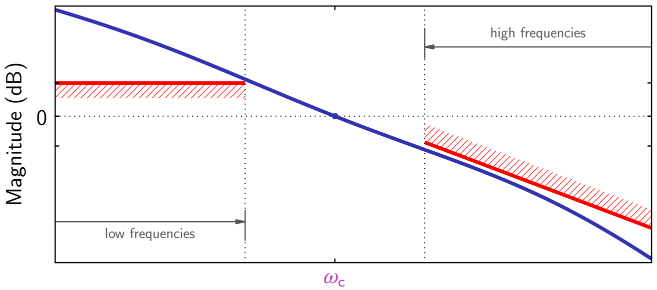
LQG:

- guarantees closed-loop stability
- not directly interpretable in terms of frequency-domain properties

Combine:

- shape loop magnitude (via frequency-dependent weights)
- solve some LQG to stabilize the shaped loop

Magnitude shaping



Conceptually simple,

- pick W such that $|P_a(j\omega)|$, where $P_a = PW$, is shaped appropriately
  possible choices
  - $W(s) = k_p$ — to fix the crossover
  - $W(s) = k_p\omega_c/s$ (I) or $W(s) = k_p(1 + k_i/s)$ (PI) — to add integral action
  - $W(s) = k_p/(\tau s + 1)$ — to add low-pass filter
- whatever C_a designed for P_a is then implemented as $C = WC_a$ for P

LQR

Bring in

$$P_a : \begin{cases} \dot{x}_a(t) = A_a x_a(t) + B_a u_a(t) + B_a d_a(t), \quad x_a(0) = 0, \\ y(t) = C_a x_a(t) + n(t), \end{cases}$$

and consider a balanced LQG for it with the cost

$$\mathcal{J} = \int_0^\infty \big(y^2(t) + u_a^2(t)\big)\,dt = \int_0^\infty \begin{bmatrix} x_a'(t) & u_a'(t) \end{bmatrix} \begin{bmatrix} C_a'C_a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_a(t) \\ u_a(t) \end{bmatrix} dt$$

and unit intensities of d_a and n. In other words,

- cost puts equal importance to y and $u_a = (1/W)u$,
- load disturbance $d_a = (1/W)d$ and noise n are equally important.

Motivation:

- LQR shall just ensure stability $\implies$ unsophisticated and balanced.

Controller

The resulted controller is

$$C(s) = W(s)C_a(s).$$

Some of its properties:

- the degree of $C(s)$, $\deg(C(s)) = \deg(P(s)) + 2\deg(W(s))$
  unless some poles or zeros of $W(s)$ are canceled by $C_a(s)$
- integrators of $W(s)$ become integrators of $C(s)$
- any nonzero roll-off in $W(s)$ increases that of $C(s)$

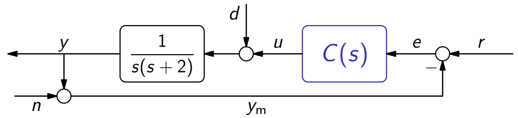
Success indicator:

$$\epsilon = \frac{1}{\sqrt{1 + \lambda_{\max}(\bar{X}_a \bar{Y}_a)}} < 1,$$

where $\bar{X}_a$ and $\bar{Y}_a$ are solutions to CAREs for P_a. The level $\epsilon \geq 0.25$ may be regarded as adequate, $\epsilon \geq 0.5$ is very good.

Example: DC motor

A DC motor from Lecture 1, controlled in closed loop



Requirements:

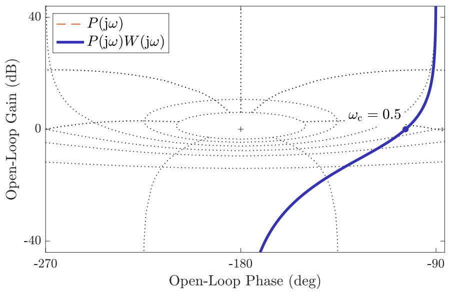
- closed-loop stability (of course)
- zero steady-state error for a step in r — always holds
- zero steady-state error for a step in d — integrator in $C(s)$
- good stability margins
- ω_c is treated as a tuning parameter

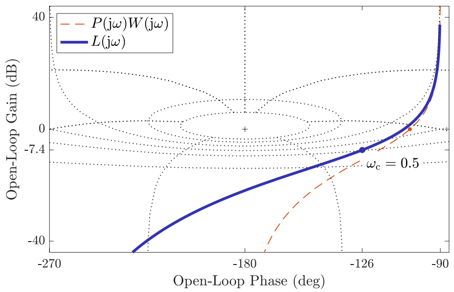
Example: $\omega_c = 0.5$, P weight

Choose

$$W(s) = k_p$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.6432$, loops are





and the controller

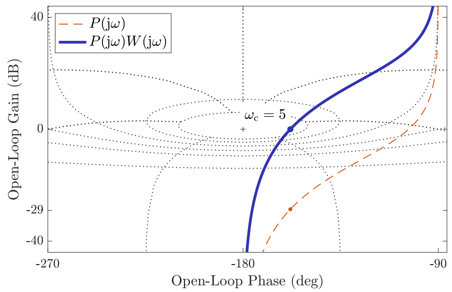
$$C(s) = \frac{0.52555(s + 2.022)}{s^2 + 2.924s + 2.275}$$

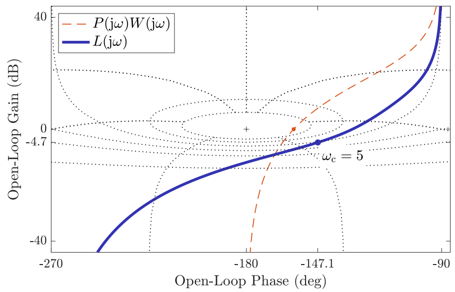
Example: $\omega_c = 5$, P weight

Choose

$$W(s) = k_p$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.4673$, loops are





and the controller

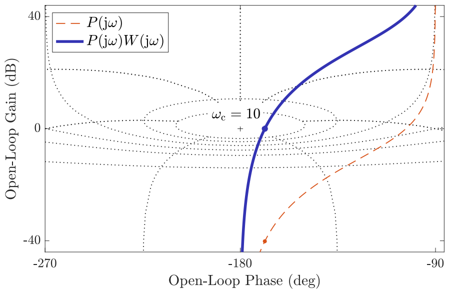
$$C(s) = \frac{239.04(s + 3.033)}{s^2 + 13.21s + 85.28}$$

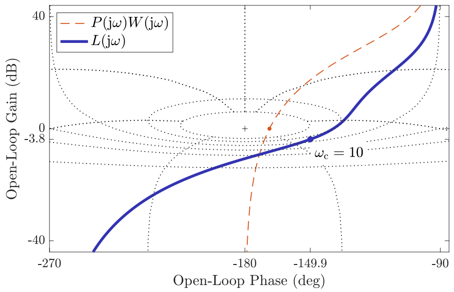
Example: $\omega_c = 10$, P weight

Choose

$$W(s) = k_p$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.4276$, loops are





and the controller

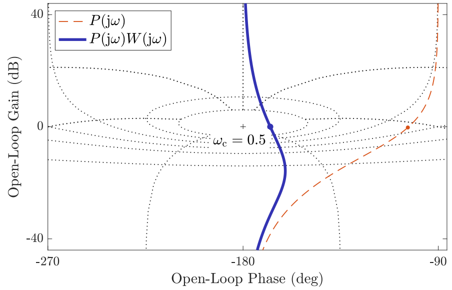
$$C(s) = \frac{2224.8(s + 4.675)}{s^2 + 26.84s + 358.2}$$

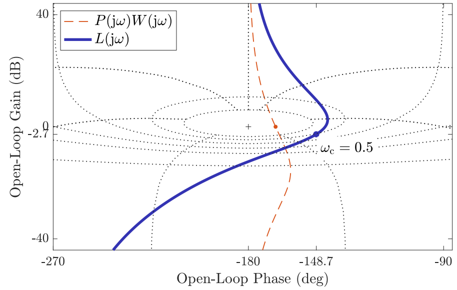
Example: $\omega_c = 0.5$, PI weight

Choose

$$W(s) = k_p\left(1 + \frac{1}{s}\right)$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.4223$, loops are





and the controller

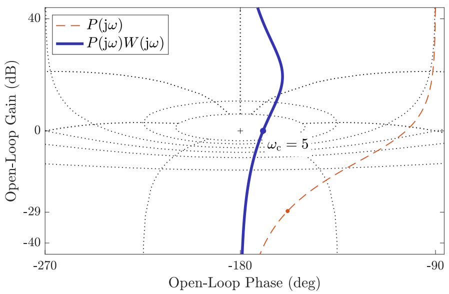
$$C(s) = \frac{0.59807(s + 2.001)(s + 1)(s + 0.1776)}{s(s + 1.821)(s^2 + 1.578s + 0.903)}$$

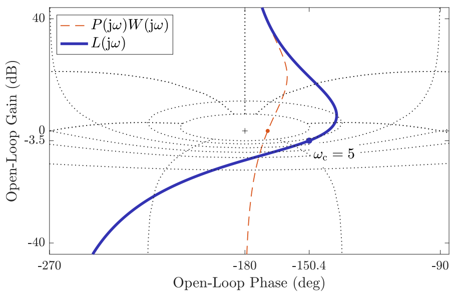
Example: $\omega_c = 5$, PI weight

Choose

$$W(s) = k_p\left(1 + \frac{1}{s}\right)$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting loops of $\epsilon = 0.4173$, loops are are





and the controller (zero of $W(s)$ at $s = -1$ is canceled by a pole of $C_a(s)$)

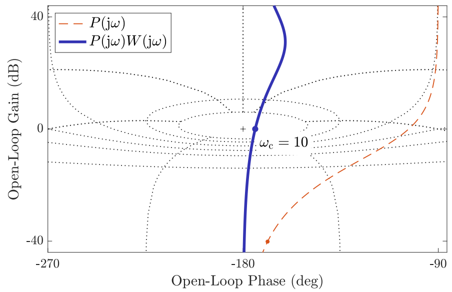
$$C(s) = \frac{311.03(s + 2.658)(s + 0.8432)}{s(s^2 + 13.93s + 95.51)}$$

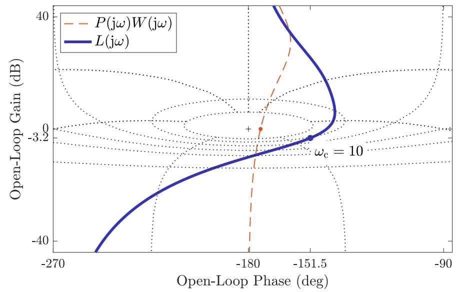
Example: $\omega_c = 10$, PI weight

Choose

$$W(s) = k_p\left(1 + \frac{1}{s}\right)$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.403$, loops are





and the controller (zero of $W(s)$ at $s = -1$ is canceled by a pole of $C_a(s)$)

$$C(s) = \frac{2556.5(s + 4.209)(s + 0.9569)}{s(s^2 + 27.7s + 382.1)}$$