Control Theory (035188)
lecture no. 8

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Setup

Plant:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0 \\ y(t) = Cx(t) \\ y_{\mathrm{m}}(t) = Cx(t) + n(t) \end{cases}$$

for known $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$, and $C \in \mathbb{R}^{1 \times n}$ such that (A, B) is stabilizable and (C, A) is detectable.

Uncertainty:

– initial condition $x_0 \in \mathbb{R}^n$
– load disturbance $d(t) \in \mathbb{R}$
– measurement noise $n(t) \in \mathbb{R}$

Control goals:

– stabilize
– reduce the effect of uncertainty on $x(t)$
– track a known reference signal $r(t)$ by $y(t)$

Outline

State feedback (no uncertainty)

State observer (only the past is uncertain)

Observer-based output feedback (only the past is uncertain)

Effect of disturbances

2DOF state feedback (for curious)

State feedback

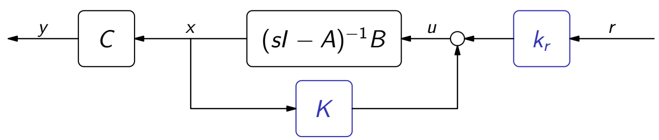
With

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \end{cases}$$

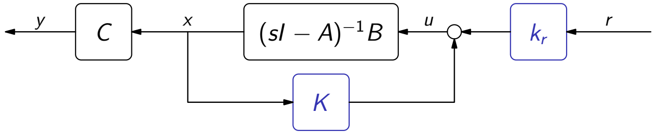
control law

$$u(t) = k_r r(t) + Kx(t)$$

is called state feedback. Equivalently,



State feedback (contd)



The closed-loop state-space realization

$$\begin{cases} \dot{x}(t) = A_K x(t) + Bk_r r(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \end{cases}$$

where $A_K := A + BK$. The closed-loop transfer function from r to y:

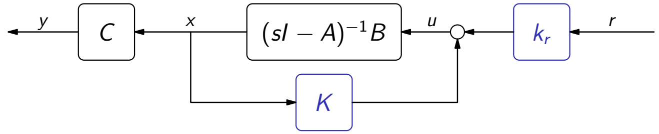
$$T_{yr}(s) = C(sI - A_K)^{-1}Bk_r$$

(assuming $x_0 = 0$). Closed-loop characteristic polynomial:

$$\chi_{\text{cl}}(s) = \det(sI - A_K).$$

If (A, B) stabilizable, then $\chi_{\text{cl}}(s)$ can be made Hurwitz by a choice of K.

State feedback and system zeros



Let $P(s) = C(sI - A)^{-1}B = \frac{N_P(s)}{D_P(s)}$, then

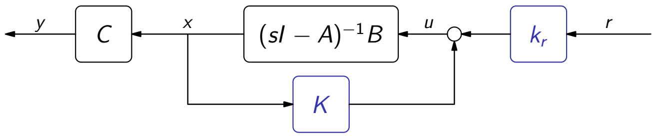
$$u = k_r r + K(sI - A)^{-1}Bu \iff u = \overbrace{\frac{1}{1 - K(sI - A)^{-1}B}}^{T_{ur}(s)} k_r r.$$

Because $[1 + C(sI - A)^{-1}B]^{-1} = 1 - C(sI - (A - BC))^{-1}B$,

$$\begin{aligned} T_{ur}(s) = \frac{1}{1 - K(sI - A)^{-1}B} k_r &= \frac{D_P(s)}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} k_r \\ = (1 + K(sI - A_K)^{-1}B)k_r &= \frac{s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{\chi_{\text{cl}}(s)} k_r \end{aligned}$$

Hence, $T_{ur}(s) = k_r D_P(s)/\chi_{\text{cl}}(s)$ (mind that $T_{ur}(\infty) = k_r$).

State feedback and system zeros (contd)



Thus,

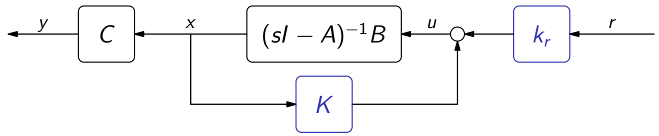
$$T_{yr}(s) = P(s)T_{ur}(s) = \frac{N_P(s)}{D_P(s)} \frac{D_P(s)}{\chi_{\text{cl}}(s)} k_r = \frac{N_P(s)}{\chi_{\text{cl}}(s)} k_r.$$

This means that

- state feedback does not move zeros

(stable zeros may be canceled by roots of $\chi_{\text{cl}}(s)$ though).

State feedback and steady-state error



Steady-state error to a step r,

$$e_{\text{ss}} := \lim_{t\to\infty} |r(t) - y(t)| = |1 - T_{yr}(0)| = |1 + CA_K^{-1}Bk_r|.$$

To render it zero, we have to choose

$$k_r = -\frac{1}{CA_K^{-1}B} = \frac{\chi_{\text{cl}}(0)}{N_P(0)}.$$

Note that

- A_K is invertible because it is Hurwitz
- k_r exists (is finite) iff $N_P(0) \neq 0$, i.e. plant has no zeros at the origin

Pole placement: companion form

Let's start with (A, B) in the companion form:

$$A = A_{\mathrm{cf}} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad \text{and} \quad B = B_{\mathrm{cf}} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Select a desired closed-loop characteristic polynomial, say

$$\chi_{\mathrm{cl}}(s) = s^n + \chi_{n-1}s^{n-1} + \cdots + \chi_1 s + \chi_0$$

for some coefficients χ_i. We already know (Lect. 7) that the state feedback gain

$$K = K_{\mathrm{cf}} := \begin{bmatrix} a_0 - \chi_0 & a_1 - \chi_1 & \cdots & a_{n-1} - \chi_{n-1} \end{bmatrix}$$

renders $\chi_{A+BK}(s) = \chi_{\mathrm{cl}}(s)$.

Pole placement: arbitrary realization

Conceptually, all we need to do is to

- transform (A, B) into the companion form by similarity transformation.

Indeed, for any controllable (A, B) there is (we'll show this by construction) nonsingular T such that

$$A_{\mathrm{cf}} = TAT^{-1} \quad \text{and} \quad B_{\mathrm{cf}} = TB.$$

Then $A = T^{-1}A_{\mathrm{cf}}T$, $B = T^{-1}B_{\mathrm{cf}}$, and

$$K = K_{\mathrm{cf}}T$$

does the job:

$$A + BK = T^{-1}A_{\mathrm{cf}}T + T^{-1}B_{\mathrm{cf}}K_{\mathrm{cf}}T = T^{-1}(A_{\mathrm{cf}} + B_{\mathrm{cf}}K_{\mathrm{cf}})T.$$

Elegant algorithm to construct required $K_{\mathrm{cf}}T$ w/o explicit calculation of T is offered by Ackermann's formula.

Ackermann's formula: preliminaries

1. by the Cayley–Hamilton theorem, $\chi_{\mathrm{ol}}(A_{\mathrm{cf}}) = 0$, so that

$$\begin{aligned} \chi_{\mathrm{cl}}(A_{\mathrm{cf}}) &= A_{\mathrm{cf}}^n + \chi_{n-1}A_{\mathrm{cf}}^{n-1} + \cdots + \chi_1 A_{\mathrm{cf}} + \chi_0 I \\ &= -a_{n-1}A_{\mathrm{cf}}^{n-1} - \cdots - a_1 A_{\mathrm{cf}} - a_0 I \\ &\quad + \chi_{n-1}A_{\mathrm{cf}}^{n-1} + \cdots + \chi_1 A_{\mathrm{cf}} + \chi_0 I \\ &= (\chi_{n-1} - a_{n-1})A_{\mathrm{cf}}^{n-1} + \cdots + (\chi_1 - a_1)A_{\mathrm{cf}} + (\chi_0 - a_0)I. \end{aligned}$$

2. if e_i is the ith standard basis in $\mathbb{R}^n$, then $\forall i = 1, \ldots, n-1$,

$$e_i' A_{\mathrm{cf}} = e_{i+1}' \quad \text{or, equivalently,} \quad e_1' A_{\mathrm{cf}}^i = e_{i+1}',$$

3. if $A_{\mathrm{cf}} = TAT^{-1}$ and $B_{\mathrm{cf}} = TB$, then

$$M_{\mathrm{c,cf}} = TM_{\mathrm{c}} \iff T = M_{\mathrm{c,cf}}M_{\mathrm{c}}^{-1}.$$

Ackermann's formula: preliminaries (contd)

4. Combining 1 and 2:

$$\begin{aligned} K_{\mathrm{cf}} &= \begin{bmatrix} a_0 - \chi_0 & a_1 - \chi_1 & \cdots & a_{n-1} - \chi_{n-1} \end{bmatrix} \\ &= (a_0 - \chi_0)e_1' + (a_1 - \chi_1)e_2' + \cdots + (a_{n-1} - \chi_{n-1})e_n' \\ &= (a_0 - \chi_0)e_1' + (a_1 - \chi_1)e_1' A_{\mathrm{cl}} + \cdots + (a_{n-1} - \chi_{n-1})e_1' A_{\mathrm{cl}}^{n-1} \\ &= -e_1' \chi_{\mathrm{cl}}(A_{\mathrm{cf}}) \end{aligned}$$

5. By 2 (and the fact that $B_{\mathrm{cf}} = e_n$):

$$e_1' M_{\mathrm{c,cf}} = e_1' \begin{bmatrix} B_{\mathrm{cf}} & A_{\mathrm{cf}}B_{\mathrm{cf}} & \cdots & A_{\mathrm{cf}}^{n-1}B_{\mathrm{cf}} \end{bmatrix} = e_n'$$

Ackermann's formula: derivation

We have:

$$
\begin{aligned}
K_{\mathrm{cf}} &= -e_1' \chi_{\mathrm{cl}}(A_{\mathrm{cf}}) && \text{by item } 4 \\
&= -e_1' \chi_{\mathrm{cl}}(TAT^{-1}) = -e_1' T \chi_{\mathrm{cl}}(A) T^{-1} \\
&= -e_1' M_{\mathrm{c,cf}} M_{\mathrm{c}}^{-1} \chi_{\mathrm{cl}}(A) T^{-1} && \text{by item } 3 \\
&= -e_n' M_{\mathrm{c}}^{-1} \chi_{\mathrm{cl}}(A) T^{-1} && \text{by item } 5
\end{aligned}
$$

Now it is time to return to the original coordinates:

$$
\begin{aligned}
K = K_{\mathrm{cf}} T &= -e_n' M_{\mathrm{c}}^{-1} \chi_{\mathrm{cl}}(A) T^{-1} T \\
&= -e_n' M_{\mathrm{c}}^{-1} \chi_{\mathrm{cl}}(A),
\end{aligned}
$$

voilà!

Ackermann's formula

The feedback gain assigning the closed-loop poles to the roots of $\chi_{\mathrm{cl}}(s)$ is

$$
K = -\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_{\mathrm{c}}^{-1} \chi_{\mathrm{cl}}(A),
$$

where

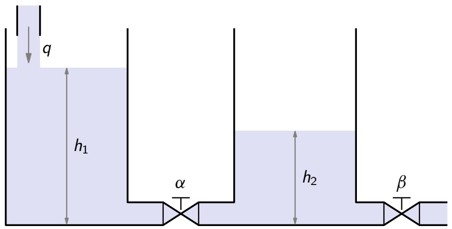
$$
M_{\mathrm{c}} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}
$$

and

$$
\chi_{\mathrm{cl}}(A) = A^n + \chi_{n-1} A^{n-1} + \cdots + \chi_1 A + \chi_0 I.
$$

This gain K is called Ackermann's formula and indeed depends only on the original (controllable) realization.

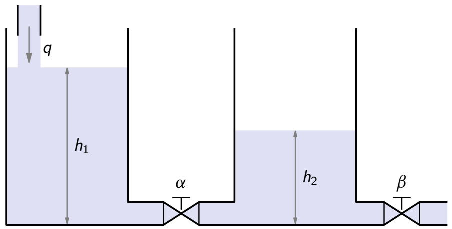
Example: two-tank system



Here:

- q is the control flow,
- h_1 and h_2 are fluid heights,
- α is the resistance to the valve between the tanks,
- β is the resistances of the output valve,
- crossing areas of each tank is σ.

Example: two-tank system (contd)



Its nonlinear dynamics (provided $h_1 > h_2$)

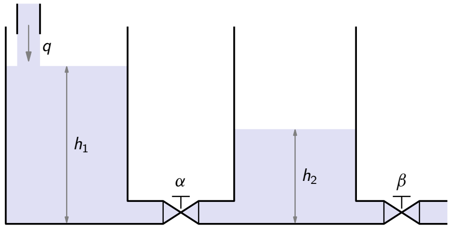
$$
\sigma \begin{bmatrix} \dot h_1(t) \\ \dot h_2(t) \end{bmatrix} = \begin{bmatrix} -\alpha\sqrt{h_1(t) - h_2(t)} + q \\ \alpha\sqrt{h_1(t) - h_2(t)} - \beta\sqrt{h_2(t)} \end{bmatrix}
$$

The equilibria are

$$
h_{2,\mathrm{eq}} = \frac{\alpha^2}{\alpha^2 + \beta^2} h_{1,\mathrm{eq}} < h_{1,\mathrm{eq}} \quad \text{and} \quad q_{\mathrm{eq}} = \sqrt{\frac{\alpha^2\beta^2}{\alpha^2 + \beta^2} h_{1,\mathrm{eq}}}
$$

for any $h_{1,\mathrm{eq}} > 0$.

Example: two-tank system (contd)



With $x_i = h_i - h_{i,\mathrm{eq}}$ and $u = q - q_{\mathrm{eq}}$, the linearized dynamics are

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \frac{\alpha\sqrt{1+(\alpha/\beta)^2}}{2\sigma\sqrt{h_{1,\mathrm{eq}}}} \begin{bmatrix} -1 & 1 \\ 1 & -1-(\beta/\alpha)^2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \frac{1}{\sigma} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

with negative real poles at

$$s_{1,2} = -\frac{\alpha\sqrt{1+(\alpha/\beta)^2}}{2\sigma\sqrt{h_{1,\mathrm{eq}}}} \frac{2\alpha^2+\beta^2 \pm \sqrt{4\alpha^4+\beta^4}}{2\alpha^2}.$$

Example: two-tank system (contd)

Let $\sigma = 1$, $\alpha = \beta = 1$ and choose

$$h_{1,\mathrm{eq}} = 1/2 \quad \Longrightarrow \quad h_{2,\mathrm{eq}} = 1/4 \quad \text{and} \quad q_{\mathrm{eq}} = 1/2.$$

If $h_1(0) = h_2(0) = 0$, then $x_1(0) = -1/2$ and $x_2(0) = -1/4$ and we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = -\begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}.$$

with modes at $s_1 = -2.618$ and $s_2 = -0.382$ and

$$P_1(s) = \frac{s+2}{(s+2.618)(s+0.382)} \quad \text{and} \quad P_2(s) = \frac{1}{(s+2.618)(s+0.382)}$$

as the transfer functions $u \mapsto x_1$ and $u \mapsto x_2$, respectively.

Our goal is to

- regulate x from $x(0)$ to $\lim_{t\to\infty} x(t) = 0$ in a desired matter

(regulator problem), which is effectively the set-point tracking of h_{eq}.

Example: state feedback

Controllability matrix

$$M_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad M_c^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let's choose

$$\chi_{\mathrm{cl}}(s) = s^2 + 2\zeta\omega_n s + \omega_n^2.$$

Ackermann's formula $K = -\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} \chi_{\mathrm{cl}}(A)$ reads then

$$\begin{aligned} K &= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}^2 + 2\zeta\omega_n \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2-2\zeta\omega_n+\omega_n^2 & -3+2\zeta\omega_n \\ -3+2\zeta\omega_n & 5-4\zeta\omega_n+\omega_n^2 \end{bmatrix} \\ &= -\begin{bmatrix} -3+2\zeta\omega_n & 5-4\zeta\omega_n+\omega_n^2 \end{bmatrix} \end{aligned}$$

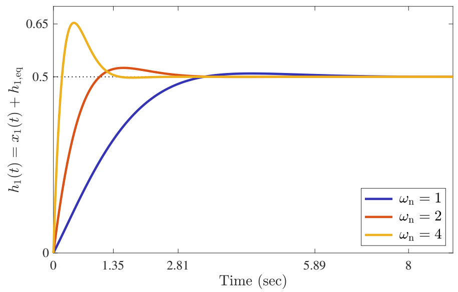
(the absolute values of both components of K grow with ω_n).

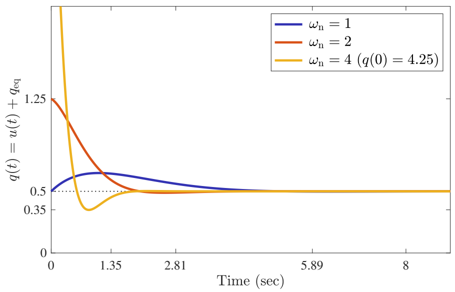
Example: simulations

The control law

$$u(t) = Kx(t) \quad \Longrightarrow \quad q(t) = Kx(t) + q_{\mathrm{eq}} - Kh_{\mathrm{eq}}.$$

With $\sigma = 1$, $\alpha = \beta = 1$, $\zeta = 0.8$, and $\omega_n = \{1, 2, 5\}$,



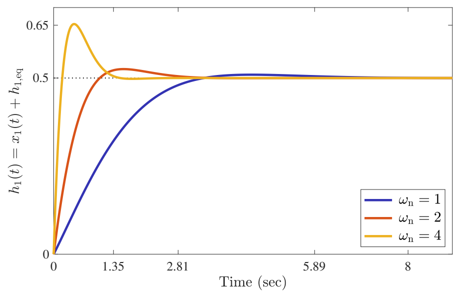


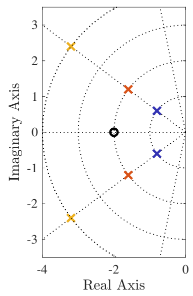
under $K = -\begin{bmatrix} -1.4 & 2.8 \end{bmatrix}$, $K = -\begin{bmatrix} 0.2 & 2.6 \end{bmatrix}$, $K = -\begin{bmatrix} 3.4 & 8.2 \end{bmatrix}$. Thus,

- faster poles $\Longrightarrow$ faster response & larger control effort
- faster poles $\Longrightarrow$ large overshoot (why?)

Example: simulations (contd)

The reason can be seen in the pole-zero map:





Because system zeros are not moved,

– when poles move left, zero at -2 becomes dominant.

Outline

State reconstruction

Consider state equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

If the state vector cannot be measured (this is what typically happens), then it could be reconstructed from measurements of $y(t)$. Such reconstructor is called state observer or simply observer.

Naïve observer

A possible approach is to construct virtual plant, like

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad \hat{x}(0) = \hat{x}_0,$$

for some initial guess $\hat{x}_0$. Define observation error $\epsilon(t) := x(t) - \hat{x}(t)$. Then

$$\dot{\epsilon}(t) = A\epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0.$$

Good news:

– if A is stable, $\lim_{t\to\infty} \epsilon(t) = 0$, i.e. $\hat{x}(t) \to x(t)$ asymptotically (no matter what $u(t)$ is, provided we know it, of course)

Bad news:

– we cannot affect error dynamics,
– if A is unstable, $\hat{x}(t)$ doesn't converge to $x(t)$.

Luenberger observer

Both problems can be resolved by the following modification, which uses y:

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), & \hat{x}(0) &= \hat{x}_0 \\ &= (A+LC)\hat{x}(t) + Bu(t) - Ly(t), & \hat{x}(0) &= \hat{x}_0\end{aligned}$$

i.e. by adding correction term with observer gain L. In this case,

$$\dot{\epsilon}(t) = (\underbrace{A+LC}_{A_L})\epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0.$$

Now we can

- affect its dynamics (more precisely, observable modes of (C, A)) and
- stabilize it, provided (C, A) is detectable.

Although $y(t) - C\hat{x}(t) = C\epsilon(t)$ depends only on a part of $\epsilon(t)$,

- detectability ensures that $C\hat{x}(t) \to y(t) \implies \hat{x}(t) \to x(t)$.

Transfer functions of Luenberger observer

The observer is a dynamical system having $u(t)$ and $y(t)$ as its inputs and $\hat{x}(t)$ as its output. Under zero initial conditions, the transfer functions from u to $\hat{x}$ and from y to $\hat{x}$ are (here $A_L := A + LC$)

$$G_{\hat{x}u}(s) = (sI - A_L)^{-1}B \quad \text{and} \quad G_{\hat{x}y}(s) = -(sI - A_L)^{-1}L,$$

respectively.

Remark: If we are interested to reconstruct only a part of the state, e.g.

$$z(t) = C_z x(t),$$

the transfer functions from u to $\hat{z} := C_z\hat{x}$ and from y to $\hat{z}$ are

$$G_{\hat{z}u}(s) = C_z(sI - A_L)^{-1}B \quad \text{and} \quad G_{\hat{z}y}(s) = -C_z(sI - A_L)^{-1}L,$$

respectively.

Luenberger observer: choice of L

Let (C, A) be observable, then for an arbitrary polynomial

$$\hat{\chi}_{\text{cl}}(s) = s^n + \hat{\chi}_{n-1}s^{n-1} + \cdots + \hat{\chi}_1 s + \hat{\chi}_0$$

there exists observer gain L such that $\hat{\chi}_{\text{cl}}(s)$ is characteristic polynomial of observer error, i.e. $\hat{\chi}_{\text{cl}}(s) = \det(sI - A_L)$.

The choice of L leading to a required $\hat{\chi}_{\text{cl}}(s)$ is

- easy if (C, A) are in the observer canonical form
- done by the counterpart of Ackermann's formula:

$$L = -\hat{\chi}_{\text{cl}}(A)M_o^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Example: two-tank system (contd)

Suppose that fluid height only in the first tank can be measured, i.e.

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t), & \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = -\begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

(here $y = h_1 - h_{1,\text{eq}}$). To reconstruct $x_2(t)$ we build state observer (virtual sensor) in the form

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t) - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}(y(t) - \hat{x}_1(t))$$

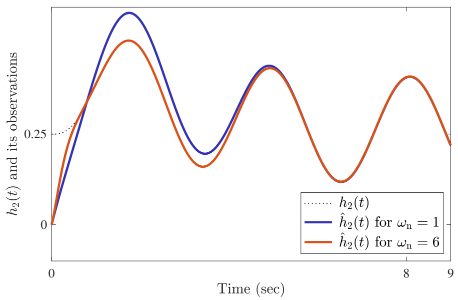
where

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = -\hat{\chi}_{\text{cl}}(A)\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} -3 + 2\hat{\zeta}\hat{\omega}_\text{n} \\ 5 - 4\hat{\zeta}\hat{\omega}_\text{n} + \hat{\omega}_\text{n} \end{bmatrix}$$

for a desired $\hat{\chi}_{\text{cl}}(s) = s^2 + 2\hat{\zeta}\hat{\omega}_\text{n}s + \hat{\omega}_\text{n}^2$.

Example: simulations

With $u(t) = q(t) - q_{\text{eq}} = 0.5\sin(2t)$, $\hat{\zeta} = 0.8$, and $\hat{\omega}_{\text{n}} = \{1, 5\}$,



under $L = \begin{bmatrix} 1.4 \\ -2.8 \end{bmatrix}$ and $L = -\begin{bmatrix} 6.6 \\ 21.8 \end{bmatrix}$.

Outline

State feedback (no uncertainty)

State observer (only the past is uncertain)

Observer-based output feedback (only the past is uncertain)

Effect of disturbances

2DOF state feedback (for curious)

Output feedback: naïve approach

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \end{cases}$$

in which the state vector $x(t)$ is not measured. Therefore, state feedback cannot be used. Under this circumstance, we may try to

– combine state feedback and state observer

instead, i.e. to use observed state in control law as if it were the true state.

This results to the following control law:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \hat{x}_0 \\ u(t) = k_r r(t) + K\hat{x}(t) \end{cases}$$

which called observer-based controller.

Observer-based controller

Observer-based control law can be rewritten as

$$\begin{cases} \dot{\hat{x}}(t) = (A + BK + LC)\hat{x}(t) - Ly(t) + Bk_r r(t), \quad \hat{x}(0) = \hat{x}_0 \\ u(t) = K\hat{x}(t) + k_r r(t) \end{cases}$$

which is a dynamical system having $y(t)$ and $r(t)$ as its inputs and $u(t)$ as its output. Under zero initial conditions, transfer function from y to u is

$$C_y(s) = -K\big(sI - (A + BK + LC)\big)^{-1} L$$

and from r to u is

$$C_r(s) = \big(1 + K\big(sI - (A + BK + LC)\big)^{-1} B\big) k_r.$$

Closed-loop system

State equation of the closed-loop system, from r to $y = Cx$ is:

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A+BK+LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} Bk_r \\ Bk_r \end{bmatrix} r(t) \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \end{cases}$$

with initial conditions $\left[\begin{smallmatrix} x_0 \\ \hat{x}_0 \end{smallmatrix}\right]$. Let's now change state vector to

$$\begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},$$

i.e. apply similarity transformation with

$$T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}.$$

Closed-loop system (contd)

We have:

$$\begin{aligned} \tilde{A}_{\rm cl} &= TA_{\rm cl}T^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & BK \\ -LC & A+BK+LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \\ &= \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \\ \tilde{B}_{\rm cl} &= TB_{\rm cl} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} Bk_r \\ Bk_r \end{bmatrix} \\ &= \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} \\ \tilde{C}_{\rm cl} &= C_{\rm cl}T^{-1} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \\ &= \begin{bmatrix} C & 0 \end{bmatrix}. \end{aligned}$$

Closed-loop system (contd)

Thus, we end up with the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} k_r r(t), \quad \begin{bmatrix} x(0) \\ \epsilon(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_0 - \hat{x}_0 \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} \end{cases}$$

Thus, closed-loop characteristic polynomial is

$$\chi_{\rm cl}(s) = \det(sI - A_K)\det(sI - A_L),$$

which is stable provided

- matrix $A_K = A + BK$ Hurwitz (i.e. state feedback is stabilizing) and
- matrix $A_L = A + LC$ Hurwitz (i.e. state observer is stable)

(separation principle).

Closed-loop transfer function from r to y

In

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} k_r r(t), \quad \begin{bmatrix} x(0) \\ \epsilon(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_0 - \hat{x}_0 \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix}. \end{cases}$$

all modes of A_L are uncontrollable (check it with PBH). This means that if initial guess $\hat{x}_0$ is correct, these modes are not excited and can be excluded. Indeed,

$$\dot{\epsilon}(t) = A_L\epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0$$

which we already know (observer). Hence, if $x_0 = \hat{x}_0$, then $\epsilon \equiv 0$ and

$$\dot{x}(t) = A_K x(t) - BK\epsilon(t) + Bk_r r(t) = A_K x(t) + Bk_r r(t), \quad x(0) = x_0$$

is independent on the dynamics of the observer.

Closed-loop transfer function from r to y (contd)

Another way to see this is via direct calculation of

$$\begin{aligned} T_{yr}(s) &= \begin{bmatrix} C & 0 \end{bmatrix} \left(s \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} k_r \\ &= C(sI - A_K)^{-1} B k_r, \end{aligned}$$

which is exactly as in the state-feedback case. It uses the relation

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

Outline

State feedback (no uncertainty)

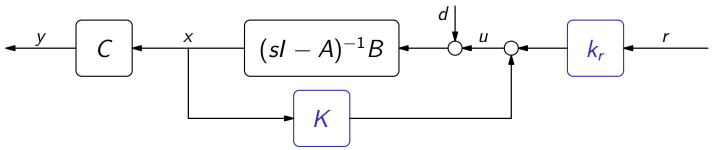
State observer (only the past is uncertain)

Observer-based output feedback (only the past is uncertain)

Effect of disturbances

2DOF state feedback (for curious)

Disturbance response of state feedback



If $\dot{x}(t) = Ax(t) + B(u(t) + d(t))$ and $u(t) = Kx(t)$, then

$$T_{yd}(s) = C(sI - A_K)^{-1} B$$

The effect of K is not immediate, although (remember Vieta's formulae)

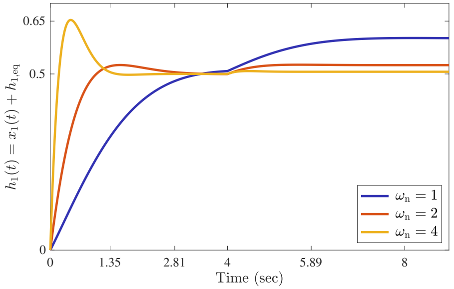
$$T_{yd}(0) = \frac{N_P(0)}{\chi_{\mathrm{cl}}(0)} = \frac{N_P(0)}{\prod_i |\lambda_i|}$$

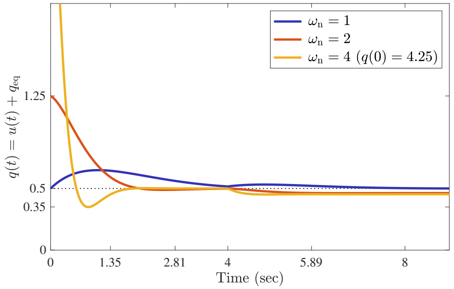
where λ_i are roots of $\chi_{\mathrm{cl}}(s)$. Hence,

- faster poles $\implies$ smaller steady-state effects of $d(t) = \mathbb{1}(t)$

Example: simulations (contd)

With $d(t) = 0.05\mathbb{1}(t-4)$,





Thus,

- faster poles $\implies$ smaller the effect of d

Luenberger observer and disturbances

If

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + d(t)), \quad x(0) = x_0 \\ y(t) = Cx(t) + n(t) \end{cases}$$

the estimator is still

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \hat{x}_0$$

(we use all information available), but the estimation error,

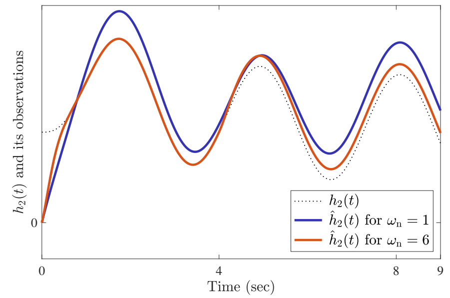
$$\dot{\epsilon}(t) = A_L\epsilon(t) + Bd(t) + Ln(t), \quad \epsilon(0) = x_0 - \hat{x}_0$$

includes both $d(t)$ and $n(t)$.

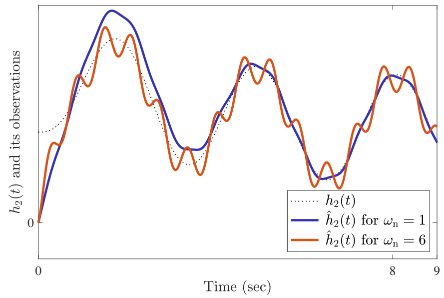
Example: simulations (contd)

Returning to our two-tank system,

$d(t) = 0.05\mathbb{1}(t-4)$ and $n(t) = 0$



$d(t) = 0$ and $n(t) = 0.025\sin(10t)$



and observations no longer converge to $h_2(t)$, with

- faster poles $\implies$ higher gain L $\implies$ smaller effect of d
- slower poles $\implies$ lower gain L $\implies$ smaller effect of n

(but be careful with generalizing that).

Closed-loop system with observer-based controller

Then the closed-loop system is

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} k_r r(t) + \begin{bmatrix} B \\ B \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ L \end{bmatrix} n(t) \\ \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} \end{cases}$$

with initial conditions $\left[\begin{smallmatrix} x_0 \\ x_0 - \hat{x}_0 \end{smallmatrix}\right]$.

In this case

$$T_{yd}(s) = \overbrace{C(sI - A_K)^{-1}B}^{\text{state-feedback } T_{yd}(s)}(1 - K(sI - A_L)^{-1}B)$$

and

$$T_{yn}(s) = -C(sI - A_K)^{-1}BK(sI - A_L)^{-1}L$$

and the effect of K and L on the closed-loop behavior is quite complicated.

Outline

State feedback (no uncertainty)

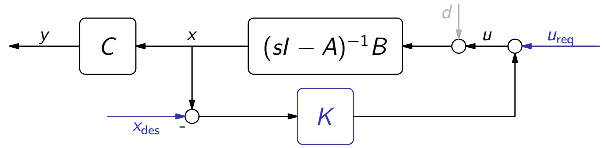
State observer (only the past is uncertain)

Observer-based output feedback (only the past is uncertain)

Effect of disturbances

2DOF state feedback (for curious)

Architecture



As usual, two requirements

1. $x_{\rm des}$ and $u_{\rm req}$ are bounded — stability
2. $x_{\rm des} = (sI - A)^{-1} B u_{\rm req}$ — consistency

In this case

$$u = u_{\rm req} + K((sI - A)^{-1} B u - x_{\rm des}) \implies u = \frac{u_{\rm req} - K x_{\rm des}}{1 - K(sI - A)^{-1} B} = u_{\rm req}$$

and $x = (sI - A)^{-1} B u = x_{\rm des}$, regardless K (provided it is stabilizing).

Generating $x_{\rm des}$: $y_{\rm des} \to u_{\rm req} = \dfrac{y_{\rm des}}{C(sI - A)^{-1} B} \to x_{\rm des} = (sI - A)^{-1} B u_{\rm req}$

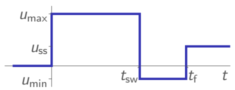
Example: time-optimal response

We may chose $y_{\rm des}$ as the

- fastest filling of tank 1 to $x_1 = x_{1,\rm f}$ under $u(t) \in [u_{\min}, u_{\max}]$

for some $-q_{\rm eq} \le u_{\min} < u_{\max}$. The optimal bang-bang

$$u_{\rm opt}(t) =$$

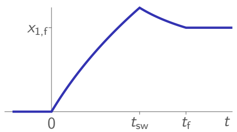


has

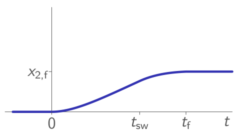
$$U_{\rm opt}(s) = \frac{u_{\max} - (u_{\max} - u_{\min}) e^{-s t_{\rm sw}} + (x_{1,\rm f}/P_1(0) - u_{\min}) e^{-s t_{\rm f}}}{s},$$

where $t_{\rm sw}$ and $t_{\rm f}$ are chosen to render $P_1(s) U_{\rm opt}(s)$ FIR. In this case

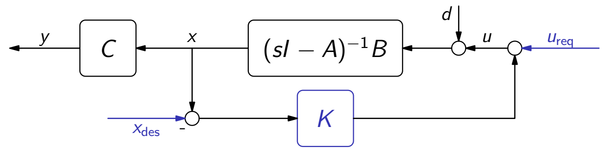
$x_{{\rm des},1}(t) =$



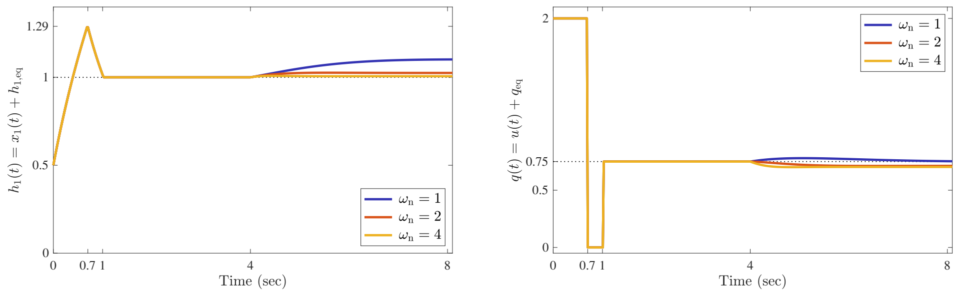
and $x_{{\rm des},2}(t) =$



Example: simulations (contd)



With $x_{1,\rm f} = 0.5$, $u_{\min} = -0.5 = -q_{\rm eq}$, $u_{\max} = 1.5$, and $u_{\rm req} = u_{\rm opt}$,



where the disturbance $d(t) = 0.05\mathbb{1}(t - 4)$.