Control Theory (00350188)
lecture no. 9

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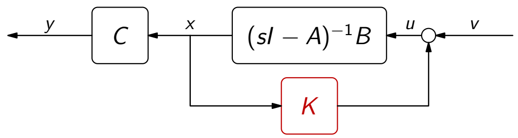
State feedback



efficient in
- stabilizing
- shaping closed-loop modes
- optimizing quadratic cost function
- ...

The elephant in the room:
- what if the state vector cannot be measured directly?

Outline

State reconstruction

Consider state equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

If the state vector cannot be measured (this is what typically happens), then it could be reconstructed from the measured y. Such reconstructor is called state observer or simply observer.

Naïve observer

A possible approach is to construct a virtual plant, like

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad \hat{x}(0) = \hat{x}_0,$$

for some initial guess $\hat{x}_0$. The observation error $\epsilon := x - \hat{x}$ satisfies

$$\dot{\epsilon}(t) = A\epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0$$

which are autonomous dynamics, driven only by the mismatch between $\hat{x}(0)$ and $x(0)$.

Good news:

- if A is Hurwitz, then $\lim_{t\to\infty} \epsilon(t) = 0$, i.e. $\hat{x}(t) \to x(t)$ asymptotically
  no matter what u is, provided we know it, of course

Bad news:

- we cannot affect error dynamics,
- if A is unstable, $\hat{x}$ doesn't converge to x.

Luenberger observer

Naïve observer ignores the

- information about x, available in the measurement y.

Consider adding a function of measured mismatch $y - C\hat{x}$ (aka innovations signal) in the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), & \hat{x}(0) = \hat{x}_0 \\ &= (A + LC)\hat{x}(t) + Bu(t) - Ly(t), & \hat{x}(0) = \hat{x}_0 \end{aligned}$$

for a gain[1] $L \in \mathbb{R}^{n\times 1}$. In this case,

$$\dot{\epsilon}(t) = \underbrace{(A + LC)}_{:= A_L}\epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0.$$

Now we potentially

- can affect the error dynamics.

Q: what freedom we have in assigning $\operatorname{spec}(A + LC)$ by the choice of L?

[1] This L is for Luenberger.

Special case: observer form

Assume that

$$A = \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & \cdots & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Choosing

$$L = \begin{bmatrix} l_{n-1} \\ \vdots \\ l_1 \\ l_0 \end{bmatrix} \implies A_L = A + LC = \begin{bmatrix} -(a_{n-1} - l_{n-1}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(a_1 - l_1) & 0 & \cdots & 1 \\ -(a_0 - l_0) & 0 & \cdots & 0 \end{bmatrix}$$

is still an observer form (companion matrix) and its characteristic polynomial

$$\chi_{A_L}(\lambda) = \lambda^n + (a_{n-1} - l_{n-1})\lambda^{n-1} + \cdots + (a_1 - l_1)\lambda + (a_0 - l_0).$$

Special case: observer form (contd)

Therefore, any desired observer characteristic polynomial, say

$$\hat{\chi}(s) = s^n + \hat{\chi}_{n-1}s^{n-1} + \cdots + \hat{\chi}_1 s + \hat{\chi}_0$$

for some coefficients $\hat{\chi}_i > 0$, can be assigned by

$$L = \begin{bmatrix} a_{n-1} - \hat{\chi}_{n-1} \\ \vdots \\ a_1 - \hat{\chi}_1 \\ a_0 - \hat{\chi}_0 \end{bmatrix} \implies A_L = \begin{bmatrix} -\hat{\chi}_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\chi}_1 & 0 & \cdots & 1 \\ -\chi_0 & 0 & \cdots & 0 \end{bmatrix}$$

Q: under what condition this can be said about an arbitrary realization?

Outline

State observer

Observability

Example: 2-mass system (observability)

Minimality

State observer: pole placement

Observer-based feedback

Observability: definition

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t) + Du(t). \end{cases}$$

This system (or the pair (C, A)) is said to be

– observable if any initial state x_0 can be reconstructed from time history of $u(t)$ and $y(t)$ in interval $[0, t_1]$ for every $t_1 > 0$ and $u(t)$.

Simplifying observation:

– Without loss of generality we can assume that $u(t) \equiv 0$. Indeed, as

$$y(t) = Ce^{At}x_0 + Du(t) + C\int_0^t e^{A(t-s)}Bu(s)ds,$$

x_0 reconstructable from time history of $y(t), u(t)$ iff it reconstructable from time history of $\tilde{y}(t) := y(t) - Du(t) - C\int_0^t e^{A(t-s)}Bu(s)ds$.

Observability and observability matrix

Matrix

$$M_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

called the observability matrix.

Theorem

Pair (C, A) is observable if and only if $\det M_o \neq 0$.

Proof

If $u = 0$, then $y(t) = Ce^{At}x_0$ and

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = M_o x_0.$$

We have:

1. If $\det M_o \neq 0$, x_0 can be obtained from $n-1$ derivatives of y at $t = 0$.
2. If $\det M_o = 0$, then $\exists v \neq 0$ such that $M_o v = 0$, i.e. that $CA^i v = 0$ for all $i = 0, \ldots, n-1$. Then, by Cayley-Hamilton,

$$CA^i v = 0, \quad \forall i \in \mathbb{Z}^+ \implies Ce^{At}v \equiv 0.$$

Therefore, if $x_0 = v$, then $y(t) = Ce^{At}x_0 \equiv 0$ and this initial condition is indistinguishable from $x(0) = 0$. □

Observability and similarity

If $\tilde{A} = TAT^{-1}$ and $\tilde{C} = CT^{-1}$ for some nonsingular T, then

$$\tilde{M}_o := \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T^{-1}$$
$$= M_o T^{-1}$$

i.e.

- observability is not affected by similarity transformations.

Ovservability: some other tests

Theorem

The following statements are equivalent:

1. (C, A) *is observable;*
2. $\det M_o \neq 0$;
3. $\det W_o(t) \neq 0$ *for all* $t > 0$, *where* $W_o(t) := \int_0^t e^{A's} C' C e^{As} ds \in \mathbb{R}^{n \times n}$;
4. $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \in \mathbb{C}^{n+1 \times n}$ *has full column rank* $\forall \lambda \in \mathbb{C}$ *(PBH test);*
5. *eigenvalues of* $A + LC$ *can be freely assigned by* $L \in \mathbb{R}^n$;
6. (A', C') *is controllable.*

The last statement shows

- duality between observability and controllability properties.

Ovservability: some useful facts

The following observations/definitions are important:

- $W_o(t)$-test leads to a derivative-free reconstruction algorithm. Let

$$\tilde{x}(t) := [W_o(t_1)]^{-1} \int_0^t e^{A's} C' y(s) ds.$$

  In this case

$$\tilde{x}(t_1) = [W_o(t_1)]^{-1} \int_0^{t_1} e^{A's} C' C e^{As} x_0 ds = x_0.$$

- If (C, A) is not observable, the PBH test fails for some $\lambda_i \in \mathbb{C}$. These λ_i are eigenvalues of A and called unobservable modes of (C, A).
- If λ is an unobservable mode of (C, A), then it is eigenvalue of $A + LC$ for any L.

Detectability

Pair (C, A) is said to be

- detectable if all its unobservable modes are stable (in open LHP).

Detectability means that there exists $L \in \mathbb{R}^n$ such that

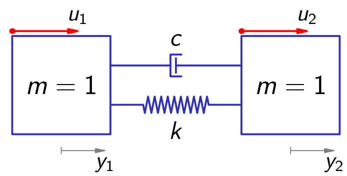
$$A_L := A + LC$$

is Hurwitz (all eigenvalues are in the open LHP).

Outline

Setup

Consider again



with

$$\begin{cases} \underbrace{\begin{bmatrix} \dot y_1(t) \\ \dot y_2(t) \\ \ddot y_1(t) \\ \ddot y_2(t) \end{bmatrix}}_{\dot x(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot y_1(t) \\ \dot y_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot y_1(t) \\ \dot y_2(t) \end{bmatrix} \end{cases}$$

Observability under $y = \gamma_1 y_1 + \gamma_2 y_2$

In this case

$$y(t) = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 & 0 \end{bmatrix} x(t)$$

Observability matrix (denoting $\delta_\gamma := \gamma_1 - \gamma_2$):

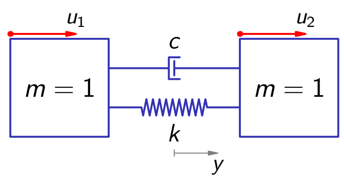
$$M_o = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_1 & \gamma_2 \\ -k\delta_\gamma & k\delta_\gamma & -c\delta_\gamma & c\delta_\gamma \\ 2ck\delta_\gamma & -2ck\delta_\gamma & -(k-2c^2)\delta_\gamma & (k-2c^2)\delta_\gamma \end{bmatrix}$$

with $\det M_o = -k^2(\gamma_1^2 - \gamma_2^2)^2$. Thus, the system is

– unobservable for $\gamma_1 = \pm\gamma_2$.

What could it mean?

Example 1: observability with $\gamma_1 = \gamma_2$, e.g. $y = \frac{y_1 + y_2}{2}$



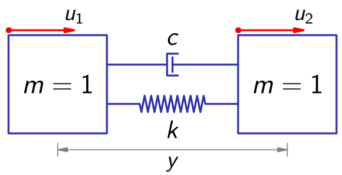
PBH test:

$$\operatorname{rank} \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -k & k & -c-\lambda & c \\ k & -k & c & -c-\lambda \\ 1 & 1 & 0 & 0 \end{bmatrix} \Bigg|_{\lambda = -c \pm \sqrt{c^2 - 2k}} = 3,$$

(rank lost at unobservable modes of A). This agrees with our intuition that

– oscillations cannot be seen via the center of mass.

Example 1: observability with $\gamma_1 = -\gamma_2$, e.g. $y = y_1 - y_2$



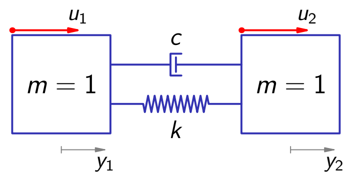
PBH test:

$$\operatorname{rank} \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -k & k & -c-\lambda & c \\ k & -k & c & -c-\lambda \\ 1 & -1 & 0 & 0 \end{bmatrix}\Bigg|_{\lambda=0} = 3,$$

(rank lost at unobservable mode of A). This agrees with our intuition that

– rigid body motion cannot be seen via relative position of the masses.

Transfer functions for $y = \gamma_1 y_1 + \gamma_2 y_2$



Transfer function from u_1 to y:

$$P_1(s) = \frac{\gamma_1 s^2 + c(\gamma_1 + \gamma_2)s + k(\gamma_1 + \gamma_2)}{s^2(s^2 + 2cs + 2k)}$$

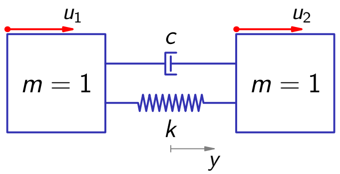
and transfer function from u_2 to y:

$$P_2(s) = \frac{\gamma_2 s^2 + c(\gamma_1 + \gamma_2)s + k(\gamma_1 + \gamma_2)}{s^2(s^2 + 2cs + 2k)}$$

(both obtained via $C(sI - A)^{-1}B$).

Transfer functions for $y = \gamma_1 y_1 + \gamma_2 y_2$ (contd)

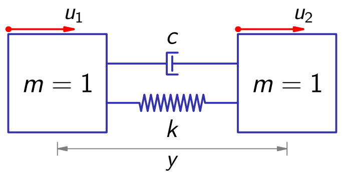
$\gamma_1 = \gamma_2$:



then

$$P_1(s) = P_2(s) = \frac{\gamma_1}{s^2}.$$

$\gamma_1 = -\gamma_2$:



then

$$P_1(s) = -P_2(s) = \frac{\gamma_1}{s^2 + 2cs + 2k}.$$

In both cases we have pole/zero cancellations (of different modes though).

Outline

Minimal state-space realization

Example

Let $G(s) = \frac{1}{s+1}$. The following are its state-space realizations:

$$\begin{cases} \dot{x} = -x + u, & x(0) = 0 \\ y = x \end{cases} \quad \text{and} \quad \begin{cases} \dot{\tilde{x}} = -\left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]\tilde{x} + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]u, & \tilde{x}(0) = 0, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix}\tilde{x}. \end{cases}$$

The first of them has state dimension $n = 1$, while the second one—$n = 2$. This indicates that there is *redundancy* in $\tilde{x}$ (it accumulates somebody else history as well).

We may be interested to avoid redundancy. To this end, the notion of

- minimal state-space realization, i.e. a realization with minimal possible dimension,

plays a key role.

Minimality criterion

Theorem

Realization

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0, \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is minimal iff it is both controllable and observable.

Explanations:

- uncontrollable part of x cannot be affected by input u,
- unobservable part of x is invisible from output y.

Important fact:

- every two minimal realizations of the same system are similar

(i.e. there is a similarity transformation between them).

Minimality and poles

Theorem

If

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0, \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is minimal, then $\lambda \in \mathbb{C}$ *is a pole of* $G(s) = D + C(sI - A)^{-1}B$ *iff it is an eigenvalue of* A.

Outline

Luenberger observer: choice of L

Let (C, A) be observable, then for an arbitrary polynomial

$$\hat{\chi}_{\rm cl}(s) = s^n + \hat{\chi}_{n-1}s^{n-1} + \cdots + \hat{\chi}_1 s + \hat{\chi}_0$$

there exists observer gain L such that $\hat{\chi}_{\rm cl}(s)$ is characteristic polynomial of observer error, i.e. $\hat{\chi}_{\rm cl}(s) = \det(sI - A_L)$.

The gain L leading to a required $\hat{\chi}_{\rm cl}(s)$ can be chosen by the counterpart of Ackermann's formula[2]:

$$L = -\hat{\chi}_{\rm cl}(A)M_{\rm o}^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

[2] Apply Ackermann's formula to $(A + LC)' = A' + C'L'$ and then transpose the result.

Example: two-tank system (contd)

Suppose that fluid height only in the first tank can be measured, i.e.

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = -\begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

(here $y = h_1 - h_{1,\rm eq}$). To reconstruct $x_2(t)$ we build state observer (virtual sensor) in the form

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t) - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}(y(t) - \hat{x}_1(t))$$

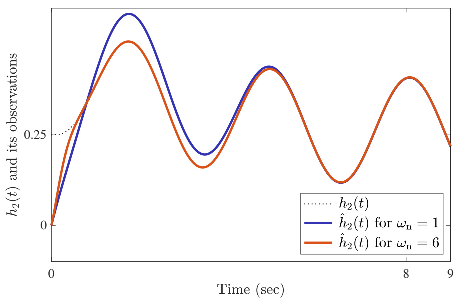
where

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = -\hat{\chi}_{\rm cl}(A)\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} -3 + 2\hat{\zeta}\hat{\omega}_{\rm n} \\ 5 - 4\hat{\zeta}\hat{\omega}_{\rm n} + \hat{\omega}_{\rm n} \end{bmatrix}$$

for a desired $\hat{\chi}_{\rm cl}(s) = s^2 + 2\hat{\zeta}\hat{\omega}_{\rm n}s + \hat{\omega}_{\rm n}^2$.

Example: simulations

With $u(t) = q(t) - q_{\rm eq} = 0.5\sin(2t)$, $\hat{\zeta} = 0.8$, and $\hat{\omega}_{\rm n} = \{1, 5\}$,



under $L = \begin{bmatrix} 1.4 \\ -2.8 \end{bmatrix}$ and $L = -\begin{bmatrix} 6.6 \\ 21.8 \end{bmatrix}$.

Outline

Output feedback: naïve approach

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \end{cases}$$

in which the state vector x is not measured. Hence, state feedback cannot be used. Under this circumstance, we may try to

– combine state feedback and state observer

instead, i.e. to use observed state in control law as if it were the true state.

This results to the following control law:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \hat{x}_0 \\ u(t) = K\hat{x}(t) + v(t) \end{cases}$$

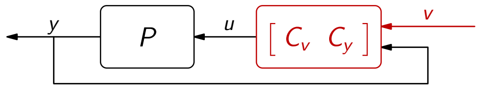
which is called observer-based controller.

Observer-based controller

Observer-based control law can be rewritten as

$$\begin{cases} \dot{\hat{x}}(t) = (A + BK + LC)\hat{x}(t) - Ly(t) + Bv(t), \quad \hat{x}(0) = \hat{x}_0 \\ u(t) = K\hat{x}(t) + v(t) \end{cases}$$

which is a system having v and y as its inputs and u as its output:



with

$$\begin{bmatrix} C_v(s) & C_y(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} + K(sI - (A + BK + LC))^{-1} \begin{bmatrix} B & -L \end{bmatrix},$$

where $C_v : v \mapsto u$ and $C_y : y \mapsto u$. Note that

– controller $-C_y$ is the state-space counterpart of the feedback controller

in the standard unity-feedback case with negative feedback.

Closed-loop system

State equation of the closed-loop system $v \mapsto y = Cx$ is:

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v(t) \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \end{cases}$$

with initial conditions $\left[\begin{smallmatrix} x_0 \\ \hat{x}_0 \end{smallmatrix}\right]$. What can we say about its modes / stability?

A key observation is that

– dynamics of the observer error $\epsilon = x - \hat{x}$ do not depend on u.

So change the state vector to

$$\begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},$$

i.e. use the similarity transformation with $T = \left[\begin{smallmatrix} I & 0 \\ I & -I \end{smallmatrix}\right] = T^{-1}$.

Closed-loop system: similarity transformation

We have:

$$\tilde{A}_{\rm cl} = TA_{\rm cl}T^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix}$$

$$\tilde{B}_{\rm cl} = TB_{\rm cl} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\tilde{C}_{\rm cl} = C_{\rm cl}T^{-1} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} C & 0 \end{bmatrix}$$

Note that

– the pair $(\tilde{A}_{\rm cl}, \tilde{B}_{\rm cl})$ has all modes of A_L uncontrollable in it.

Indeed,

$$\begin{bmatrix} 0 & \tilde{\eta}_2' \end{bmatrix} \begin{bmatrix} A_K - \lambda I & -BK & B \\ 0 & A_L - \lambda I & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\eta}_2'(A_L - \lambda I) & 0 \end{bmatrix} = 0$$

for every right (nonzero) eigenvector $\tilde{\eta}_2$ of A_L, so PBH yields the conclusion.

The separation

Thus, we end up with the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t), & \begin{bmatrix} x(0) \\ \epsilon(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_0 - \hat{x}_0 \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} \end{cases}$$

The roots of the closed-loop characteristic polynomial

$$\chi_{\text{cl}}(s) = \det(sI - A_K)\det(sI - A_L)$$

are the union of the state-feedback and observer modes. Thus, all we need to do to stabilize the system is to

- design stabilizing state feedback — i.e. its gain K
- design stable observer — i.e. its gain L

separately. This is known as the separation principle.

Closed-loop system $v \mapsto y$

The "ϵ" part of the closed-loop behavior

$$\dot{\epsilon}(t) = A_L \epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0 \quad \Longrightarrow \quad \epsilon(t) = e^{A_L t}(x_0 - \hat{x}_0)$$

The "x" part is then

$$\dot{x}(t) = A_K x(t) + Bv(t) - BK\epsilon(t) = A_K x(t) + B\big(v(t) - K e^{A_L t}(x_0 - \hat{x}_0)\big)$$

i.e. including an observer is

- equivalent to adding an exponentially decaying signal to v.

Moreover, if $x_0 = \hat{x}_0$, then $\epsilon = 0$ and

$$\dot{x}(t) = A_K x(t) + Bv(t), \quad x(0) = x_0$$

exactly like in the case of measured state.