Control Theory (00350188)
lecture no. 10

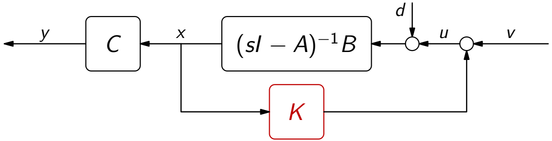
Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT

Outline

Outline

Disturbance response of state feedback



If $\dot{x} = Ax + B(u + d)$ and $u = Kx + v$, then

$$T_{yd}(s) = C(sI - A_K)^{-1}B \quad \text{and} \quad T_{ud}(s) = K(sI - A_K)^{-1}B$$

The effect of K is not immediate, although (remember Vieta's formulae)

$$|T_{yd}(0)| = \frac{|N_P(0)|}{|\chi_{\text{cl}}(0)|} = \frac{|N_P(0)|}{\prod_i |\lambda_i|}$$

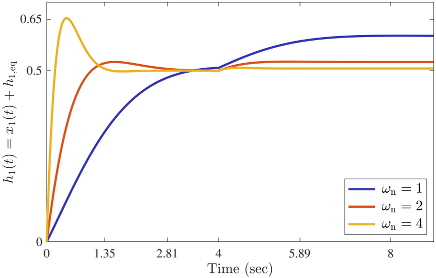
where λ_i are roots of $\chi_{\text{cl}}(s)$. Hence,

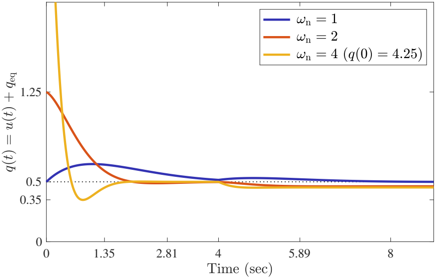
- faster poles $\implies$ smaller steady-state effects of $d = \mathbb{1}$

Two-tank example: state feedback and disturbances

Input disturbance could be caused by an external leakage in the first tank.

With $d(t) = 0.05\mathbb{1}(t-4)$,





Thus,

- faster poles $\implies$ smaller the effect of d

State observer and disturbances

If

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0 \\ y_{\mathrm{m}}(t) = Cx(t) + n(t) \end{cases}$$

the estimator is still

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y_{\mathrm{m}}(t) - C\hat{x}(t)), \quad \hat{x}(0) = \hat{x}_0$$

(we use all information available), but the estimation error,

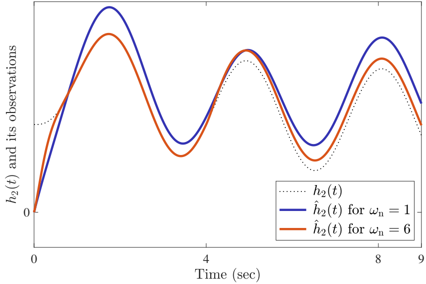
$$\dot{\epsilon}(t) = A_L\epsilon(t) + Bd(t) + Ln(t), \quad \epsilon(0) = x_0 - \hat{x}_0$$

includes both d and n.

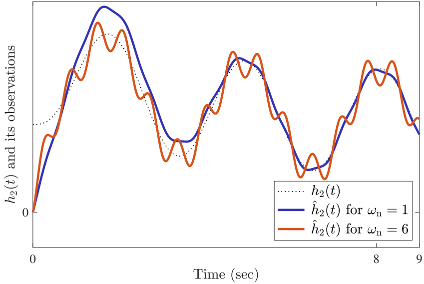
Two-tank example: state observer and disturbances

Returning to our two-tank system,

$d(t) = 0.05\mathbb{1}(t-4)$ and $n(t) = 0$



$d(t) = 0$ and $n(t) = 0.025\sin(10t)$



and observations no longer converge to $h_2(t)$, with

- faster poles $\implies$ higher gain L $\implies$ smaller effect of d
- slower poles $\implies$ lower gain L $\implies$ smaller effect of n

(but be careful with generalizing that).

Closed-loop system with observer-based controller

Taking into account that $\epsilon = x - \hat{x}$, it can be shown that

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t) + \begin{bmatrix} B \\ B \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ L \end{bmatrix} n(t) \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} C & 0 \\ K & -K \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \end{cases}$$

with initial conditions $\left[\begin{smallmatrix} x_0 \\ x_0 - \hat{x}_0 \end{smallmatrix}\right]$.

Hence (check it),

$$T_{yd}(s) = \overbrace{C(sI - A_K)^{-1}B}^{\text{state-feedback } T_{yd}(s)} \left(1 - K(sI - A_L)^{-1}B\right)$$

and

$$T_{yn}(s) = -C(sI - A_K)^{-1}BK(sI - A_L)^{-1}L$$

and effects of K and L on the closed-loop behavior are not transparent.

Closed-loop system with observer-based controller

Taking into account that $\epsilon = x - \hat{x}$, it can be shown that

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t) + \begin{bmatrix} B \\ B \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ L \end{bmatrix} n(t) \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} C & 0 \\ K & -K \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \end{cases}$$

with initial conditions $\left[\begin{smallmatrix} x_0 \\ x_0 - \hat{x}_0 \end{smallmatrix}\right]$.

Hence (check it),

$$T_{yd}(s) = \overbrace{C(sI - A_K)^{-1}B}^{\text{state-feedback } T_{yd}(s)}(1 - K(sI - A_L)^{-1}B)$$

and

$$T_{yn}(s) = -C(sI - A_K)^{-1}BK(sI - A_L)^{-1}L$$

and effects of K and L on the closed-loop behavior are not transparent.

Outline

Effects of disturbances on state feedback and observers

Disturbance observers

Observer-based feedback with disturbance observers

Idea

Consider state reconstruction for

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0, \\ y(t) = Cx(t), \end{cases}$$

If d is

- measurable, $\dot{\epsilon}(t) = A_L \epsilon(t)$ and hence $\epsilon(t) \to 0$
- unmeasurable, $\dot{\epsilon}(t) = A_L \epsilon(t) + Bd(t)$ and hence $\epsilon(t) \not\to 0$ in general

To overcome this problem, we may try to

- observe not only x, but also d,

feasible only if some information about d, like the waveform of its dominant components, is available. This information is normally

- cast as a model of the disturbance signal, aka exosystem.

Idea

Consider state reconstruction for

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0, \\ y(t) = Cx(t), \end{cases}$$

If d is

- measurable, $\dot{\epsilon}(t) = A_L \epsilon(t)$ and hence $\epsilon(t) \to 0$
- unmeasurable, $\dot{\epsilon}(t) = A_L \epsilon(t) + Bd(t)$ and hence $\epsilon(t) \not\to 0$ in general

To overcome this problem, we may try to

- observe not only x, but also d,

feasible only if some information about d, like the waveform of its dominant components, is available. This information is normally

- cast as a model of the disturbance signal, aka exosystem.

Idea

Consider state reconstruction for

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0, \\ y(t) = Cx(t), \end{cases}$$

If d is

- measurable, $\dot{\epsilon}(t) = A_L \epsilon(t)$ and hence $\epsilon(t) \to 0$
- unmeasurable, $\dot{\epsilon}(t) = A_L \epsilon(t) + Bd(t)$ and hence $\epsilon(t) \not\to 0$ in general

To overcome this problem, we may try to

- observe not only x, but also d,

feasible only if some information about d, like the waveform of its dominant components, is available. This information is normally

- cast as a model of the disturbance signal, aka exosystem.

Disturbance generators (exosystem)

Possible model of (unmeasurable) d:

$$\begin{cases} \dot{x}_d(t) = A_d x_d(t), & x_d(0) = x_{d,0}, \\ d(t) = C_d x_d(t), \end{cases}$$

for known A_d and C_d, reflecting our knowledge about d, and unknown $x_{d,0}$, reflecting uncertainty in d. This system

– called disturbance generator

and typically A_d has all its eigenvalues on the jω-axis, generating persistent signals. This model describes the family of signals,

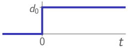
$$d(t) = C_d e^{A_d t} x_{d,0} \iff D(s) = C_d (sI - A_d)^{-1} x_{d,0}$$

for some unknown $x_{d,0}$.

Examples of disturbance generators: step

If

$$d(t) = d_0 \cdot \mathbb{1}(t) =$$



for some unknown d_0, then

$$D(s) = \frac{d_0}{s}.$$

The corresponding signal generator is

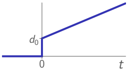
$$\begin{cases} \dot{x}_d(t) = 0 \cdot x_d(t), \quad x_d(0) = d_0, \\ d(t) = 1 \cdot x_d(t), \end{cases}$$

i.e. $A_d = 0$ and $C_d = 1$.

Examples of disturbance generators: ramp

If

$$d(t) = (d_0 + d_r \cdot t) \cdot \mathbb{1}(t) =$$



for some unknown d_0 and d_r, then

$$D(s) = \frac{d_0 s + d_r}{s^2},$$

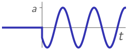
The corresponding signal generator is

$$\begin{cases} \dot{x}_d(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_d(t), \quad x_d(0) = \begin{bmatrix} d_0 \\ d_r \end{bmatrix}, \\ d(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(t), \end{cases}$$

i.e. $A_d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Examples of disturbance generators: harmonic signal

If

$$d(t) = a \sin(\omega t + \phi) \cdot \mathbb{1}(t) =$$ 

for some known ω and unknown a and ϕ, then

$$D(s) = \frac{a \sin(\phi)\, s + a\omega \cos(\phi)}{s^2 + \omega^2},$$

The corresponding signal generator[1] is

$$\begin{cases} \dot{x}_d(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x_d(t), \quad x_d(0) = \begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix} a, \\ d(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(t), \end{cases}$$

i.e. $A_d = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$ and $C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

[1] Take the observer form and apply the similarity transformation with $T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1/\omega \end{smallmatrix}\right]$.

Augmented system: plant + disturbance

Now we have two systems (assume minimality of both):

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)) \\ y(t) = Cx(t) \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_d(t) = A_d x_d(t) \\ d(t) = C_d x_d(t) \end{cases}$$

with corresponding initial conditions. This can be written as

$$P_{\text{a}} : \begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \xi(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad \xi(0) = \begin{bmatrix} x_0 \\ x_{d,0} \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \xi(t), \end{cases}$$

with $\xi := \left[\begin{smallmatrix} x \\ x_d \end{smallmatrix}\right]$, with uncontrollable modes of A_d. Important is that

– the combined system has no unmeasurable inputs,

only unknown initial conditions. Hence, a Luenberger observer can be built to asymptotically estimate both x and x_d, if the realization is detectable.

Augmented system: plant + disturbance

Now we have two systems (assume minimality of both):

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)) \\ y(t) = Cx(t) \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_d(t) = A_d x_d(t) \\ d(t) = C_d x_d(t) \end{cases}$$

with corresponding initial conditions. This can be written as

$$P_a : \begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \xi(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad \xi(0) = \begin{bmatrix} x_0 \\ x_{d,0} \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \xi(t), \end{cases}$$

with $\xi := \left[\begin{smallmatrix} x \\ x_d \end{smallmatrix}\right]$, with uncontrollable modes of A_d. Important is that

- the combined system has no unmeasurable inputs,

only unknown initial conditions. Hence, a Luenberger observer can be built to asymptotically estimate both x and x_d, if the realization is detectable.

Augmented system: observability

A key question:

- is the pair $\left(\begin{bmatrix} C & 0 \end{bmatrix}, \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \right)$ observable (at least, detectable)?

If $\lambda \in \text{spec}(A) \cup \text{spec}(A_d)$ is an unobservable mode, then by PBH

$$\begin{bmatrix} A - \lambda I & BC_d \\ 0 & A_d - \lambda I \\ C & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0, \quad \text{for some } \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \neq 0.$$

Equivalently,

$$\begin{cases} (\lambda I - A_d)\eta_2 = 0 \\ (\lambda I - A)\eta_1 = BC_d \eta_2 \\ C\eta_1 = 0 \end{cases}$$

Two cases:

1. $\lambda \notin \text{spec}(A_d) \implies \eta_2 = 0 \xrightarrow{\text{obs. of } (C,A)} \eta_1 = 0 \implies$ contradiction

Augmented system: observability

A key question:

– is the pair $\left(\begin{bmatrix} C & 0 \end{bmatrix}, \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \right)$ observable (at least, detectable)?

If $\lambda \in \mathrm{spec}(A) \cup \mathrm{spec}(A_d)$ is an unobservable mode, then by PBH

$$\begin{bmatrix} A - \lambda I & BC_d \\ 0 & A_d - \lambda I \\ C & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0, \quad \text{for some } \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \neq 0.$$

Equivalently,

$$\begin{cases} (\lambda I - A_d)\eta_2 = 0 \\ (\lambda I - A)\eta_1 = BC_d\eta_2 \\ C\eta_1 = 0 \end{cases}$$

Two cases:

1. $\lambda \notin \mathrm{spec}(A_d) \implies \eta_2 = 0 \overset{\text{obs. of } (C,A)}{\implies} \eta_1 = 0 \implies$ contradiction

Augmented system: observability (contd)

2. if $\lambda \in \operatorname{spec}(A_d) \setminus \operatorname{spec}(A)$, then

$$\begin{cases} (\lambda I - A_d)\eta_2 = 0 \\ (\lambda I - A)\eta_1 = BC_d\eta_2 \\ C\eta_1 = 0 \end{cases} \overset{\eta_1 = (\lambda I - A)^{-1}BC_d\eta_2}{\Longrightarrow} \begin{cases} (\lambda I - A_d)\eta_2 = 0 \\ \underbrace{C(\lambda I - A)^{-1}B}_{P(\lambda)}C_d\eta_2 = 0 \end{cases}$$

Thus, η_2 is an eigenvector of A_d and $C_d\eta_2 \neq 0$ (by the observability of (C_d, A_d)). Hence, $P(\lambda)C_d\eta_2 = 0 \iff P(\lambda) = 0$.

Therefore,

– $\left(\begin{bmatrix} C & 0 \end{bmatrix}, \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix}\right)$ is observable iff $P(s)$ has no zeros in $\operatorname{spec}(A_d)$

which is logical.

Augmented system: observability (contd)

2. if $\lambda \in \operatorname{spec}(A_d) \setminus \operatorname{spec}(A)$, then

$$\begin{cases} (\lambda I - A_d)\eta_2 = 0 \\ (\lambda I - A)\eta_1 = BC_d\eta_2 \\ C\eta_1 = 0 \end{cases} \overset{\eta_1=(\lambda I-A)^{-1}BC_d\eta_2}{\Longrightarrow} \begin{cases} (\lambda I - A_d)\eta_2 = 0 \\ \underbrace{C(\lambda I - A)^{-1}B}_{P(\lambda)}C_d\eta_2 = 0 \end{cases}$$

Thus, η_2 is an eigenvector of A_d and $C_d\eta_2 \neq 0$ (by the observability of (C_d, A_d)). Hence, $P(\lambda)C_d\eta_2 = 0 \iff P(\lambda) = 0$.

Therefore[2],

– $\left(\begin{bmatrix} C & 0 \end{bmatrix}, \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \right)$ is observable iff $P(s)$ has no zeros in $\operatorname{spec}(A_d)$

which is logical.

[2] This is true also if $\operatorname{spec}(A) \cap \operatorname{spec}(A_d) \neq \varnothing$, but proving is beyond our toolset.

Observer for combined system

Straightforward use of known formulae:

$$\begin{aligned}\dot{\hat{\xi}}(t) &= \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} \left(y(t) - \begin{bmatrix} C & 0 \end{bmatrix} \hat{\xi}(t) \right) \\ &= \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} y(t)\end{aligned}$$

with $\hat{\xi}(0) = \hat{\xi}_0$. In this case error $\epsilon(t) := \xi(t) - \hat{\xi}(t)$ satisfies

$$\dot{\epsilon}(t) = \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \epsilon(t), \quad \epsilon(0) = \xi_0 - \hat{\xi}_0,$$

and asymptotically converges to zero if L and L_d are chosen properly.

Because $\xi = \begin{bmatrix} x \\ x_d \end{bmatrix}$,

– $\hat{\xi}$ reconstructs both x (plant state) and x_d (disturbance state).

Observer for combined system

Straightforward use of known formulae:

$$\begin{aligned}\dot{\hat{\xi}}(t) &= \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} \big(y(t) - \begin{bmatrix} C & 0 \end{bmatrix} \hat{\xi}(t)\big) \\ &= \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} y(t)\end{aligned}$$

with $\hat{\xi}(0) = \hat{\xi}_0$. In this case error $\epsilon(t) := \xi(t) - \hat{\xi}(t)$ satisfies

$$\dot{\epsilon}(t) = \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \epsilon(t), \quad \epsilon(0) = \xi_0 - \hat{\xi}_0,$$

and asymptotically converges to zero if L and L_d are chosen properly.

Because $\xi = \begin{bmatrix} x \\ x_d \end{bmatrix}$,

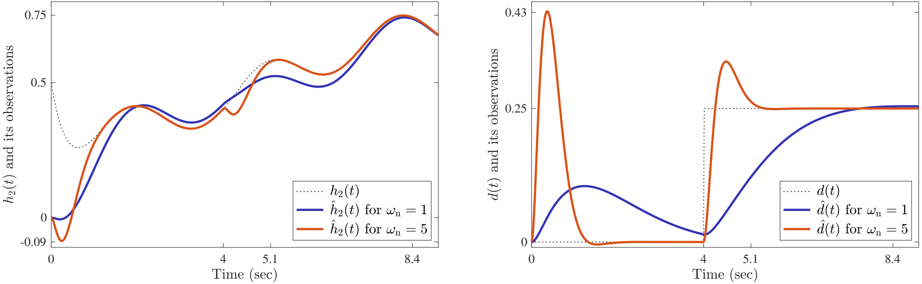
– $\hat{\xi}$ reconstructs both x (plant state) and x_d (disturbance state).

Two-tank example

With $q(t) = 0.5(\sin(2t) + 1)$, $d(t) = 0.25\mathbb{1}(t - 4)$, and

$$\hat{\chi}_{\mathsf{cl}}(s) = (s^2 + 2\hat{\zeta}\hat{\omega}_{\mathsf{n}} s + \hat{\omega}_{\mathsf{n}}^2)(s + 7) \quad \text{for } \hat{\zeta} = 0.8 \text{ and } \hat{\omega}_{\mathsf{n}} = \{1, 5\}$$

as the observer characteristic polynomial, we end up with



Outline

Effects of disturbances on state feedback and observers

Disturbance observers

Observer-based feedback with disturbance observers

Idea

Consider controller design for

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0, \\ y(t) = Cx(t), \end{cases}$$

If both x and d were measurable, we could use

$$u(t) = Kx(t) - d(t) + v(t)$$

to stabilize the system and reject d.

We know what to do when

- x is not measurable $\implies$ observer-based feedback.

What if we use the same idea with a disturbance observer?

Idea

Consider controller design for

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0, \\ y(t) = Cx(t), \end{cases}$$

If both x and d were measurable, we could use

$$u(t) = Kx(t) - d(t) + v(t)$$

to stabilize the system and reject d.

We know what to do when

- x is not measurable $\implies$ observer-based feedback.

What if we use the same idea with a disturbance observer?

Controller

If $v = 0$, then

$$\begin{cases} \dot{\hat{\xi}}(t) = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} \hat{\xi}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} \left(y(t) - \begin{bmatrix} C & 0 \end{bmatrix} \hat{\xi}(t) \right) \\ u(t) = \begin{bmatrix} K & -C_d \end{bmatrix} \hat{\xi}(t) \end{cases}$$

where

$$A + BK \quad \text{and} \quad \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} A + LC & BC_d \\ L_d C & A_d \end{bmatrix}$$

are Hurwitz. The state relation reads

$$\begin{aligned} \dot{\hat{\xi}}(t) &= \left(\begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K & -C_d \end{bmatrix} + \begin{bmatrix} L \\ L_d \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} \right) \hat{\xi}(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} y(t) \\ &= \begin{bmatrix} A + BK + LC & 0 \\ L_d C & A_d \end{bmatrix} \hat{\xi}(t) - \begin{bmatrix} L \\ L_d \end{bmatrix} y(t) \end{aligned}$$

Closed-loop dynamics

Combining the plant and controller, the closed-loop state

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{x}}_d(t) \end{bmatrix} = \begin{bmatrix} A & BK & -BC_d \\ -LC & A + BK + LC & 0 \\ -L_d C & L_d C & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{x}_d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} d(t)$$

With the standard (by now) trick of replacing $\hat{x} \to \epsilon_x := x - \hat{x}$,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ -\dot{\hat{x}}_d(t) \end{bmatrix} = \begin{bmatrix} A + BK & -BK & BC_d \\ 0 & A + LC & BC_d \\ 0 & L_d C & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon_x(t) \\ -\hat{x}_d(t) \end{bmatrix} + \begin{bmatrix} B \\ B \\ 0 \end{bmatrix} d(t)$$

which are stable.

Closed-loop dynamics

Combining the plant and controller, the closed-loop state

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{x}}_d(t) \end{bmatrix} = \begin{bmatrix} A & BK & -BC_d \\ -LC & A+BK+LC & 0 \\ -L_dC & L_dC & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ \hat{x}_d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} d(t)$$

With the standard (by now) trick of replacing $\hat{x} \to \epsilon_x := x - \hat{x}$,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ -\dot{\hat{x}}_d(t) \end{bmatrix} = \begin{bmatrix} A+BK & -BK & BC_d \\ 0 & A+LC & BC_d \\ 0 & L_dC & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon_x(t) \\ -\hat{x}_d(t) \end{bmatrix} + \begin{bmatrix} B \\ B \\ 0 \end{bmatrix} d(t)$$

which are stable.

Disturbance response

If d is indeed generated by its model, then

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ -\dot{\hat{x}}_d(t) \\ \dot{x}_d(t) \end{bmatrix} = \begin{bmatrix} A+BK & -BK & BC_d & BC_d \\ 0 & A+LC & BC_d & BC_d \\ 0 & L_dC & A_d & 0 \\ 0 & 0 & 0 & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon_x(t) \\ -\hat{x}_d(t) \\ x_d(t) \end{bmatrix}$$

with some initial conditions. Introducing $\epsilon_d := x_d - \hat{x}_d$, these dynamics read

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ \dot{\epsilon}_d(t) \\ \hline \dot{x}_d(t) \end{bmatrix} = \left[\begin{array}{ccc|c} A+BK & -BK & BC_d & 0 \\ 0 & A+LC & BC_d & 0 \\ 0 & L_dC & A_d & 0 \\ \hline 0 & 0 & 0 & A_d \end{array}\right] \begin{bmatrix} x(t) \\ \epsilon_x(t) \\ \epsilon_d(t) \\ \hline x_d(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \epsilon_x(0) \\ \epsilon_d(0) \\ \hline x_d(0) \end{bmatrix} = \cdots$$

Therefore,

– x is decoupled from $x_d \implies y = Cx$ is decoupled from $d = C_d x_x$

meaning perfect asymptotic rejection of disturbances from a given class.

Disturbance response

If d is indeed generated by its model, then

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ -\dot{\hat{x}}_d(t) \\ \dot{x}_d(t) \end{bmatrix} = \begin{bmatrix} A+BK & -BK & BC_d & BC_d \\ 0 & A+LC & BC_d & BC_d \\ 0 & L_dC & A_d & 0 \\ 0 & 0 & 0 & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon_x(t) \\ -\hat{x}_d(t) \\ x_d(t) \end{bmatrix}$$

with some initial conditions. Introducing $\epsilon_d := x_d - \hat{x}_d$, these dynamics read

$$\left[\begin{array}{c} \dot{x}(t) \\ \dot{\epsilon}_x(t) \\ \dot{\epsilon}_d(t) \\ \hdashline \dot{x}_d(t) \end{array}\right] = \left[\begin{array}{ccc:c} A+BK & -BK & BC_d & 0 \\ 0 & A+LC & BC_d & 0 \\ 0 & L_dC & A_d & 0 \\ \hdashline 0 & 0 & 0 & A_d \end{array}\right] \left[\begin{array}{c} x(t) \\ \epsilon_x(t) \\ \epsilon_d(t) \\ \hdashline x_d(t) \end{array}\right], \quad \left[\begin{array}{c} x(0) \\ \epsilon_x(0) \\ \epsilon_d(0) \\ \hdashline x_d(0) \end{array}\right] = \dots$$

Therefore,

- x is decoupled from $x_d \implies y = Cx$ is decoupled from $d = C_d x_x$

meaning perfect asymptotic rejection of disturbances from a given class.

Controller structure

Controller $C_y : y \mapsto u$ has the transfer function

$$C_y(s) = -\begin{bmatrix} K & -C_d \end{bmatrix} \left(sI - \begin{bmatrix} A + BK + LC & 0 \\ L_d C & A_d \end{bmatrix} \right)^{-1} \begin{bmatrix} L \\ L_d \end{bmatrix}$$

whose "A" matrix has all eigenvalues of A_d as its eigenvalues. Moreover, it can be shown that

- eigenvalues of A_d are always poles of $C_y(s)$

(to this end we need to prove that all eigenvalues of A_d are both controllable and observable in the realization above, which is true).

This is a version of the Internal Model Principle, roughly saying that

- disturbance model should be a part of the controller.

We are supposed to know it well for the case of $A_d = 0$ (integral action)...

Controller structure

Controller $C_y : y \mapsto u$ has the transfer function

$$C_y(s) = -\begin{bmatrix} K & -C_d \end{bmatrix} \left(sI - \begin{bmatrix} A + BK + LC & 0 \\ L_d C & A_d \end{bmatrix} \right)^{-1} \begin{bmatrix} L \\ L_d \end{bmatrix}$$

whose "A" matrix has all eigenvalues of A_d as its eigenvalues. Moreover, it can be shown that

- eigenvalues of A_d are always poles of $C_y(s)$

(to this end we need to prove that all eigenvalues of A_d are both controllable and observable in the realization above, which is true).

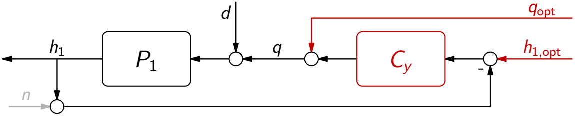
This is a version of the Internal Model Principle, roughly saying that

- disturbance model should be a part of the controller.

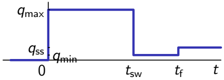
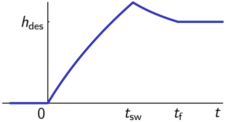
We are supposed to know it well for the case of $A_d = 0$ (integral action) . . .

Two-tank example

Use the 2DOF control architecture



for the time-optimal

$q_{opt}(t) =$  and $h_{1,opt}(t) =$ 

under given bounds q_{min} and q_{max}.

Assuming that $d = d_0 \mathbb{1}$ for an unknown d_0, we design an

- observer-based C_y with the disturbance model

which therefore contains an integral action.

Two-tank example (contd)

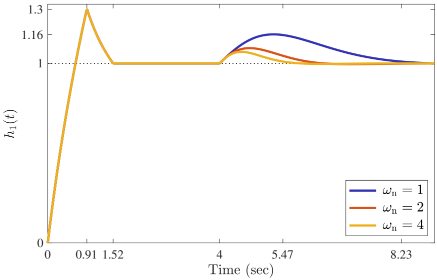
With $q_{\min} = 0.2$, $q_{\max} = 2$, $d(t) = 0.25\mathbb{1}(t-4)$,

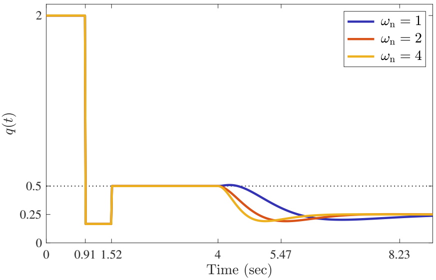
$$\chi_{\mathrm{cl}}(s) = s^2 + 2\zeta\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2 \quad \text{for } \zeta = 0.8 \text{ and } \omega_{\mathrm{n}} = \{1, 2, 4\}$$

as the state-feedback characteristic polynomial (independent of W_d), and

$$\hat{\chi}_{\mathrm{cl}}(s) = (s^2 + 2\hat{\zeta}\hat{\omega}_{\mathrm{n}}s + \hat{\omega}_{\mathrm{n}}^2)(s+7) \quad \text{for } \hat{\zeta} = 0.8 \text{ and } \hat{\omega}_{\mathrm{n}} = 2$$

as the observer characteristic polynomial, we end up with





Two-tank example (contd)

With $q_{\min} = 0.2$, $q_{\max} = 2$, $d(t) = 0.25\mathbb{1}(t-4)$,

$$\chi_{\mathrm{cl}}(s) = s^2 + 2\zeta\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2 \quad \text{for } \zeta = 0.8 \text{ and } \omega_{\mathrm{n}} = \{1, 2, 4\}$$

as the state-feedback characteristic polynomial (independent of W_d), and

$$\hat{\chi}_{\mathrm{cl}}(s) = (s^2 + 2\hat{\zeta}\hat{\omega}_{\mathrm{n}}s + \hat{\omega}_{\mathrm{n}}^2)(s+7) \quad \text{for } \hat{\zeta} = 0.8 \text{ and } \hat{\omega}_{\mathrm{n}} = 5$$

as the observer characteristic polynomial, we end up with